NATIONAL JUNIOR COLLEGE
SENIOR HIGH 2
Higher 2

NAME

CLASS 2ma2 REGISTRATION NUMBER

MATHEMATICS 9758

Preliminary Examination **12 September 2022**

Paper 2 **3 hours**

Candidates answer on the Question Paper.

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name, class and registration number in the boxes above.
Please write clearly and use capital letters.

Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing and/or scientific calculator is expected, where appropriate.
All relevant working, statements and reasons must be shown in order to obtain full credit for your solution.

You are reminded of the need for clear presentation in your answers.
Up to 2 marks may be deducted for improper presentation.

The number of marks is given in the brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

Question Number	Marks Possible	Marks Obtained
1	7	
2	9	
3	13	
4	11	
5	3	
6	10	
7	7	
8	7	
9	10	
10	11	
11	12	
Presentation Deduction		– 1 / – 2
TOTAL	100	

Section A: Pure Mathematics [40 marks]

1 It is given that $f'(x) = f(x)\ln(\sec ax)$ and $f(0) = \frac{1}{2}$, where a is a real non-zero constant.

(i) Show that $f''(x)f(x) = [f'(x)]^2 + a[f(x)]^2 \tan ax$. Hence, find the Maclaurin series of $f(x)$ in terms of a, up to and including the term in x^3. [4]

Assume for the remainder of this question that $a = 3$.

The function $g(x)$ is an anti-derivative of $f(x)[1 + \ln(\cos 3x)]$, such that $g(0) = 0$.

(ii) Deduce the Maclaurin series of $g(x)$ up to and including the term in x^3. [3]

2 The function f is defined by $f(x) = \begin{cases} 3x^2 + 2, & \text{for } x \in \mathbb{R}, 0 < x \le 1, \\ \frac{1}{x}, & \text{for } x \in \mathbb{R}, x > 1. \end{cases}$

(i) Sketch the graph of $y = f(x)$. [2]

(ii) Show that f^2 exists and define f^2 in a similar form. [4]

(iii) Find f^{-1} in a form similar to f. [3]

3 A curve C has equation $y^2 - x^2 = 9$.

(i) Sketch C, labelling clearly the coordinates of the turning points, the equations of the asymptotes, and the angle between the asymptotes. [3]

(ii) Show that $\frac{d}{d\theta}(\sec\theta\tan\theta) = 2\sec^3\theta - \sec\theta$. [2]

The region bounded by C, the positive x- and y-axes, and the line $x = 4$ is denoted by S.

(iii) Use the substitution $x = 3\tan\theta$ and the result in part **(ii)** to show that the exact area of S can be expressed in the form $p + q\ln r$ for some rational constants p, q and r to be determined. [6]

(iv) Without using integration, find the exact value of $\int_3^5 \sqrt{x^2 - 9}\, dx$ in a form similar to your answer in part **(iii)**. [2]

4 A curve C has equation

$$y = \frac{a^2}{x} - \frac{1}{x-k},$$

where a and k are positive constants, with $a > 1$.

(i) Find the x-coordinates of the stationary points of C in terms of a and k. [3]

It is given that the point P is the stationary point of C with the larger x-coordinate, and L is the locus of P as a varies.

(ii) Find a cartesian equation of L as a varies, expressing y in terms of x and k. [4]

Assume now that $k = 4$. It is further given that Q has the same equation as C with $a = 3$.

(iii) Sketch Q and L on the same diagram, for $x > 4$. Label clearly the coordinates of any turning points and any points where the curves cross the x-axis, as well as the equations of any asymptotes. [4]

Section B: Probability and Statistics [60 marks]

5 The random variable W has the distribution $\mathrm{B}(n, p)$ and a mode m. By considering the inequality $\mathrm{P}(W = m) \geq \mathrm{P}(W = m+1)$, show that

$$m \geq \mathrm{E}(W) + p - 1.$$ [3]

6 Every weekday morning, Jo arrives at the bus stop nearest to her home to catch bus 156 to Bishan MRT station. The waiting time for bus 156 follows a normal distribution with mean 9 minutes and standard deviation 1.5 minutes, and the time taken for the journey from the bus stop to Bishan MRT station follows a normal distribution with mean 10 minutes and standard deviation 1.1 minutes.

(i) Find the probability that the mean waiting time for bus 156 on 50 randomly chosen weekdays exceeds 9.3 minutes. [3]

(ii) Explain why your calculations in part **(i)** would still be valid even if the waiting time for bus 156 on a weekday is not known to be normally distributed. [2]

(iii) If the train at Bishan MRT station leaves at 7.00am, find the latest time (correct to the nearest minute) that Jo has to reach the bus stop if she misses the train less than 30% of the time. [2]

(iv) In the following year, the waiting time for bus 156 is increased by 30% due to COVID-19 infections among bus drivers. Find the probability that the waiting time for a randomly chosen bus 156 in the following year differs from the original waiting time for a randomly chosen bus 156 by at most 2 minutes. [3]

7 On average, 7.8% of markers produced by a factory are faulty. Each day, the factory's quality manager picks a random sample of n markers and inspects the number of faulty markers found in the sample. The number of faulty markers found in the sample is the random variable X and whether the markers are faulty are independent of one another.

(i) Explain what is meant by a random sample in this context. [1]

(ii) Given that the probability that more than $(n-4)$ markers in the random sample are found to be non-faulty is less than 0.3, find the least possible value of n. [2]

Assume for the remainder of this question that $n = 10$.

The samples in 50 randomly chosen days are collected and the number of faulty markers in each sample is recorded.

(iii) Find the probability that exactly 35 of the samples each contains at least one faulty marker, given that none of the 50 samples contain more than 2 faulty markers each. [4]

8 The table shows the probability distribution of a discrete random variable Y.

y	-1	0	1	2	3
$\mathrm{P}(Y=y)$	$\frac{32}{81}$	a	$\frac{8}{27}$	b	c

(i) Given that $\mathrm{E}(\lvert Y-3\rvert)=\frac{232}{81}$, show that $2b+3c=\frac{19}{81}$. [3]

(ii) Given further that $\mathrm{Var}(Y)=\frac{7736}{6561}$, find the values of a, b and c in rational form. [3]

(iii) Justify if Y follows a binomial distribution. [1]

9 Osoi is frequently late for school. His mean travelling time to school is 24.5 minutes. The Year Head asks Osoi to take an alternative route to school and to record his travelling time, t minutes, from home to school every morning on the alternative route. The results for 72 randomly chosen mornings when Osoi took the alternative route are summarised as follows:

$$\sum(t-20)=215; \qquad \sum(t-20)^2=3234$$

(i) Calculate unbiased estimates of the population mean and variance of the travelling time for the alternative route. [2]

(ii) Using a 5% level of significance, test whether Osoi's mean travelling time from home to school has shortened after taking the alternative route to school. You should state your hypotheses and define any symbols that you use. [5]

Suppose the Year Head wants to investigate if the alternative route to school has made a difference to Osoi's mean travelling time in the morning instead of whether it has shortened his mean travelling time in the morning, at the same level of significance.

(iii) Find the critical region for this test. [3]

10 A financial magazine publishes an annual ranking of the financial services companies in the world. The ranking is based on sales, profit, assets and market value. A random sample of 7 pairs of assets, \$ s billion, and profit, \$ p billion, is shown in the table below.

Assets, s	326.7	394.5	491.9	2832.2	4159.9	4301.7	4914.7
Profit, p	2	2.6	3.4	17.9	31.3	39.3	65.8

(i) Draw a scatter diagram for these values, labelling the axes clearly. [1]

It is thought that the profit p can be modelled by one of the following models.

Model I: $p = as + b$ or Model II: $p = as^2 + b$,

where a and b are constants.

(ii) Find the value of the product moment correlation coefficient between

(a) s and p, [1]

(b) s^2 and p. [1]

(iii) Use your answers to parts **(i)** and **(ii)** to explain why Model II is a better model than Model I. [2]

(iv) Find the equation of the regression line of p on s^2. [1]

(v) Using the regression line in part **(iv)**, estimate the value of p if $s = 300$. Comment on the reliability of the estimate you have obtained. [2]

(vi) The original data set contains 8 pairs of data with the regression line $p = 0.010423445s - 4.043538369$. Find the value of p for the missing pair if the corresponding value of s is 3689. [3]

11 During Senior High Orientation, one of the bonding games to be played is Avalon. In each round of this game, a student is randomly selected within the Orientation Group to be the "Avalon". The student chosen to be the "Avalon" wins if the group is unsuccessful in guessing who the Avalon is within a given number of attempts.

In a particular Orientation Group, there are a total of 5 Orientation Group Leaders (OGLs), consisting of 2 boys and 3 girls, and 16 orientees, consisting of 6 boys and 10 girls. The OGLs appoints one particular girl from the 16 orientees to be the Chairperson of the Orientation Group.

The Orientation Group plays a total of five rounds of Avalon. In each of the first two rounds, the Avalon is randomly selected from the group of 5 OGLs. In each of the subsequent three rounds, the Avalon is randomly selected from the group of 16 orientees. It is possible for the same person to be randomly selected to be the Avalon in multiple rounds of the game.

(i) Find the probability that in the five rounds, the Avalons selected are all distinct. [2]

(ii) Find the probability that in the five rounds, a girl is selected in the first round and a boy is selected in the second round. [2]

(iii) For the five rounds, events A and B are defined as follows.

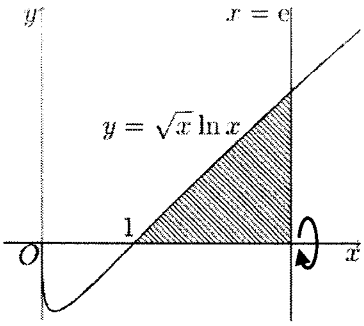
A: The same OGL is selected in both of the first two rounds.
B: Exactly two orientees are selected in the last three rounds.

Find the probability that A or B occurs, but not both. [4]

(iv) Find the probability that in the five rounds, the Chairperson of the Orientation Group is selected exactly once. [2]

(v) Find the probability that in the five rounds, all the Avalons selected are girls, given that the Chairperson of the Orientation Group is selected exactly once. [2]

Question 1 (Integration by Parts, Volume of Revolution)

(i)	$I_n = \int_1^e x(\ln x)^n \, dx$ $= \left[\frac{x^2}{2}(\ln x)^n\right]_1^e - \int_1^e \frac{x^2}{2} \cdot n(\ln x)^{n-1} \frac{1}{x} \, dx$ $= \frac{e^2}{2} - \frac{n}{2}\int_1^e x(\ln x)^{n-1} \, dx$ $= \frac{e^2}{2} - \frac{n}{2} I_{n-1}$ (shown)	
(ii)	Volume $= \pi \int_1^e x(\ln x)^2 \, dx$ $= \pi I_2$ $= \pi\left(\frac{e^2}{2} - \frac{2}{2} I_1\right)$ $= \pi\left[\frac{e^2}{2} - \left(\frac{e^2}{2} - \frac{1}{2} I_0\right)\right]$ $= \frac{\pi}{2} I_0 = \frac{\pi}{2}\left[\frac{x^2}{2}\right]_1^e$ $= \frac{\pi}{4}(e^2 - 1)$ units^3  	

Question 2 (Inequalities)

(i)

$$\frac{px^2-1}{x^2+(1-p)x-p}\ge 1$$

$$\Rightarrow \frac{px^2-1}{x^2+(1-p)x-p}-1\ge 0$$

$$\Rightarrow \frac{px^2-1-\left[x^2+(1-p)x-p\right]}{x^2+(1-p)x-p}\ge 0$$

$$\Rightarrow \frac{px^2-1-x^2+(p-1)x+p}{x^2-px+x-p}p\ge 0$$

$$\Rightarrow \frac{(p-1)x^2+(p-1)x+(p-1)}{(x+1)(x-p)}\ge 0$$

$$\Rightarrow \frac{(p-1)\left(x^2+x+1\right)}{(x+1)(x-p)}\ge 0$$

$$\Rightarrow \frac{x^2+x+1}{(x+1)(x-p)}\ge 0 \quad \text{since } p>1\Rightarrow p-1>0$$

$$\Rightarrow \frac{1}{(x+1)(x-p)}\ge 0 \quad \text{since } x^2+x+1=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}>0$$

$$\Rightarrow (x+1)(x-p)>0 \quad \text{since denominator cannot be zero}$$

$$\Rightarrow x<-1 \quad \text{or} \quad x>p$$

(ii) We observe that

$$\frac{px^2-1}{x^2+(p-1)|x|-p}=\frac{p(-|x|)^2-1}{(-|x|)^2+(1-p)(-|x|)-p}.$$

Hence we replace x by $-|x|$ in the answer obtained in **(i)**, i.e.

$$-|x|<-1 \quad \text{or} \quad -|x|>p \quad \Rightarrow \quad |x|>1 \quad \text{or} \quad |x|<-p \text{ (N.A.)}$$

$$\Rightarrow \quad x>1 \quad \text{or} \quad x<-1$$

Question 3 (Vectors I – Abstract Vectors)

(i)	Given that *OAB* is an equilateral triangle, $\lvert\mathbf{a}\rvert = \lvert\mathbf{b}\rvert$ and $\angle AOB = \frac{\pi}{3}$. $\mathbf{a}\cdot\mathbf{b} = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\angle AOB$ $= \lvert\mathbf{a}\rvert^2\cos\left(\frac{\pi}{3}\right)$ $= \frac{1}{2}\lvert\mathbf{a}\rvert^2$	
(ii)	$\overrightarrow{AC}$ $= \overrightarrow{OC} - \overrightarrow{OA}$ $= 3\mathbf{a} - 4\mathbf{b} - \mathbf{a}$ $= 2\mathbf{a} - 4\mathbf{b}$ $\overrightarrow{OA}\cdot\overrightarrow{AC}$ $= \mathbf{a}\cdot(2\mathbf{a} - 4\mathbf{b})$ $= 2\mathbf{a}\cdot\mathbf{a} - 4\mathbf{a}\cdot\mathbf{b}$ $= 2\lvert\mathbf{a}\rvert^2 - 4\left(\frac{1}{2}\lvert\mathbf{a}\rvert^2\right)$ $= 2\lvert\mathbf{a}\rvert^2 - 2\lvert\mathbf{a}\rvert^2$ $= 0$ Since $\overrightarrow{OA}\cdot\overrightarrow{AC} = 0$, $\therefore$ *OA* and *AC* are perpendicular.	
(iii)	$\mathbf{r}\times(\mathbf{b}-\mathbf{a}) = \mathbf{a}\times(\mathbf{b}-\mathbf{a})$ $\mathbf{r}\times(\mathbf{b}-\mathbf{a}) - \mathbf{a}\times(\mathbf{b}-\mathbf{a}) = \mathbf{0}$ $(\mathbf{r}-\mathbf{a})\times(\mathbf{b}-\mathbf{a}) = \mathbf{0}$ $\mathbf{r}-\mathbf{a} = \lambda(\mathbf{b}-\mathbf{a})$ $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b}-\mathbf{a}) = \overrightarrow{OA} + \lambda\overrightarrow{AB}$ The equation represents the line *AB* and $\mathbf{r}$ represents the position vector of a variable point on this line.	

Question 4 (Complex Numbers – Algebra)

(i) **Method 1 (recommended in this case as use of GC is allowed)**

Let $w = a + \mathrm{i}b$ and $u = c + \mathrm{i}d$, where a, b, c and d are real. $\therefore w^* = a - \mathrm{i}b$ and $u^* = c - \mathrm{i}d$

Since $w^* - 2\mathrm{i}u = 8$,

$a - \mathrm{i}b - 2\mathrm{i}(c + \mathrm{i}d) = 8$

$a + 2d - \mathrm{i}(b + 2c) = 8$

Comparing real and imaginary parts,

$a + 2d = 8$ ---------- (1)

$b + 2c = 0$ ---------- (2)

Since $(2\mathrm{i} - 1)w + 2u^* = 4$,

$(2\mathrm{i} - 1)(a + \mathrm{i}b) + 2(c - \mathrm{i}d) = 4$

$2\mathrm{i}a - 2b - a - \mathrm{i}b + 2c - 2\mathrm{i}d = 4$

$-a - 2b + 2c + \mathrm{i}(2a - b - 2d) = 4$

Comparing real and imaginary parts,

$-a - 2b + 2c = 4$ ---------- (3)

$2a - b - 2d = 0$ ---------- (4)

Using GC to solve the 4 linear equations,

$a = 2, b = -2, c = 1, d = 3.$

$\therefore w = 2 - 2\mathrm{i}$ and $u = 1 + 3\mathrm{i}$

Method 2

Since $w^* - 2\mathrm{i}u = 8$,

$w^* = 8 + 2\mathrm{i}u \Rightarrow w = 8^* + (2\mathrm{i}u)^* = 8 - 2\mathrm{i}u^*$ (1)

Substituting (1) into $(2\mathrm{i} - 1)w + 2u^* = 4$,

$(2\mathrm{i} - 1)(8 - 2\mathrm{i}u^*) + 2u^* = 4$

$16\mathrm{i} - 8 + 4u^* + 2\mathrm{i}u^* + 2u^* = 4$

$(6 + 2\mathrm{i})u^* = 12 - 16\mathrm{i}$

$u^* = \dfrac{12 - 16\mathrm{i}}{6 + 2\mathrm{i}} = 1 - 3\mathrm{i}$

$u = 1 + 3\mathrm{i}$

Thus, from equation (1),

$w = 8 - 2\mathrm{i}(1 - 3\mathrm{i}) = 8 - 2\mathrm{i} - 6 = 2 - 2\mathrm{i}$

(ii) **<u>Method 1 (Conjugate Root Theorem)</u>** Since the coefficients of the equation are all real, $\therefore w^*$ and u^* are also roots of the equation. $(z-w)(z-w^*)(z-u)(z-u^*)=0$ $\left[z^2-(w+w^*)z+ww^*\right]\left[z^2-(u+u^*)z+uu^*\right]=0$ $\left[z^2-(2\text{Re}(w))z+\lvert w\rvert^2\right]\left[z^2-(2\text{Re}(u))z+\lvert u\rvert^2\right]=0$ $\left[z^2-2(2)z+2^2+2^2\right]\left[z^2-2(1)z+1^2+3^2\right]=0$ $(z^2-4z+8)(z^2-2z+10)=0$ $\therefore c=8,\ d=2$	
<u>Method 2</u> $(z^2-4z+c)(z^2-dz+10)=0$ $z^2-4z+c=0$ or $z^2-dz+10=0$ $z=\dfrac{4\pm\sqrt{4^2-4c}}{2}$ or $z=\dfrac{d\pm\sqrt{d^2-40}}{2}$ $z=2\pm\sqrt{4-c}$ or $z=\dfrac{d\pm\sqrt{d^2-40}}{2}$ Comparing, $2-\sqrt{4-c}=w=2-2\text{i}$ $\Rightarrow 4-c=2^2 \Rightarrow c=4+2^2=8$ Therefore, $\dfrac{d}{2}=\text{Re}(u)=1\Rightarrow d=2$	

Question 5 (Maxima & Minima Problems)

(i) $\frac{4}{3}\pi r^3+\pi r^2 x=600$ $\pi r^2\left(\frac{4}{3}r+x\right)=600$ $\frac{4}{3}r+x=\frac{600}{\pi r^2}$ $x=\frac{600}{\pi r^2}-\frac{4}{3}r$ $V=\frac{4}{3}\pi\left(r+\frac{1}{4}\right)^3+\pi\left(r+\frac{1}{4}\right)^2 x-600$ $=\frac{4}{3}\pi\left(r+\frac{1}{4}\right)^3+\pi\left(r+\frac{1}{4}\right)^2\left(\frac{600}{\pi r^2}-\frac{4}{3}r\right)-600$ $=\left(r+\frac{1}{4}\right)^2\left[\frac{4\pi}{3}\left(r+\frac{1}{4}\right)+\frac{600}{r^2}-\frac{4\pi}{3}r\right]-600$ $=\left(r+\frac{1}{4}\right)^2\left(\frac{\pi}{3}+\frac{600}{r^2}\right)-600$	
(ii) For V to be minimum, $\frac{dV}{dr}=0$. $\frac{dV}{dr}=2\left(r+\frac{1}{4}\right)\left(\frac{\pi}{3}+\frac{600}{r^2}\right)+\left(r+\frac{1}{4}\right)^2\left[\frac{600(-2)}{r^3}\right]$ $=\left(2r+\frac{1}{2}\right)\left(\frac{\pi}{3}+\frac{600}{r^2}\right)-\frac{1200}{r^3}\left(r^2+\frac{1}{2}r+\frac{1}{16}\right)$ $=\frac{2\pi r}{3}+\frac{1200}{r}+\frac{\pi}{6}+\frac{300}{r^2}-\frac{1200}{r}-\frac{600}{r^2}-\frac{75}{r^3}$ $=\frac{2\pi r}{3}+\frac{\pi}{6}-\frac{300}{r^2}-\frac{75}{r^3}$ $\therefore \frac{2\pi r}{3}+\frac{\pi}{6}-\frac{300}{r^2}-\frac{75}{r^3}=0$ Using GC, $r=5.2322$ $\frac{d^2V}{dr^2}=\frac{2\pi}{3}-\frac{300(-2)}{r^3}-\frac{75(-3)}{r^4}$ $=\frac{2\pi}{3}+\frac{600}{r^3}+\frac{225}{r^4}$ Since $r>0$, $\frac{600}{r^3}>0$ and $\frac{225}{r^4}>0$, so $\frac{d^2V}{dr^2}>0$ for all real $r>0$. $\therefore$ V is minimum when $r=5.23$ (3sf)	

Question 6 (Sequences and Series)

(i) $u_k - u_{k+1} = \frac{1}{k!} - \frac{1}{(k+1)!} = \frac{k+1-1}{(k+1)!} = \frac{k}{(k+1)!}$ $\frac{3}{4!} + \frac{4}{5!} + \frac{5}{6!} + \ldots + \frac{3n+2}{(3n+3)!}$ $= \sum_{r=3}^{3n+2} \frac{r}{(r+1)!}$ $= \sum_{r=3}^{3n+2} \left[\frac{1}{r!} - \frac{1}{(r+1)!}\right]$ $= \sum_{r=3}^{3n+2} \left[\frac{1}{r!} - \frac{1}{(r+1)!}\right]$ $= \frac{1}{3!} - \frac{1}{4!}$ $+ \frac{1}{4!} - \frac{1}{5!}$ $\vdots$ $+ \frac{1}{(3n+1)!} - \frac{1}{(3n+2)!}$ $+ \frac{1}{(3n+2)!} - \frac{1}{(3n+3)!}$ $= \frac{1}{3!} - \frac{1}{(3n+3)!}$ $= \frac{1}{6} - \frac{1}{(3n+3)!}$	

(ii) Replace r with $h+1$: $\sum_{r=5}^{3n+3}\frac{r-1}{r!}=\sum_{h=4}^{3n+2}\frac{h}{(h+1)!}$ $=\sum_{h=3}^{3n+2}\frac{h}{(h+1)!}-\frac{3}{4!}$ $=\frac{1}{6}-\frac{1}{(3n+3)!}-\frac{1}{8}$ $=\frac{1}{24}-\frac{1}{(3n+3)!}$ $\sum_{r=5}^{3n+3}\frac{3}{r!}<\sum_{r=5}^{3n+3}\frac{r-1}{r!}$ $=\frac{1}{24}-\frac{1}{(3n+3)!}$ $<\frac{1}{24}$	

Question 7 (Arithmetic and Geometric Series)

(a) $160+(n-1)(-8)>0$ $168-8n>0$ $n<21$ Thus, Amy can cut off at most 20 pieces.	
(b) $S_\infty \le 1000 \Rightarrow \frac{160}{1-p} \le 1000$ $\Rightarrow 160 \le 1000(1-p)$ $\Rightarrow 0<p\le 0.84$ Largest value of p is 0.84. For the total length of ribbon that Bala cuts off to **not** exceed 9.5 m, $S_n \le 950 \Rightarrow \frac{160\left(1-0.84^n\right)}{1-0.84} \le 950$ $\Rightarrow 160\left(1-0.84^n\right) \le 152$ $\Rightarrow 0.84^n \ge 0.05$ $\Rightarrow n \le \frac{\ln 0.05}{\ln 0.84}$ $\Rightarrow 0<n\le 17.18$ So, Bala can cut off at most 17 pieces of string (since n is an integer).	
(c) Total length of the original long roll of ribbon $=\frac{20}{2}\left[2(160)+19(-8)\right]+\frac{160\left(1-0.84^{17}\right)}{1-0.84}+4.5$ $=1680+948.388+4.5$ $=2632.888$ $\approx 2633 \text{ cm } (\text{nearest cm})$	

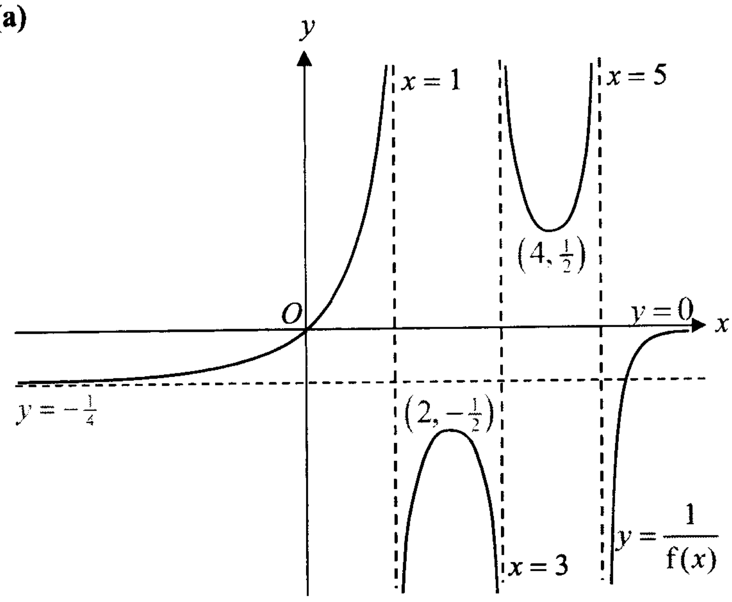
Question 8 (Complex Numbers – Geometrical Forms)

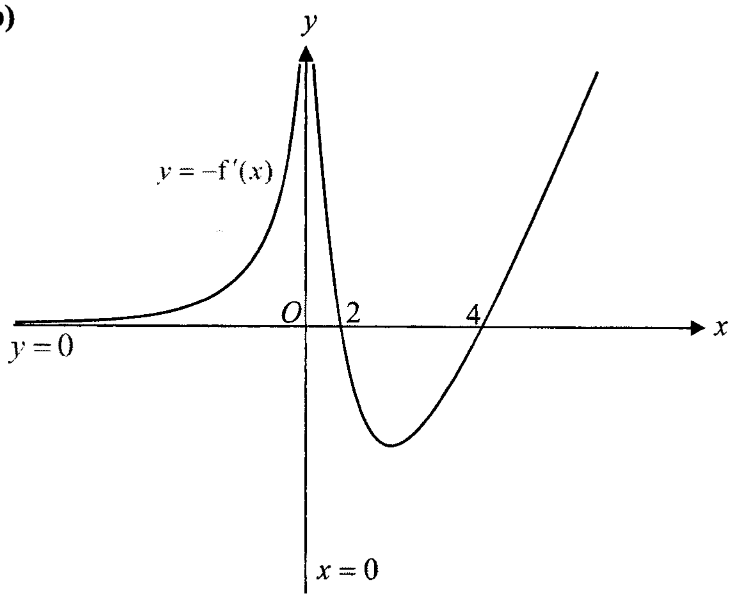
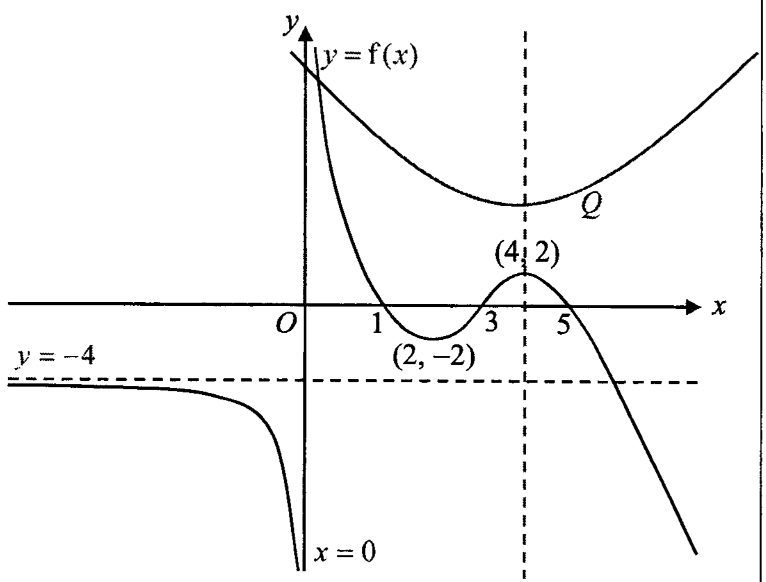
(i) $w^* = \left(\left(-\mathrm{i}\sqrt{3}\right)z\right)^* = \mathrm{i}\sqrt{3}z^*$ $= \sqrt{3}\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}\right)}2\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{4}\right)} = 2\sqrt{3}\mathrm{e}^{\mathrm{i}\left(\frac{3\pi}{4}\right)}$ $z + w^* = 2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{4}\right)} + 2\sqrt{3}\mathrm{e}^{\mathrm{i}\left(\frac{3\pi}{4}\right)}$ $= 2\mathrm{e}^{\mathrm{i}\left(\frac{3\pi}{4}\right)}\left(-\mathrm{e}^{\mathrm{i}\pi} + \sqrt{3}\right)$ $= 2\left(\sqrt{3}-1\right)\mathrm{e}^{\mathrm{i}\left(\frac{3\pi}{4}\right)}$	
(ii) Since $(z+w^*)^n$ is purely imaginary, then $\arg\left((z+w^*)^n\right) = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots$ $n\arg(z+w^*) = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots$ $\frac{3n\pi}{4} = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \pm\frac{7\pi}{2}, \ldots$ $\frac{3n}{4} = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{7}{2}, \ldots$ $= \frac{1}{2}(2k+1)$, where $k \in \mathbb{Z}$ $n = \frac{2}{3}(2k+1)$, where $k \in \mathbb{Z}$	

(iii) $\arg\left(\dfrac{z+w^*}{z^*w}\right)=\arg(z+w^*)-\arg(z^*w)$ $=\arg(z+w^*)-\arg(z^*)-\arg(w)$ $=\dfrac{3\pi}{4}-\dfrac{\pi}{4}+\dfrac{3\pi}{4}$ $=\dfrac{5}{4}\pi\equiv-\dfrac{3}{4}\pi$ Thus, $\operatorname{Im}(v)=\operatorname{Re}(v)$ $\left\|\dfrac{z+w^*}{z^*w}\right\|=\dfrac{\|z+w^*\|}{\|z\|\|w\|}$ $=\dfrac{\|z+w^*\|}{\|z\|\|(-\mathrm{i}\sqrt{3})z\|}$ $=\dfrac{\|z+w^*\|}{\sqrt{3}\|z\|^2}$ $=\dfrac{2(\sqrt{3}-1)}{4\sqrt{3}}$ $=\dfrac{\sqrt{3}-1}{2\sqrt{3}}=\dfrac{1}{2}-\dfrac{\sqrt{3}}{6}$	

Question 9 (Transformations of Graphs)

(a)

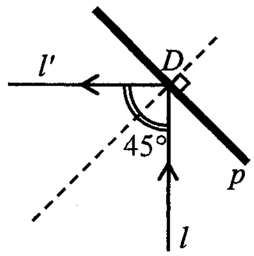


(b) 	
l.p. $y=p(x-4)^2+q$  For the two curves to intersect exactly once, $q>2$.	

Question 10 (Binomial Series, Geometric Series)

(i) $f(x)=(a+x)^n=a^n\left(1+\frac{x}{a}\right)^n$ $\approx a^n\left[1+n\left(\frac{x}{a}\right)+\frac{n(n-1)}{2!}\left(\frac{x}{a}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{x}{a}\right)^3+\frac{n(n-1)(n-2)(n-3)}{4!}\left(\frac{x}{a}\right)^4\right]$ $=a^n+na^{n-1}x+\frac{1}{2}n(n-1)a^{n-2}x^2+\frac{1}{6}n(n-1)(n-2)a^{n-3}x^3$ $+\frac{1}{24}n(n-1)(n-2)(n-3)a^{n-4}x^4$ Since the 1st, 3rd and 5th terms in the series expansion of $f(x)$ are consecutive terms of a geometric series, $\frac{\frac{1}{2}n(n-1)a^{n-2}x^2}{a^n}=\frac{\frac{1}{24}n(n-1)(n-2)(n-3)a^{n-4}x^4}{\frac{1}{2}n(n-1)a^{n-2}x^2}$ $\frac{n(n-1)x^2}{2a^2}=\frac{(n-2)(n-3)x^2}{12a^2}$ $6n(n-1)=n^2-5n+6\ (\because a,x\neq 0)$ $5n^2-n-6=0$ $(5n-6)(n+1)=0$ $n=\frac{6}{5}$ or $n=-1$	
(ii) For the Maclaurin series of $f(x)$ to converge, $\left\|\frac{x}{a}\right\|<1\Rightarrow\|x\|<a$. The common ratio of G is $\frac{-1(-1-1)x^2}{2a^2}=\frac{x^2}{a^2}$. Thus, for G to be convergent, $\left\|\frac{x^2}{a^2}\right\|<1\Rightarrow\|x\|<a$. Thus, the range of values of x for the Maclaurin series of $f(x)$ to converge is equal to the range of values of x for G to converge.	
(iii) Sum to infinity of $G=\frac{a^{-1}}{1-\frac{x^2}{a^2}}$ $=\frac{a}{a^2-x^2}$	
(iv) $0<\|x\|<a\Rightarrow 0<x^2<a^2$ $\Rightarrow -a^2<-x^2<0$ $\Rightarrow 0<a^2-x^2<a^2$ $\Rightarrow \frac{1}{a^2-x^2}>\frac{1}{a^2}$ $\Rightarrow S=\frac{a}{a^2-x^2}>\frac{1}{a}$	

Question 11 (Vectors II – Lines and Planes)

(i) Since $\begin{pmatrix} 2/3 \\ c \\ 2/3 \end{pmatrix}$ is a unit vector, $\left\|\begin{pmatrix} 2/3 \\ c \\ 2/3 \end{pmatrix}\right\| = 1$. Therefore, $\sqrt{\frac{4}{9}+c^2+\frac{4}{9}} = 1 \Rightarrow \frac{8}{9}+c^2 = 1$ $\Rightarrow c^2 = \frac{1}{9} \Rightarrow c = -\frac{1}{3}$ $\left(\text{reject } c = \frac{1}{3} \text{ since } c < 0\right)$ Thus, l has direction vector $2\mathbf{i}-\mathbf{j}+2\mathbf{k}$. $\cos 45° = \left\| \frac{\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ b \\ -1 \end{pmatrix}}{\sqrt{2^2+(-1)^2+2^2}\sqrt{b^2+1}} \right\|$ $\frac{1}{\sqrt{2}} = \left\| \frac{-b-2}{3\sqrt{b^2+1}} \right\|$  $\sqrt{\frac{b^2+1}{2}} = \left\| \frac{b+2}{3} \right\|$ $\frac{b^2+1}{2} = \left\| \frac{b+2}{3} \right\|^2 = \frac{b^2+4b+4}{9}$ $9b^2+9 = 2b^2+8b+8$ $7b^2-8b+1=0$ $(b-1)(7b-1)=0$ $b=1$ or $b=\frac{1}{7}$ (rej $\because$ $b \in \mathbb{Z}$)	

(ii) Since D lies on l, $\overrightarrow{OD}=\begin{pmatrix}5+2\lambda\\1-\lambda\\3+2\lambda\end{pmatrix}$ for some $\lambda\in\mathbb{R}$. p has equation $y-z=4\Rightarrow \mathbf{r}\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix}=4$ Since D lies on p, $\begin{pmatrix}5+2\lambda\\1-\lambda\\3+2\lambda\end{pmatrix}\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix}=4.$ $\begin{pmatrix}5+2\lambda\\1-\lambda\\3+2\lambda\end{pmatrix}\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix}=1.\ \Rightarrow 1-\lambda-3-2\lambda=4$ $\Rightarrow 3\lambda=-6$ $\Rightarrow \lambda=-2$ So $\overrightarrow{OD}=\begin{pmatrix}5+2(-2)\\1-(-2)\\3+2(-2)\end{pmatrix}=\begin{pmatrix}1\\3\\-1\end{pmatrix}$ Coordinates of D are $(1,3,-1)$.	

(iii) Let L be the normal to p passing through D and N be the foot of perpendicular of S onto L. L has equation $$\mathbf{r}=\begin{pmatrix}1\\3\\-1\end{pmatrix}+\mu\begin{pmatrix}0\\1\\-1\end{pmatrix},\ \mu\in\mathbb{R}.$$ <u>**Method 1**</u> N lies on L: $\overrightarrow{ON}=\begin{pmatrix}1\\3+\mu\\-1-\mu\end{pmatrix}$ for some $\mu\in\mathbb{R}$. $\therefore\ \overrightarrow{SN}=\begin{pmatrix}1\\3+\mu\\-1-\mu\end{pmatrix}-\begin{pmatrix}5\\1\\3\end{pmatrix}=\begin{pmatrix}-4\\2+\mu\\-4-\mu\end{pmatrix}$ Since $SN\perp$ normal to p, $\overrightarrow{SN}\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix}=0$. Thus, $\begin{pmatrix}-4\\2+\mu\\-4-\mu\end{pmatrix}\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix}=0 \Rightarrow 2+\mu+4+\mu$ $\Rightarrow 2\mu=-6$ $\Rightarrow \mu=-3$ So $\overrightarrow{ON}=\begin{pmatrix}1\\3+(-3)\\-1-(-3)\end{pmatrix}=\begin{pmatrix}1\\0\\2\end{pmatrix}$	

Method 2

$\overrightarrow{DN}$ is the projection vector of $\overrightarrow{DS}$ onto the normal to p. Thus,

$$\overrightarrow{DN} = \left(\frac{\overrightarrow{DS} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}{1^2 + (-1)^2 + 0^2} \right) \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$= \left(\frac{1}{2} \begin{pmatrix} 4 \\ -2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right) \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \frac{-2-4}{2} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 3 \end{pmatrix}$$

$$\therefore \overrightarrow{ON} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ -3 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

By Ratio Theorem, $\overrightarrow{ON} = \dfrac{\overrightarrow{OS} + \overrightarrow{OS'}}{2}$. Thus,

$$\overrightarrow{OS'} = 2\overrightarrow{ON} - \overrightarrow{OS} = 2\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$$

Thus, S' has coordinates $(-3, -1, 1)$.

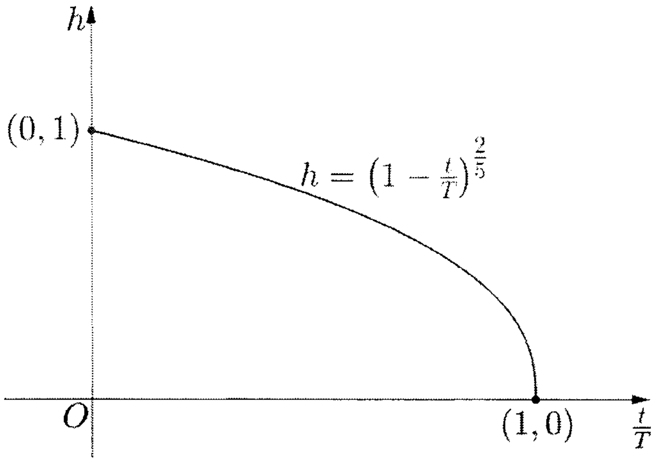
$$\overrightarrow{DS'} = \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \\ 2 \end{pmatrix}$$

Thus, l' has eqn $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}, \alpha \in \mathbb{R}$.

Method 3 A vector that is perpendicular to both l and L is $$\begin{pmatrix}2\\-1\\2\end{pmatrix}\times\begin{pmatrix}0\\1\\-1\end{pmatrix}=\begin{pmatrix}-1\\2\\2\end{pmatrix}$$ Thus, a vector that is parallel to l' is $$\begin{pmatrix}2\\-1\\2\end{pmatrix}\times\begin{pmatrix}-1\\2\\2\end{pmatrix}=\begin{pmatrix}-6\\-6\\3\end{pmatrix}=-3\begin{pmatrix}2\\2\\-1\end{pmatrix}$$ Thus, l' has eqn $\mathbf{r}=\begin{pmatrix}1\\3\\-1\end{pmatrix}+\beta\begin{pmatrix}2\\2\\-1\end{pmatrix}, \beta\in\mathbb{R}$. S' lies on l': $\overrightarrow{OS'}=\begin{pmatrix}1+2\beta\\3+2\beta\\-1-\beta\end{pmatrix}$ for some $\beta\in\mathbb{R}$. $\overrightarrow{SS'}=\begin{pmatrix}1+2\beta\\3+2\beta\\-1-\beta\end{pmatrix}-\begin{pmatrix}5\\1\\3\end{pmatrix}=\begin{pmatrix}-4+2\beta\\2+2\beta\\-4-\beta\end{pmatrix}$ Since $\overrightarrow{SS'}\perp L$, $\overrightarrow{SS'}\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix}=0$. Thus, $\begin{pmatrix}-4+2\beta\\2+2\beta\\-4-\beta\end{pmatrix}\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix}=0 \quad \Rightarrow 2+2\beta+4+\beta=0$ $\Rightarrow 3\beta+6=0$ $\Rightarrow \beta=-2$ So $\overrightarrow{OS'}=\begin{pmatrix}1+2(-2)\\3+2(-2)\\-1-(-2)\end{pmatrix}=\begin{pmatrix}-3\\-1\\1\end{pmatrix}$ Thus, S' has coordinates $(-3,-1,1)$.	

Question 12 (Differential Equations)

(i) By similar triangles, radius of the water surface at time t s is $\frac{0.5}{1}h=\frac{h}{2}$ m $W=\frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$ (by similar triangles) $=\frac{1}{12}\pi h^3$	
(ii) Since no water is added to the funnel, $\frac{\mathrm{d}W}{\mathrm{d}t}=-\pi a^2 v$ $=-\pi a^2\sqrt{2gh}$ $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{12}\pi h^3\right)=-\pi a^2\sqrt{2gh}$ $\frac{1}{4}\pi h^2\frac{\mathrm{d}h}{\mathrm{d}t}=-\pi a^2\sqrt{2g}\cdot h^{\frac{1}{2}}$ $\frac{\mathrm{d}h}{\mathrm{d}t}=-\left(4a^2\sqrt{2g}\right)h^{-\frac{3}{2}}$ or $h^{\frac{3}{2}}\frac{\mathrm{d}h}{\mathrm{d}t}=-4a^2\sqrt{2g}$	
(iii) $\int h^{\frac{3}{2}}\frac{\mathrm{d}h}{\mathrm{d}t}\,\mathrm{d}t=-\int 4a^2\sqrt{2g}\,\mathrm{d}t$ $\frac{2}{5}h^{\frac{5}{2}}=-\left(4a^2\sqrt{2g}\right)t+c$ Since $h=1$ when $t=0$, $c=\frac{2}{5}$. Therefore, $\frac{2}{5}h^{\frac{5}{2}}=-\left(4a^2\sqrt{2g}\right)t+\frac{2}{5}$ $h^{\frac{5}{2}}=1-\left(10a^2\sqrt{2g}\right)t$ $h=\left[1-\left(10a^2\sqrt{2g}\right)t\right]^{\frac{2}{5}}$, $k=10$ and $p=\frac{2}{5}$ (shown)	
(iv) For the funnel to become empty, $h=0$. $1-\left(10a^2\sqrt{2g}\right)T=0$ $T=\frac{1}{10a^2\sqrt{2g}}$	

(v) $h=\left[1-\left(10a^2\sqrt{2g}\right)t\right]^{\frac{2}{5}}=\left(1-\frac{t}{T}\right)^{\frac{2}{5}}$ 	

Question 1 (Power Series)

(i) $f'(x) = f(x)\ln(\sec ax) \quad (1)$

Differentiating (1) both sides implicitly wrt x,

$$f''(x) = f'(x)\ln(\sec ax) + f(x)\left(\frac{a\sec ax \tan ax}{\sec ax}\right)$$

$$f''(x) = f'(x)\left[\frac{f'(x)}{f(x)}\right] + f(x)\left(a\tan ax\right)$$

$$f''(x)f(x) = \left[f'(x)\right]^2 + a\left[f(x)\right]^2 \tan(ax)(\text{shown})$$

$$f'''(x)f(x) + f''(x)f'(x)$$
$$= 2f'(x)f''(x) + \left[f(x)\right]^2 a^2 \sec^2 ax$$
$$+ 2a f(x)f'(x)\tan ax$$

$$f'''(x)f(x) - f''(x)f'(x)$$
$$= \left[f(x)\right]^2 a^2 \sec^2 ax + 2a f(x)f'(x)\tan ax$$

$$f(0) = \frac{1}{2},\ f'(0) = 0,\ f''(0) = 0,\ f'''(0) = \frac{1}{2}a^2$$

$$\therefore f(x) = \frac{1}{2} + \frac{\left(\frac{1}{2}a^2\right)}{3!}x^3 + \cdots = \frac{1}{2} + \frac{1}{12}a^2x^3 + \cdots$$

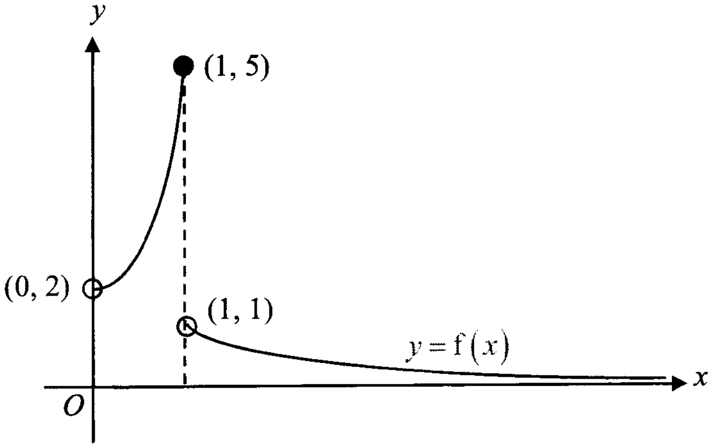
(ii)

$$f(x)\left[1 + \ln(\cos 3x)\right] = f(x)\left[1 - \ln(\sec 3x)\right]$$
$$= f(x) - f(x)\ln(\sec 3x)$$
$$= f(x) - f'(x)$$
$$= \frac{1}{2} - \frac{1}{4}(3)^2x^2 + \cdots$$
$$= \frac{1}{2} - \frac{9}{4}x^2 + \cdots$$

$$g(x) = \int \frac{1}{2} - \frac{9}{4}x^2 + \cdots \, dx$$
$$= c + \frac{1}{2}x - \frac{3}{4}x^3 + \cdots$$

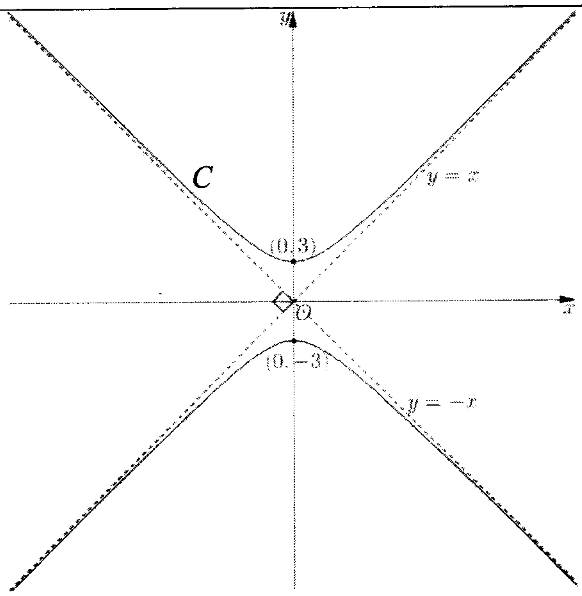
Since $g(0) = 0$, $c = 0$. $\therefore g(x) = \frac{1}{2}x - \frac{3}{4}x^3 + \cdots$

Question 2 (Functions)

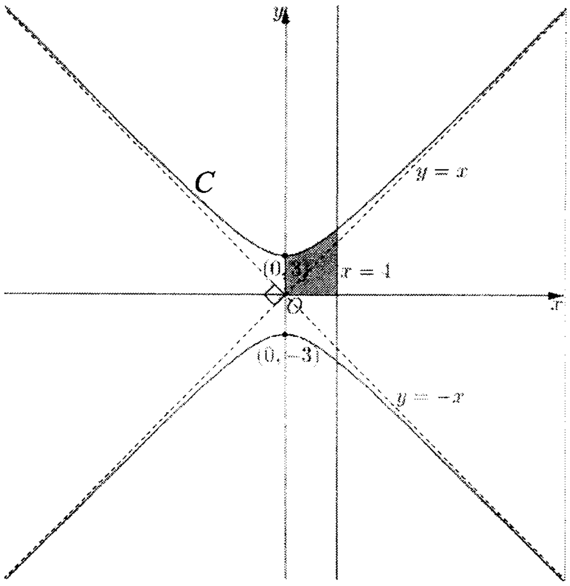
(i) 	
(ii) $D_f = (0, \infty)$, $R_f = (0,1) \cup (2,5]$ Since $R_f \subseteq D_f$, If $0 < x \leq 1$, $f^2(x) = f(3x^2+2) = \dfrac{1}{3x^2+2}$ If $x > 1$, $f^2(x) = f\left(\dfrac{1}{x}\right) = 3\left(\dfrac{1}{x}\right)^2 + 2 = \dfrac{3}{x^2} + 2$ $\therefore f^2(x) = \begin{cases} \dfrac{1}{3x^2+2}, & \text{for } x \in \mathbb{R}, 0 < x \leq 1, \\ \dfrac{3}{x^2}+2, & \text{for } x \in \mathbb{R}, x > 1. \end{cases}$	

(iii) From the graph in part **(i)**, we see that for $2 < y \le 5$, $y = \mathrm{f}(x) = 3x^2 + 2$ $3x^2 = y - 2$ $x^2 = \dfrac{y-2}{3}$ $x = \sqrt{\dfrac{y-2}{3}}\ (\because x > 0)$ and for $0 < y < 1$, $y = \mathrm{f}(x) = \dfrac{1}{x} \Rightarrow x = \dfrac{1}{y}$ Thus, $\mathrm{f}^{-1}(x) = \begin{cases} \dfrac{1}{x}, & \text{for } x \in \mathbb{R}, 0 < x < 1, \\ \sqrt{\dfrac{x-2}{3}}, & \text{for } x \in \mathbb{R}, 2 < x \le 5. \end{cases}$	

Question 3 (Conics, Area, Integration by Substitution)

<table>
<tr><td>(i) 
</td><td></td></tr>
<tr><td>(ii) $\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sec\theta\tan\theta\right)=\sec\theta\tan\theta\cdot\tan\theta+\sec\theta\cdot\sec^2\theta$
$=\sec\theta\tan^2\theta+\sec^3\theta$
$=\sec\theta\left(\sec^2\theta-1\right)+\sec^3\theta$
$=2\sec^3\theta-\sec\theta$ (shown)</td><td></td></tr>
</table>

(iii)



$$\text{Area} = \int_0^4 y \, dx$$

$$= \int_0^4 \sqrt{x^2+9} \, dx$$

$$= \int_0^{\tan^{-1}\frac{4}{3}} \sqrt{9\tan^2\theta+9}\left(3\sec^2\theta\right) d\theta$$

$$= \int_0^{\tan^{-1}\frac{4}{3}} 3\sec\theta \cdot 3\sec^2\theta \, d\theta$$

$$= 9\int_0^{\tan^{-1}\frac{4}{3}} \sec^3\theta \, d\theta$$

$$= \frac{9}{2}\int_0^{\tan^{-1}\frac{4}{3}} \left(2\sec^3\theta - \sec\theta\right) + \sec\theta \, d\theta$$

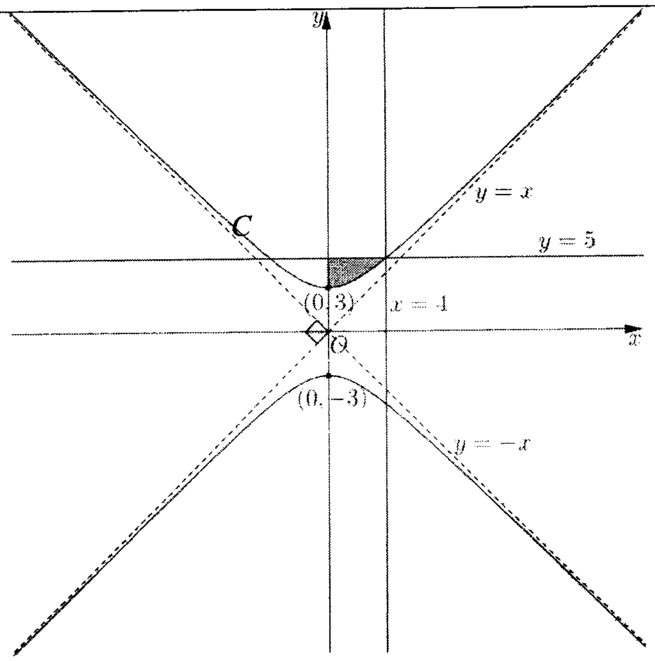
$$= \frac{9}{2}\left[\sec\theta\tan\theta + \ln\left|\sec\theta+\tan\theta\right|\right]_0^{\tan^{-1}\frac{4}{3}}$$

$$= \frac{9}{2}\left[\left(\frac{\sqrt{4^2+3^2}}{3}\right)\left(\frac{4}{3}\right) + \ln\left|\frac{\sqrt{4^2+3^2}}{3}+\frac{4}{3}\right|\right]$$

$$= \frac{9}{2}\left(\frac{20}{9} + \ln 3\right)$$

$$= 10 + \frac{9}{2}\ln 3 \text{ units}^2$$

(iv)

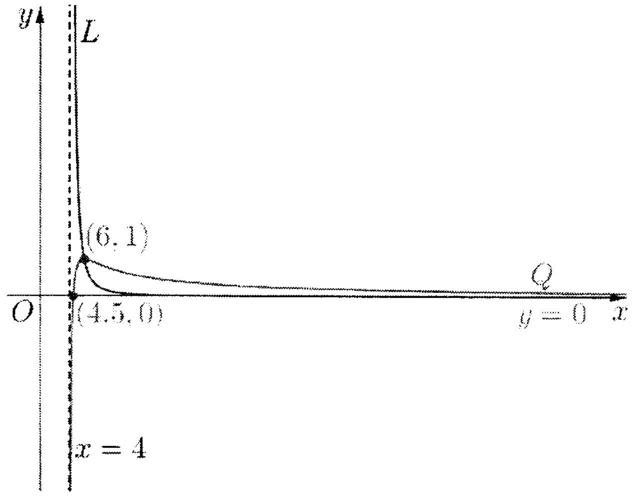


$$\int_3^5 \sqrt{x^2-9}\,\mathrm{d}x = \int_3^5 \sqrt{y^2-9}\,\mathrm{d}y$$
$$= 4\cdot 5 - \text{area of } S$$
$$= 20 - \left(10 + \frac{9}{2}\ln 3\right)$$
$$= 10 - \frac{9}{2}\ln 3$$

Question 4 (Curve Sketching)

(i) $y=\frac{a^2}{x}-\frac{1}{x-k} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{a^2}{x^2}+\frac{1}{(x-k)^2}$ At the stationary points, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$. Thus, $-\frac{a^2}{x^2}+\frac{1}{(x-k)^2}=0$ $\frac{1}{(x-k)^2}=\frac{a^2}{x^2}$ $\left(\frac{x}{x-k}\right)^2=a^2$ $\frac{x}{x-k}=\pm a$ $\frac{x}{x-k}=a$ or $\frac{x}{x-k}=-a$ $x=ax-ak$ or $x=-ax+ak$ $(a-1)x=ak$ or $(a+1)x=ak$ $x=\frac{ak}{a-1}$ or $x=\frac{ak}{a+1}$	

(ii) The larger x-coordinate of the two stationary points is $x=\frac{ak}{a-1}$. The y-coordinate of this point is $y=\frac{a^2}{\left(\frac{ak}{a-1}\right)}-\frac{1}{\frac{ak}{a-1}-k}$ $=\frac{a(a-1)}{k}-\frac{a-1}{ak-k(a-1)}$ $=\frac{a^2-a}{k}-\frac{a-1}{k}$ $=\frac{a^2-2a+1}{k}$ $=\frac{(a-1)^2}{k}$ So the locus of P has parametric equations $\begin{cases} x=\frac{ak}{a-1} & (1) \\ y=\frac{(a-1)^2}{k} & (2) \end{cases}$ From (1), $x=\frac{ak}{a-1} \Rightarrow ax-x=ak$ $\Rightarrow ax-ak=x$ $\Rightarrow a(x-k)=x$ $\Rightarrow a=\frac{x}{x-k}$ (3) Substituting (3) into (2), $y=\frac{(a-1)^2}{k}$ $y=\frac{\left(\frac{x}{x-k}-1\right)^2}{k}$ $y=\frac{\left(\frac{x-x+k}{x-k}\right)^2}{k}$ $y=\frac{k}{(x-k)^2}$	

(iii) 	

Question 5 (Binomial Distribution – Mode)

$W \sim \mathrm{B}(n,p)$ $\mathrm{P}(W=m) \geq \mathrm{P}(W=m+1)$ $\Rightarrow \binom{n}{m}(p)^m(1-p)^{n-m} \geq \binom{n}{m+1}(p)^{m+1}(1-p)^{n-m-1}$ $\Rightarrow \frac{n!}{m!(n-m)!}(1-p) \geq \frac{n!}{(m+1)!(n-m-1)!}(p)$ $\Rightarrow (m+1)!(n-m-1)!(1-p) \geq m!(n-m)!(p)$ $\Rightarrow (1-p)(m+1) \geq p(n-m)$ $\Rightarrow m+1-mp-p \geq np-mp$ $\Rightarrow m \geq np+p-1$ $\Rightarrow m \geq \mathrm{E}(W)+p-1$ (since $\mathrm{E}(W)=np$)(shown)	

Question 6 (Normal Distribution)

(i)	Let X be the the waiting time for a randomly chosen bus 55 on a weekday. $X \sim \mathrm{N}\left(9, \frac{1.5^2}{50}\right)$ Let $\overline{X} = \frac{X_1 + X_2 + \cdots + X_{50}}{50} \sim \mathrm{N}\left(9, \frac{1.5^2}{50}\right)$ $\mathrm{P}(\overline{X} > 9.3) = 0.0786$	
(ii)	Since the number of days, 50, is large, the sample mean waiting time follows a normal distribution approximately by Central Limit Theorem. Therefore the calculations in part **(i)** will still be valid.	
(iii)	Let Y be the journey time (in minutes) to the MRT station for a randomly chosen bus on a weekday, and let t be the time taken (in minutes) such that she misses the train less than 30% of the time. $Y \sim \mathrm{N}\left(10,\ 1.1^2\right)$ $X + Y \sim \mathrm{N}(19, 3.46)$ $\mathrm{P}(X + Y > t) < 0.3$ $t > 19.975$ Least $t = 20$ mins Latest time to leave home is 6.40 am	
(iv)	Let $W = 1.3X_1 - X_2 \sim \mathrm{N}\left(2.7, 1.3^2\left(1.5^2\right) + 1.5^2\right)$ $\mathrm{P}(-2 \le W \le 2) = 0.3599 \approx 0.360$	

Question 7 (Binomial Distribution – Finding Probabilities)

(i) In this context, a random sample is one such that every marker produced has equal probability of being selected in this sample and the markers are selected independently of one another.

(ii) $X \sim \mathrm{B}(n, 0.078)$

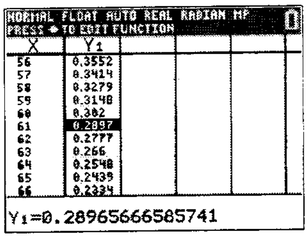
If $n - X$ markers are not faulty, X markers are faulty. Thus,

$$\mathrm{P}(n - X > n - 4) < 0.3$$

$$\mathrm{P}(X < 4) < 0.3$$

$$\mathrm{P}(X \le 3) < 0.3$$

Using GC and from the table,


NORMAL FLOAT AUTO REAL RADIAN MP
PRESS ◆ TO EDIT FUNCTION

X	Y1
56	0.3552
57	0.3414
58	0.3279
59	0.3148
60	0.302
61	0.2897
62	0.2777
63	0.266
64	0.2548
65	0.2439
66	0.2334

Y1=0.28965666585741

Least value of n is 61.

(iii) $X \sim \mathrm{B}(10, 0.078)$

$\mathrm{P}(X = 0) = 0.4439246$

$\mathrm{P}(X \le 2) = 0.962450$

$\mathrm{P}(1 \le X \le 2) = \mathrm{P}(X \le 2) - \mathrm{P}(X = 0) = 0.518526$

Thus, the required probability is

$$\frac{\binom{50}{35}\left(\mathrm{P}(1 \le X \le 2)\right)^{35}\left(\mathrm{P}(X = 0)\right)^{15}}{\left(\mathrm{P}(X \le 2)\right)^{50}}$$

$$= \frac{\binom{50}{35}(0.518526)^{35}(0.4439246)^{15}}{(0.962450)^{50}}$$

$= 0.0081291$

$= 0.00813$ (to 3 s.f.)

Alternatively, let Y be the number of samples with at least one faulty marker, out of 50 samples with at most 2 faulty markers each. Then Y is binomially distributed with probability of success

$$p' = \frac{\mathrm{P}(1 \le X \le 2)}{\mathrm{P}(X \le 2)} = \frac{0.518526}{0.962450} = 0.538756$$

Thus, $Y \sim \mathrm{B}(50, 0.538756)$. So,

$\mathrm{P}(Y = 35) = 0.0081288$

$= 0.00813$ (to 3 s.f.)

Question 8 (Discrete Random Variables)

(i)	Since sum of all probabilities = 1, $\frac{32}{81}+a+\frac{8}{27}+b+c=1 \Rightarrow a+b+c=1-\frac{32}{81}-\frac{8}{27}=\frac{25}{81}$ --- (1) Since $\mathrm{E}(\lvert Y-3\rvert)=\frac{232}{81}$, $\lvert -1-3\rvert\frac{32}{81}+\lvert 0-3\rvert a+\lvert 1-3\rvert\frac{8}{27}+\lvert 2-3\rvert b+\lvert 3-3\rvert c$ $=4\left(\frac{32}{81}\right)+3a+2\left(\frac{8}{27}\right)+b$ $4\left(\frac{32}{81}\right)+3a+2\left(\frac{8}{27}\right)+b=\frac{232}{81} \Rightarrow 3a+b=\frac{56}{81}$ --- (2) $3\times(1)-(2)$: $3(a+b+c)-(3a+b)=3\times\frac{25}{81}-\frac{56}{81}$ $2b+3c=\frac{19}{81}$ (shown) --- (3) **Alternatively**, we observe that the possible values of Y are all not more than 3. Thus, $\mathrm{E}(\lvert Y-3\rvert)=\frac{232}{81} \Rightarrow \mathrm{E}(3-Y)=\frac{232}{81} \Rightarrow 3-\mathrm{E}(Y)=\frac{232}{81}$ $\Rightarrow \mathrm{E}(Y)=3-\frac{232}{81}=\frac{11}{81}$ Hence, we have $(-1)\times\frac{32}{81}+0\times a+1\times\frac{8}{27}+2b+3c=\frac{11}{81}$ $-\frac{8}{81}+2b+3c=\frac{11}{81}$ $2b+3c=\frac{19}{81}$ (shown) --- (3)	

(ii) $\mathrm{E}(Y)=-\frac{32}{81}+\frac{8}{27}+2b+3c$ $=-\frac{8}{81}+2b+3c$ $=-\frac{8}{81}+\frac{19}{81}$ (from **(i)**) $=\frac{11}{81}$ Since $\mathrm{Var}(Y)=\frac{7736}{6561}$, $\mathrm{E}(Y^2)-(\mathrm{E}(Y))^2=\frac{7736}{6561}$ $\frac{32}{81}+\frac{8}{27}+4b+9c-\left(\frac{11}{81}\right)^2=\frac{7736}{6561}$ $4b+9c=\frac{7736}{6561}-\frac{56}{81}+\left(\frac{11}{81}\right)^2$ $4b+9c=\frac{41}{81}$ --- (4) By using the Simultaneous Equation Solver in the GC to solve equations (1), (3) and (4) simultaneously, $a=\frac{16}{81}$, $b=\frac{8}{81}$, $c=\frac{1}{81}$	
(iii) Y does not follow a binomial distribution since it can possibly take a value of -1 which is not permissible in a binomial distribution that counts the number of success.	

Question 9 (Hypothesis Testing)

(i) Let $x = t - 20$. Then $\sum x = 215$; $\sum x^2 = 3234$.

$\bar{x} = \bar{t} - 20$

$\Rightarrow \bar{t} = \bar{x} + 20$

$\Rightarrow \bar{t} = \frac{215}{72} + 20 = \frac{1655}{72} = 22.986 = 23.0$ (to 3 s.f.)

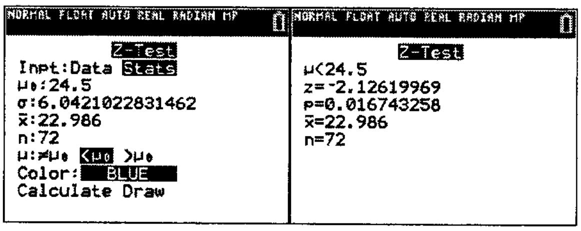
$s^2 = \frac{1}{71}\left(3234 - \frac{215^2}{72}\right) = 36.507 = 36.5$ (to 3 s.f.)

(ii) Let μ be Osoi's population mean travelling time (in minutes) from home to school on a randomly chosen morning after he started taking the alternative route

To test $H_0 : \mu = 24.5$ against $H_1 : \mu < 24.5$

Level of significance, $\alpha = 0.05$ (lower-tailed)

Under H_0, $Z = \frac{\bar{T} - 24.5}{\sqrt{\frac{36.507}{72}}} \sim N(0,1)$ approximately by Central Limit Theorem, since $n = 72$ is large (> 30).

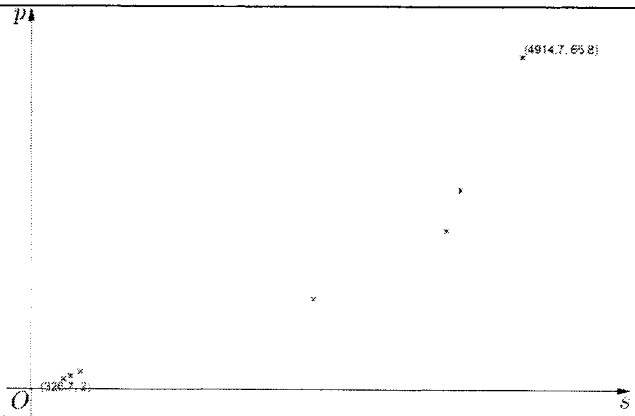
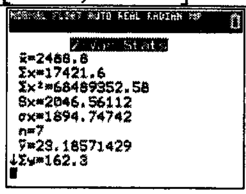


Method 1: p-value	Method 2: test statistic
p-value = 0.0167	Test statistic, $z_{calc} = -2.13$
$\alpha = 0.05$	Critical value, $z_{crit} = z_{0.05} = -1.645$
$\therefore p < \alpha$	$\therefore z_{calc} < z_{crit}$

Thus, we reject H_0. We conclude that there is **sufficient evidence** at the **5% level of significance** to claim that **Osoi's population mean travelling time from home to school in the morning has shortened after taking the alternative route to school.**

(iii) Test $H_0 : \mu = 24.5$ against $H_1 : \mu \neq 24.5$ Level of significance, $\alpha = 0.05$ (two-tailed) Standardised critical region: $z \le -1.960$ or $z \ge 1.960$ Under H_0, $Z = \dfrac{\overline{T} - 24.5}{\sqrt{\dfrac{36.507}{72}}} \sim N(0,1)$ approximately by Central Limit Theorem, since $n = 72$ is large (> 30). The observed test statistic value is $\dfrac{\overline{t} - 24.5}{\sqrt{\dfrac{36.507}{72}}}$. Thus the critical region for this test is given by $\left\lvert \dfrac{\overline{t} - 24.5}{\sqrt{\dfrac{36.507}{72}}} \right\rvert \ge 1.960$ $\lvert \overline{t} - 24.5 \rvert \ge 1.960\sqrt{\dfrac{36.507}{72}}$ $\overline{t} \le 24.5 - 1.960\sqrt{\dfrac{36.507}{72}}$ or $\overline{t} \ge 24.5 + 1.960\sqrt{\dfrac{36.507}{72}}$ $\overline{t} \le 23.1$ or $\overline{t} \ge 25.9$	

Question 10 (Correlation and Regression)

(i)		
(ii) (a)	The product moment correlation coefficient between s and $p = 0.928578571 = 0.929$ (3 s.f)	
(ii) (b)	The product moment correlation coefficient between s^2 and $p = 0.970297308 = 0.970$ (3 s.f.)	
(iii)	The scatter plot appears to follow a curve that is increasing with an increasing gradient (concave upwards). Also, the r-value for model II, 0.970, has an **absolute** value that is closer to 1 than the r-value for model I, 0.929. Thus, Model II is better.	
(iv)	Regression line of p on s^2: $p=\left(2.284063863\times10^{-6}\right)s^2+0.8379921133$ $p=\left(2.28\times10^{-6}\right)s^2+0.838$	
(v)	For $s=300$, $p=\left(2.284063863\times10^{-6}\right)(300)^2+0.8379921133$ $=1.043557858$ $=1.04$ (to 3s.f.) The estimate is not reliable as the given value of s, 300, does not lie in the data range for s, [326.7, 4914.7]	
(vi)	$p=0.010423445s-4.043538369$  $\bar{s}=\dfrac{17421.6+3689}{8}=2638.825$ $(\bar{s},\bar{p})$ lies on the regression line: $\bar{p}=0.010423445(2638.825)-4.043538369$ $=23.46210888$ Thus, missing $p=23.46210888\times8-162.3=25.39687107=25.4$ (3sf)	

Question 11 (Permutations and Combinations, Probability)

(i)	P(selections are all distinct) $=\dfrac{\left[\binom{5}{2}\times 2!\right]\times\left[\binom{16}{3}\times 3!\right]}{5^2\times 16^3}$ or $\dfrac{5\times 4}{5^2}\times\dfrac{16\times 15\times 14}{16^3}$ $=\dfrac{21}{32}$	
(ii)	P(girl in 1st round, boy in 2nd round) $=\dfrac{\binom{3}{1}\binom{2}{1}}{5^2}$ $=\dfrac{6}{25}$	
(iii)	$P(A)=\dfrac{\binom{5}{1}}{5^2}=\dfrac{1}{5}$ $P(B)=\dfrac{\binom{16}{1}\binom{15}{1}\times\dfrac{3!}{2!}}{16^3}=\dfrac{45}{256}$ Since the outcomes of the selections of the OGLs and the orientees bear no influence on each other, the two events are independent. So, $P(A\cap B)=P(A)\times P(B)$ P(A occurs or B occurs but not both) $=P(A\cup B)-P(A\cap B)$ $=P(A)+P(B)-P(A\cap B)-P(A\cap B)$ $=P(A)+P(B)-2P(A\cap B)$ $=P(A)+P(B)-2P(A)\times P(B)$ $=\dfrac{1}{5}+\dfrac{45}{256}-2\left(\dfrac{1}{5}\right)\left(\dfrac{45}{256}\right)$ $=\dfrac{391}{1280}$	

Alternatively, $P(\text{exactly one OGL and exactly 1 or 3 orientees})$ $=\frac{1}{5}\left[\left(\frac{1}{16}\right)^2+\frac{15}{16}\times\frac{14}{16}\right]=\frac{211}{1280}$ $P(\text{two OGLs and exactly 2 orientees})$ $=\frac{4}{5}\binom{3}{1}\left(\frac{1}{16}\times\frac{15}{16}\right)=\frac{9}{64}$ Required Probability $=\frac{211}{1280}+\frac{9}{64}=\frac{391}{1280}$	
(iv) $P(\text{Chairperson selected exactly once})$ $=\frac{1\times 15^2\times\binom{3}{1}}{16^3}$ $=\frac{675}{4096}$	
(v) $P(\text{all girls} \mid \text{Chairperson selected exactly once})$ $=\frac{n(\text{all girls, Chairperson selected exactly once})}{n(\text{Chairperson selected exactly once})}$ $=\frac{1\times 9^2\times\binom{3}{1}\times 3^2}{1\times 15^2\times\binom{3}{1}\times 5^2}$ $=\frac{81}{625}$	