2022 TJC Preliminary Examination H2 Mathematics Paper 1

1 Find

(a) $\int \sin^{-1} 2x \, dx$, [3]

(b) $\int \frac{1}{x^2 - nx + n^2} \, dx$, where n is a constant. [2]

2 **Do not use a calculator in answering this question.**

The equation $3z^2 - (5 + i)z = k$, where k is a real constant, has a root $1 + i$.

(i) Find the value of k. [2]

(ii) Find the other complex root. [3]

3 **(i)** On the same axes, sketch the graphs of $y = |x + a|$ and $y = |ax + 1|$ where $0 < a < 1$, and hence solve the inequality $|x + a| < |ax + 1|$. [4]

(ii) Deduce the solution to the inequality $\left|\frac{1}{x} + a\right| < \left|\frac{a}{x} + 1\right|$. [2]

4 It is given that $\ln y = 1 + \tan^{-1}(2x)$.

(i) Show that $(1 + 4x^2)\frac{d^2y}{dx^2} + (8x - 2)\frac{dy}{dx} = 0$. [2]

(ii) Find the Maclaurin series for y, up to and including the term in x^3, giving exact coefficients for each term. [4]

(iii) Hence find the Maclaurin series for $e^{\tan^{-1}(2x)}$, up to and including the term in x^3. [2]

5 With reference to the origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel.

(i) The point C has position vector $\mathbf{c}$ given by $\mathbf{c} = 9\mathbf{a} - 6\mathbf{b}$. Find, in terms of $\mathbf{a}$ and $\mathbf{b}$, the position vector of the point where the lines OC and AB meet. [4]

(ii) The point D has position vector $\mathbf{d}$ given by $\mathbf{d} = t\mathbf{a} + (1 - t)\mathbf{b}$ where t is a constant. Given that $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = 4\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$, and the angle between OA and OD is 60°, form a quadratic equation in t and hence solve for the exact values of t. [4]

6 The sum of the first r terms of a sequence of real numbers $u_1, u_2, u_3, \ldots$ is denoted by S_r.

It is given that $S_{r+1} = \dfrac{S_r}{1+(2r-1)S_r}$, for $r \geq 1$, and $u_1 = 1$.

(i) Show that $\dfrac{1}{S_{r+1}} - \dfrac{1}{S_r} = 2r - 1$. [1]

(ii) By considering $\displaystyle\sum_{r=1}^{n-1}\left(\frac{1}{S_{r+1}} - \frac{1}{S_r}\right)$ and using the result in part **(i)**, show that

$$S_n = \frac{1}{(n-1)^2 + 1}, \text{ for } n \geq 2.$$ [5]

(iii) Explain whether the sequence $S_1, S_2, S_3, \ldots$ is converging. [1]

(iv) Show that u_n is negative for all $n \geq 2$. [2]

7 It is given that

$$f(x) = \begin{cases} \sqrt{4x - x^2} & \text{for } 2 \leq x \leq 4, \\ x - 4 & \text{for } 4 < x \leq 6, \end{cases}$$

and that $f(x) = f(x+4)$ for all real values of x.

(i) Sketch the graph of $y = f(x)$ for $-3 \leq x \leq 7$. [3]

(ii) If the domain of f is restricted to $k \leq x < 4$, state the smallest value of k for which the function f^{-1} exists. For this value of k, find $f^{-1}(x)$ and state its domain. [4]

Another function g is defined by $g(x) = \ln x, \ x \in \mathbb{R}, \ e^3 < x < e^5$.

(iii) Taking the domain of f to be [2, 6], show that fg exists. Find also the exact range of fg. [3]

8 The curve C_1 has equation $\dfrac{x^2}{4} - y^2 = 1$. The curve C_2 has equation with $x^2 = a^2(1 - y^2)$ where $a > 0, \ a \neq 1$.

(a) State a sequence of transformations which transforms the graph with equation $(x+1)^2 + y^2 = 1$ onto the graph of C_2. [2]

(b) **(i)** Sketch C_1, labelling clearly the coordinates of the points of intersection with the axes and the equations of any asymptotes. [2]

(ii) State the value of a such that C_1 and C_2 intersect at exactly 2 points. [1]

It is now given that a is the value found in **(b)(ii)**.

(iii) Sketch C_2 on the same diagram as C_1. [1]

(iv) By using the substitution $x = a\cos\theta$, find the exact area bounded by C_2 and the lines $x = 2$ and $y = 1$. [6]

9 An oil company wants to drill holes to reach an oil deposit 8000 metres below ground level. The company has 2 drilling teams, A and B. The teams have been tasked to drill a hole each on separate sites above the large oil deposit.

Team A decides to drill 190 metres on Day 1. On subsequent days, the team will drill $r\%$ of the depth drilled on the previous day, where $0 < r < 100$.

(a) Find the range of values of r that will result in team A never reaching the oil deposit. [3]

For the rest of the question, let $r = 99$.

(b) On which day will team A reach the oil deposit? [3]

Team B has a different plan. It decides to drill 180 metres on Day 1. On subsequent days, the team will drill 1 metre less than the depth drilled on the previous day. The 2 teams start drilling on the same day.

(c) Find the first day that the depth drilled by team A on that day is less than the depth drilled by team B. [3]

(d) Determine which team will reach the oil deposit first. [3]

10 **(a)** Show that $\dfrac{v}{(16+v)(9-v)} = \dfrac{a}{(16+v)} + \dfrac{b}{(9-v)}$ where a and b are constants to be determined. [2]

(b) A cyclist is riding in one direction along a straight horizontal road. She starts with zero speed, and t seconds later, her speed v metres per second satisfies the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{(16+v)(9-v)}{320v}.$$

(i) Find t in terms of v. [4]

(ii) Find the cyclist's theoretical maximum speed. Hence find the time she takes to reach a speed equal to half her theoretical maximum speed. [3]

It is now given that when v is small, an approximate solution to the above differential equation is

$$v = \sqrt{\frac{9t}{10}}.$$

It is known that $v = \dfrac{\mathrm{d}x}{\mathrm{d}t}$ where x metres is the distance travelled by the cyclist after t seconds.

(iii) Find x in terms of t, and hence find the time the cyclist takes to travel 10 metres. [3]

11 A parabola G has equation $3y^2 = 2x - 1$. The point A on G has y-coordinate p, where $p > 0$. The tangent to G at A intersects the y-axis at the point B. The point C is a point on the y-axis such that AC is parallel to the x-axis.

(i) Show that the equation of the tangent to G at A can be expressed as

$$6py = 2x + 3p^2 - 1.$$ [3]

(ii) Show that the area of ΔABC is given by $\frac{1}{24}\left(9p^3 + 6p + \frac{1}{p}\right)$. [2]

(iii) Without the use of a calculator, find the minimum value of the area of ΔABC, proving that it is a minimum. [6]

(iv) Determine the type of triangle ΔABC is when its area is a minimum. [2]

2022 TJC Preliminary Examination H2 Mathematics Paper 2

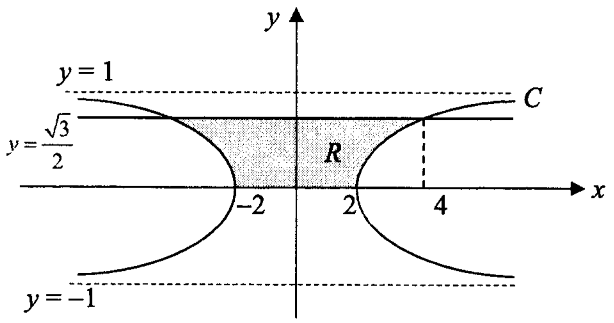
Section A: Pure Mathematics [40 marks]

1 A movie theatre supports a charity event by providing discounted movie tickets at the following prices based on age group.

Age group	Discounted ticket price
≤ 12 years	\$6
13 to 49 years	\$9
≥ 50 years (senior citizen)	\$4

The event organiser plans to spend \$700 to sponsor 100 participants of which at least 35 of them are senior citizens. Find the possible number of participants for each age group. [4]

2

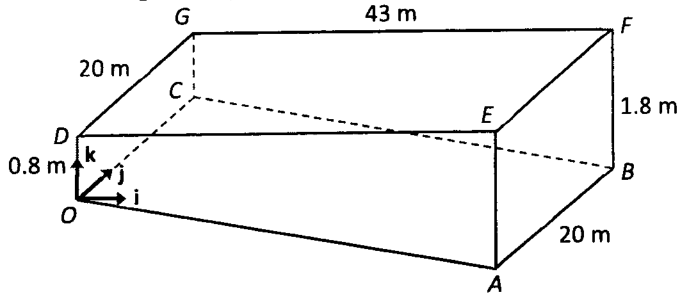


The diagram shows the curve C with equation $y^2 = 1 - \frac{4}{x^2}$. The shaded region R is enclosed by C, the x-axis and the line $y = \frac{\sqrt{3}}{2}$.

(i) Find the exact volume of the solid generated when R is rotated through π radians about the y-axis. [3]

(ii) Find the exact volume of the solid generated when R is rotated through 2π radians about the x-axis. [4]

3 The diagram **(not drawn to scale)** shows a swimming pool, with a horizontal rectangular surface *DEFG*, where $DG = 20$ m and $FG = 43$ m. The walls of the pool are perpendicular to the surface *DEFG*. The sloping floor *OABC* is a rectangle with $OC = 20$ m. The shallowest and the deepest ends of the pool are at a depth of 0.8 m and 1.8 m respectively. The point O is taken as the origin, with unit vectors **i**, **j** and **k** in the directions of $\overrightarrow{DE}$, $\overrightarrow{OC}$ and $\overrightarrow{OD}$ respectively.



(i) Show that a vector equation of the plane *OABC* is $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 0 \\ 43 \end{pmatrix} = 0$. [1]

(ii) Given that the point P with coordinates (39.6, 8, 0.8) lies on the plane *DEFG*, find the exact position vector of the foot of perpendicular from P to the plane *OABC*. [4]

(iii) Point M is the midpoint of AB. Find the acute angle between the line PM and the plane *OABC*. [3]

4 The complex numbers u, v and w are given by

$$u = \frac{1}{2}(1 + \cos\theta + \mathrm{i}\sin\theta), \quad \text{where } -\pi < \theta < \pi,$$
$$v = u^{-1},$$
$$w = u^{-2}.$$

(i) Show that $|u| = \cos\left(\frac{\theta}{2}\right)$, and $\arg u = \frac{\theta}{2}$. [2]

(ii) Using the result in part **(i)**, write down $|v|$ and $\arg v$ in term of θ. Show that the real part of v is 1. [2]

(iii) Write down $|w|$ and $\arg w$ in term of θ. Show that the real part of w is $2 - |w|$. [3]

(iv) Given that $\theta \neq 0$, find the possible values of $\arg\left(\frac{v-w}{v}\right)$. [3]

5 The curve C has parametric equations

$$x = \cos 3\theta, \quad y = \sin\theta, \quad \text{for } 0 \le \theta \le \frac{\pi}{6}.$$

(i) Sketch C, showing clearly the coordinates of the end points. [1]

It is given that C cuts the y- and x-axis at points A and B respectively.

(ii) A particle moves along C from A to B with its x-coordinate increasing at a constant rate of 0.1 units per second. Find the rate of decrease of its y-coordinate when $x = \frac{1}{2}$. [3]

(iii) Find the equation of the tangent to C at A, giving your answer in the form $y = a + bx$, where a and b are exact values to be determined. [2]

(iv) Find the exact area of the finite region between C, the tangent in **(iii)** and the x-axis. [5]

Section B: Probability and Statistics [60 marks]

6 A group of 3 married couples, 2 men and 2 women are to sit in a row of 10 adjacent seats to watch a movie. Find the number of different seating arrangements if

(i) the men and women are to sit at alternate seats, [2]

(ii) two particular men and one particular woman were to not turn up for the movie and people of the same gender are to sit together. [2]

After the movie, all the 10 people sit at a round table for dinner.

(iii) Find the number of different seating arrangements if each married couple are to sit together and the seats are numbered. [3]

7 A biased tetrahedral die has its faces marked with numbers 2, 3, 4 and 5. Let X be the number on the face in contact with the table when the die is thrown. The probability of X being 2, 3, 4 and 5 are $\frac{1}{10}$, α, β and $\frac{2}{5}$ respectively. A player throws the die and his score, Y, is given by $Y = |X - 3|$.

(i) Given that $\text{Var}(Y) = 0.56$, find the values of α and β. [6]

(ii) Find the probability that a player's mean score in 50 throws is more than 1.3. [2]

8 A study is carried out to investigate the time to failure of the batteries of Talsa electric cars after travelling different distances. The table shows the time to failure, t days, for 9 Talsa car batteries after travelling distances of s thousand kilometres.

s	105	110	114	120	128	132	143	155	164
t	118	102	90	79	78	55	68	62	58

(i) Sketch a scatter diagram of t against s. [1]

(ii) Circle the point on the scatter diagram that does not seem to follow the trend and label it as P. Suggest a possible reason for it. [2]

Omit point P identified in part (ii) for the rest of this question.

Below are three models proposed to describe the relationship between s and t.

Model I : $t = a + bs^2$ where $a > 0$ and $b < 0$

Model II: $t = a + b\sqrt{s}$ where $a > 0$ and $b < 0$

Model III: $t = a + be^{-\sqrt{s}}$ where $a > 0$ and $b > 0$

(iii) Comment on why model I is not a suitable model for the given set of data. Explain, with justification, why model III is a better choice than model II. [3]

(iv) Find the equation of the least square regression line for model III, giving the values of a and b correct to two decimal places. Hence, estimate the time to failure of a battery of a Talsa electric car after travelling 135 000 km. Give two reasons why you would expect this estimate to be reliable. [3]

9 It is found that $100p\%$ (where $0 < p < 1$) of all standard size packets of potato chips produced by a snack company contain a winning coupon. The potato chips are sold as a family pack which contains 10 randomly chosen standard size packets of potato chips.

(i) Find the range of values of p such that the most likely number of winning coupons in a family pack is 2. [3]

For the rest of the question, let $p = 0.2$.

(ii) A birthday party organizer buys N family packs. Find the least value of N if he wants to be more than 99% sure that there are at least 30 family packs with at least 1 winning coupon. [4]

The family packs are delivered to supermarkets in cartons. Each carton contains 12 randomly chosen family packs.

(iii) (a) Find the probability that a randomly chosen carton contains exactly 24 winning coupons. [2]

(b) Find the probability that every family pack in a randomly chosen carton contains exactly 2 winning coupons each. [1]

(c) Explain why the answer for **(b)** is smaller than that for **(a)**. [1]

10 A workshop employs craftspeople to make wooden souvenirs for sale at a tourist attraction. Typically, the time taken to make a wooden souvenir by a skilled craftsperson follows a normal distribution with a mean of 1.2 hours and a standard deviation of 10 minutes. A workday at the workshop is 8 hours long (with a lunch break given separately). It may be assumed that a skilled craftsperson will start making the next souvenir once the previous one is completed.

(i) Find the probability that a skilled craftsperson is able to make 7 wooden souvenirs in a workday. [2]

(ii) Two skilled craftspeople start work at the same time. Find the probability that they make their first wooden souvenir of the workday within 15 minutes of each other. [3]

(iii) Find the probability that a skilled craftsperson is unable to make the 7th souvenir in a workday. Hence find the probability that a skilled craftsperson makes exactly 6 wooden souvenirs in a workday. [3]

(iv) The time taken by an apprentice of a particular skilled craftsperson to make a wooden souvenir is 1.5 times of the time taken by the skilled craftsperson. Find the probability that the total time taken to make 3 wooden souvenirs by the apprentice is longer than 4 times the time taken by the skilled craftsperson to make a wooden souvenir.
State the assumption needed for your working. [4]

11 The tensile strength of a thread is the maximum stress it can withstand before it breaks and is measured in Newtons (N).

(i) Company A manufactures cotton threads whose tensile strength is normally distributed with mean 9.0 N. A random sample of 50 pieces of threads was taken and the tensile strength, x N, of each thread was tested. The following data was obtained:

$$\sum x = 441.6, \ \sum (x-\bar{x})^2 = 20.9.$$

(a) Explain what is meant in this context by the term 'a random sample'. [1]

(b) Test at the 2% level of significance, whether the population mean tensile strength of the threads is 9.0 N. State clearly the test statistic used. [6]

(ii) Company B claims to manufacture nylon thread with mean tensile strength of k N. 60 pieces of nylon threads were randomly chosen to be tested. The sample mean and standard deviation were found to be 35.2 N and 3.3 N respectively.

(a) Find the range of values of k for which there is sufficient evidence for the company to have overstated the mean tensile strength at the 2% level of significance. [4]

(b) Explain, in this context, the meaning of 'at the 2% level of significance'. [1]

(c) Explain if it is necessary to make any assumptions about the distribution of the tensile strengths of the nylon threads. [1]

2022 TJC Preliminary Examination H2 Mathematics Paper 1 [Suggested Solutions]

Question 1 [Solution]

(i) $\int \sin^{-1} 2x \, dx = \int 1.\sin^{-1} 2x \, dx$

$$= x\sin^{-1} 2x - \int x.\frac{2}{\sqrt{1-4x^2}} \, dx$$

$$= x\sin^{-1} 2x + \frac{1}{4}\int (-8x)(1-4x^2)^{-\frac{1}{2}} \, dx$$

$$= x\sin^{-1} 2x + \frac{1}{4}.\frac{(1-4x^2)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= x\sin^{-1} 2x + \frac{1}{2}\sqrt{1-4x^2} + c$$

(ii) $\int \frac{1}{x^2 - nx + n^2} \, dx = \int \frac{1}{\left(x - \frac{1}{2}n\right)^2 - \frac{1}{4}n^2 + n^2} \, dx$

$$= \int \frac{1}{\left(x - \frac{1}{2}n\right)^2 + \frac{3}{4}n^2} \, dx$$

$$= \frac{2}{\sqrt{3}n}\tan^{-1}\left(\frac{x - \frac{1}{2}n}{\frac{\sqrt{3}}{2}n}\right) + c \quad \text{or} \quad \frac{2}{\sqrt{3}n}\tan^{-1}\left(\frac{2x-n}{\sqrt{3}n}\right) + c$$

Question 2 [Solution]

(i) $3(1 + i)^2 - (5 + i)(1 + i) = k$

$3(1 + 2i - 1) - (5 + 5i + i - 1) = k$

$k = -4$

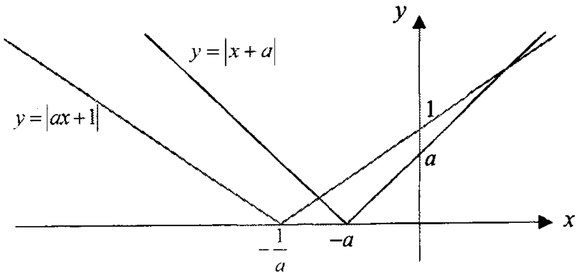
(ii) Using sum of roots : $1 + i + \beta = \frac{5+i}{3}$

$$\therefore \beta = \frac{5+i}{3} - 1 - i = \frac{2}{3} - \frac{2}{3}i$$

Thus the other root is $z = \frac{2}{3} - \frac{2}{3}i$

Question 3 [Solution]

(i)



At the intersection points, $|x+a| = |ax+1|$

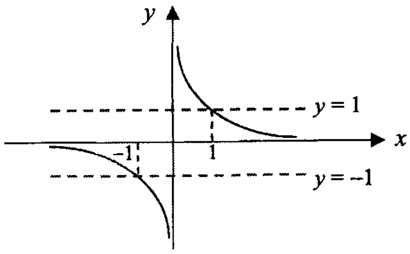
$x + a = ax + 1$ or $x + a = -(ax+1)$

$x = 1$ or $x = -1$

From the graph, the solution is $-1 < x < 1$

(ii) Replacing x by $\frac{1}{x}$, $-1 < \frac{1}{x} < 1$

From the graph of $y = \frac{1}{x}$,



The solution is $x < -1$ or $x > 1$

	Question 4 [Solution]

(i)

$\ln y = 1 + \tan^{-1}(2x)$

Differentiating w.r.t. x,

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{1+(2x)^2} = \frac{2}{1+4x^2}$$

$$\Rightarrow \quad (1+4x^2)\frac{dy}{dx} = 2y$$

Differentiating w.r.t. x,

$$(1+4x^2)\frac{d^2y}{dx^2} + 8x\frac{dy}{dx} = 2\frac{dy}{dx}$$

$$(1+4x^2)\frac{d^2y}{dx^2} + (8x-2)\frac{dy}{dx} = 0 \qquad \text{(shown)}$$

(ii) Differentiating w.r.t. x,

$$(1+4x^2)\frac{d^3y}{dx^3}+8x\frac{d^2y}{dx^2}+(8x-2)\frac{d^2y}{dx^2}+8\frac{dy}{dx}=0$$

i.e. $(1+4x^2)\frac{d^3y}{dx^3}+(16x-2)\frac{d^2y}{dx^2}+8\frac{dy}{dx}=0$

When $x=0$, $\quad \ln y = 1+\tan^{-1}(0)=1 \quad \Rightarrow \quad y=e$

$$(1+0)\frac{dy}{dx}=2e \quad \Rightarrow \quad \frac{dy}{dx}=2e$$

$$(1+0)\frac{d^2y}{dx^2}-2(2e)=0 \quad \Rightarrow \quad \frac{d^2y}{dx^2}=4e$$

$$(1+0)\frac{d^3y}{dx^3}+(0-2)(4e)+8(2e)=0 \quad \Rightarrow \quad \frac{d^2y}{dx^2}=-8e$$

Maclaurin series is $y=e+2ex+4e\left(\frac{x^2}{2!}\right)-8e\left(\frac{x^3}{3!}\right)+\ldots$

i.e., $y=e+2ex+2ex^2-\frac{4}{3}ex^3+\ldots$

(iii)

$$\ln y = 1+\tan^{-1}(2x)$$

$$y=e^{1+\tan^{-1}(2x)}=e\,e^{\tan^{-1}(2x)} \Rightarrow e^{\tan^{-1}(2x)}=\frac{y}{e}$$

Thus Maclaurin series is $e^{\tan^{-1}(2x)}=1+2x+2x^2-\frac{4}{3}x^3+\ldots$

Question 5 [Solution]

(i) Line AB: $\mathbf{r}=\mathbf{a}+\lambda(\mathbf{b}-\mathbf{a}),\ \lambda\in\mathbb{R}$

Line OC: $\mathbf{r}=\mu(9\mathbf{a}-6\mathbf{b}),\ \mu\in\mathbb{R}$

At the intersection point,

$$\mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})=\mu(9\mathbf{a}-6\mathbf{b})$$

$$(1-\lambda-9\mu)\mathbf{a}=(-6\mu-\lambda)\mathbf{b}$$

Since **a** and **b** are non-zero and non-parallel,

$$1-\lambda-9\mu=0 \quad \text{--- (1)}$$

$$-6\mu-\lambda=0 \quad \text{--- (2)}$$

Solving (1) and (2), $\mu=\frac{1}{3},\ \lambda=-2$

Position vector of the intersection point is $\mathbf{r}=\frac{1}{3}(9\mathbf{a}-6\mathbf{b})=3\mathbf{a}-2\mathbf{b}$

(ii) $\mathbf{d}=t\begin{pmatrix}1\\1\\0\end{pmatrix}+(1-t)\begin{pmatrix}4\\-2\\6\end{pmatrix}=\begin{pmatrix}-3t+4\\3t-2\\-6t+6\end{pmatrix}$

$$\cos 60^0 = \frac{\mathbf{a}\cdot\mathbf{d}}{|\mathbf{a}||\mathbf{d}|} = \frac{\begin{pmatrix}1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}-3t+4\\3t-2\\-6t+6\end{pmatrix}}{\sqrt{1^2+1^2}\sqrt{(3t-4)^2+(3t-2)^2+(6t-6)^2}}$$

$$\frac{1}{2} = \frac{2}{\sqrt{2}\sqrt{54t^2-108t+56}}$$

$$\sqrt{2}\sqrt{54t^2-108t+56} = 4$$

$$54t^2-108t+56 = 8$$

$$54t^2-108t+48 = 0$$

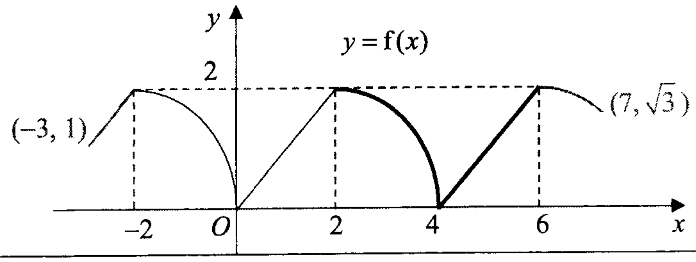
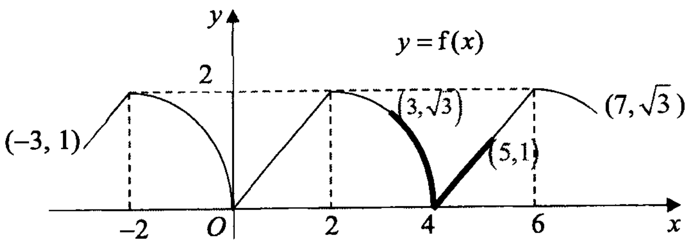
$$9t^2-18t+8 = 0$$

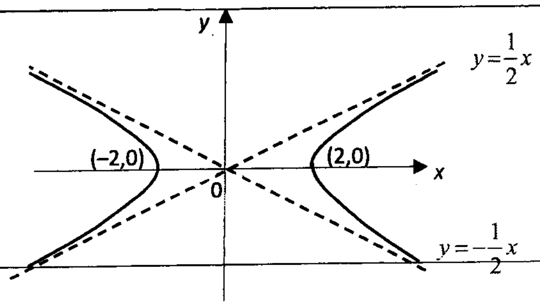
$$(3t-2)(3t-4) = 0$$

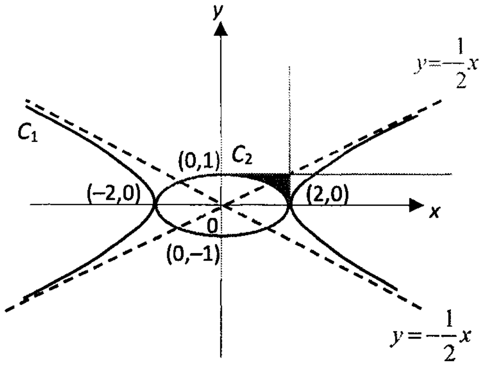
$$t = \frac{2}{3} \text{ or } t = \frac{4}{3}$$

	Question 6 [Solution]
(i)	For $r \geq 1$, $\frac{1}{S_{r+1}} - \frac{1}{S_r} = \frac{1+(2r-1)S_r}{S_r} - \frac{1}{S_r}$ $= \frac{1}{S_r} + (2r-1) - \frac{1}{S_r}$ $= 2r-1$
(ii)	$\sum_{r=1}^{n-1}\left(\frac{1}{S_{r+1}} - \frac{1}{S_r}\right) = \sum_{r=1}^{n-1}(2r-1)$ $\text{RHS} = \sum_{r=1}^{n-1}(2r-1) = \frac{n-1}{2}(1+2(n-1)-1)$ $= (n-1)^2$ $\text{LHS} = \sum_{r=1}^{n-1}\left(\frac{1}{S_{r+1}} - \frac{1}{S_r}\right) = \frac{1}{S_2} - \frac{1}{S_1}$ $+ \frac{1}{S_3} - \frac{1}{S_2}$ $+ \frac{1}{S_4} - \frac{1}{S_3}$ $+ \ldots\ldots$ $+ \frac{1}{S_{n-1}} - \frac{1}{S_{n-2}}$ $+ \frac{1}{S_n} - \frac{1}{S_{n-1}}$ $= \frac{1}{S_n} - \frac{1}{S_1} = \frac{1}{S_n} - 1$

	Equating, $\frac{1}{S_n}-1=(n-1)^2$ $\frac{1}{S_n}=(n-1)^2+1 \Rightarrow S_n=\frac{1}{(n-1)^2+1}$ for $n\geq 2$
(iii)	As $n\to\infty$, $(n-1)^2+1\to\infty$, $S_n=\frac{1}{(n-1)^2+1}\to 0$ Thus the sequence converges to 0.
(iv)	For $n\geq 2$, $u_n=S_n-S_{n-1}=\frac{1}{(n-1)^2+1}-\frac{1}{(n-2)^2+1}$ **Method 1** Since $(n-1)^2+1>(n-2)^2+1$ for $n\geq 2$, $\frac{1}{(n-1)^2+1}<\frac{1}{(n-2)^2+1}$ $u_n=\frac{1}{(n-1)^2+1}-\frac{1}{(n-2)^2+1}<0$ i.e. for $n\geq 2$, $u_n<0$ **Method 2** $u_n=\frac{1}{(n-1)^2+1}-\frac{1}{(n-2)^2+1}$ $=\frac{(n^2-4n+4+1)-(n^2-2n+1+1)}{((n-1)^2+1)((n-2)^2+1)}$ $=\frac{3-2n}{((n-1)^2+1)((n-2)^2+1)}$ For $n\geq 2$, $3-2n<0$ and $((n-1)^2+1)((n-2)^2+1)>0$ So $u_n<0$

	Question 7 [Solution]
(i)	
(ii)	Smallest value of $k = 2$ Let $y = \sqrt{4x - x^2}$ for $2 \le x < 4$ $y^2 = 4 - (x-2)^2$ $(x-2)^2 = 4 - y^2$ $x = 2 \pm \sqrt{4 - y^2}$ Since $x \ge 2$, $x = 2 + \sqrt{4 - y^2}$ $f^{-1}(x) = 2 + \sqrt{4 - x^2}$, $0 < x \le 2$
(iii)	$R_g = (3, 5)$, $D_f = [2, 6]$ Since $R_g \subseteq D_f$, fg exists.  $D_g = (e^3, e^5) \xrightarrow{g} (3,\ 5) \xrightarrow{f} [0, \sqrt{3}) = R_{fg}$

	Question 8 [Solution]
(a)	$C_2 : x^2 = a^2(1 - y^2) \Rightarrow \left(\frac{x}{a}\right)^2 + y^2 = 1$ $(x+1)^2 + y^2 = 1 \xrightarrow{T} x^2 + y^2 = 1 \xrightarrow{S} \left(\frac{x}{a}\right)^2 + y^2 = 1$ 1. Translation of 1 unit in the positive direction of the x-axis. 2. Scaling parallel to the x-axis by factor a.
(b)(i)	$C_1 : \frac{x^2}{4} - y^2 = 1$ 

(ii)	$C_2: \left(\frac{x}{a}\right)^2 + y^2 = 1$ is an ellipse with centre $(0,0)$. For C_1 and C_2 to intersect exactly twice, $a = 2$.
(iii)	
(iv)	Exact area $=2(1)-\int_0^2 \sqrt{1-\frac{x^2}{4}}\,\mathrm{d}x$ $=2(1)-\int_{\frac{\pi}{2}}^{0} \sqrt{1-\frac{4\cos^2\theta}{4}}\,(-2\sin\theta)\,\mathrm{d}\theta$ $x = 2\cos\theta$, $\frac{\mathrm{d}x}{\mathrm{d}\theta} = -2\sin\theta$ $=2(1)-\int_{\frac{\pi}{2}}^{0} \sqrt{\sin^2\theta}\,(-2\sin\theta)\,\mathrm{d}\theta$ $=2(1)+2\int_{\frac{\pi}{2}}^{0} \sin^2\theta\,\mathrm{d}\theta$ $=2(1)+\int_{\frac{\pi}{2}}^{0} (1-\cos 2\theta)\,\mathrm{d}\theta$ $=2(1)+\left[\theta-\frac{\sin 2\theta}{2}\right]_{\frac{\pi}{2}}^{0}$ $=2(1)+\left[0-\left(\frac{\pi}{2}-0\right)\right]$ $=\left(2-\frac{\pi}{2}\right)\text{units}^2$

	Question 9 [Solution]
(a)	Team A

Day	Depth drilled (m)
1	190
2	$190\left(\frac{r}{100}\right)$
3	$190\left(\frac{r}{100}\right)^2$
...	
n	$190\left(\frac{r}{100}\right)^{n-1}$

GP: first term = 190 and common ratio $=\frac{r}{100}$

Since $0<r<100$, $0<\frac{r}{100}<1$, sum to infinity S_∞ exists.

If team A never reaches the oil deposit,

$$S_\infty=\frac{190}{1-\frac{r}{100}}<8000$$

$$\frac{190}{8000}<1-\frac{r}{100}$$

$$\frac{r}{100}<\frac{781}{800}$$

$$0<r<\frac{781}{8} \quad (\text{or } 0<r<97.625)$$

(b) Team A

GP: first term = 190 and common ratio = 0.99

Consider $S_{n,A}=\frac{190(1-0.99^n)}{1-0.99}=8000$

$$1-0.99^n=\frac{80}{190}$$

$$n=\frac{\ln\left(\frac{11}{19}\right)}{\ln 0.99}=54.38$$

Team A will reach the oil deposit on Day 55.

Alternatively, $S_{n,A}=\frac{190(1-0.99^n)}{1-0.99}\le 8000$

From GC,

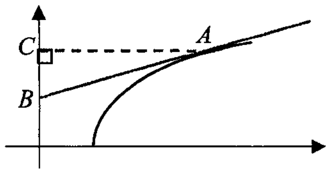
Day	Depth drilled (m)
54	7957.8
55	8068.3
...	...

Team A will reach the oil deposit on Day 55.

(c)	Team B AP: first term = 180 and common difference = −1 Depth drilled on day n for Team B, $b_n = 180+(n-1)(-1) = 181-n$ Depth drilled on day n for Team A, $a_n = 190(0.99^{n-1})$ $a_n < b_n$ $190(0.99^{n-1}) - 181 + n < 0$ From GC, <table><tr><td>n</td><td>$a_n - b_n$</td></tr><tr><td>13</td><td>$0.413 > 0$</td></tr><tr><td>14</td><td>$-0.271 < 0$</td></tr></table>Thus the first day is Day 14.
(d)	Team B AP: first term = 180 and common difference = −1 Consider $S_{n,B} = \frac{n}{2}(2(180)+(n-1)(-1)) \geq 8000$ $n(361-n) \geq 16000$ $n^2 - 361n + 16000 \leq 0$ Using GC, <table><tr><td>n</td><td>$n^2-361n+16000$</td></tr><tr><td>51</td><td>$190 > 0$</td></tr><tr><td>52</td><td>$-68 < 0$</td></tr></table>Team B will reach the oil deposit on Day 52. Thus **team B** will reach the oil deposit first.

	Question 10 [Solution]
(a)	$\frac{v}{(16+v)(9-v)} = \frac{a}{(16+v)} + \frac{b}{(9-v)}$ $v = a(9-v) + b(16+v)$ Subst $v = 9$, $b = \frac{9}{25}$ Subst $v = -16$, $a = -\frac{16}{25}$ $\therefore \frac{v}{(16+v)(9-v)} = -\frac{16}{25}\frac{1}{(16+v)} + \frac{9}{25}\frac{1}{(9-v)}$
(b)(i)	$\frac{dv}{dt} = \frac{(16+v)(9-v)}{320v}$

	$\int \frac{v}{(16+v)(9-v)}\,\mathrm{d}v = \frac{1}{320}\int 1\,\mathrm{d}t$ $-\frac{16}{25}\int \frac{1}{(16+v)}\,\mathrm{d}v + \frac{9}{25}\int \frac{1}{(9-v)}\,\mathrm{d}v = \frac{1}{320}t + c$ using (a) $-16\ln\lvert 16+v\rvert \;-9\ln\lvert 9-v\rvert = \frac{5}{64}t + c$ When $t=0$, $v=0$, $c=-16\ln 16-9\ln 9$ or $-\ln\left(9^9\right)\left(16^{16}\right)$ $-16\ln\lvert 16+v\rvert \;-9\ln\lvert 9-v\rvert = \frac{5}{64}t - 16\ln 16 - 9\ln 9$ $t=\frac{64}{5}\left(9\ln\left\lvert\frac{9}{9-v}\right\rvert + 16\ln\left\lvert\frac{16}{16+v}\right\rvert\right)$ or $t=\frac{64}{5}\ln\left(\frac{9^9 16^{16}}{\lvert 9-v\rvert^9\lvert 16+v\rvert^{16}}\right)$
(ii)	Her theoretical maximum speed occurs when $\frac{\mathrm{d}v}{\mathrm{d}t}=0$ $\frac{(16+v)(9-v)}{320v}=0$ $v=9$ or $v=-16$ (rejected $\because v\geq 0$) Her theoretical maximum speed is 9 m s^{-1}. When $v=4.5$, $t=\frac{64}{5}\left(9\ln\left\lvert\frac{9}{9-4.5}\right\rvert + 16\ln\left\lvert\frac{16}{16+4.5}\right\rvert\right)=29.1$ The time taken is 29.1 seconds.
(iii)	$v=\frac{\mathrm{d}x}{\mathrm{d}t}=\sqrt{\frac{9t}{10}}$ $x=\int\sqrt{\frac{9t}{10}}\,\mathrm{d}t$ $=\frac{3}{\sqrt{10}}\left(\frac{2}{3}t^{\frac{3}{2}}\right)+d$ $=\frac{2}{\sqrt{10}}t^{\frac{3}{2}}+d$ When $t=0, x=0,\ d=0$ $\therefore\ x=\frac{2}{\sqrt{10}}t^{\frac{3}{2}}$ When $x=10$, $10=\frac{2}{\sqrt{10}}t^{\frac{3}{2}} \Rightarrow t=6.30$ The time taken is approximately 6.30 seconds.

	Question 11 [Solution]
(i)	$3y^2 = 2x - 1$ $6y\frac{dy}{dx} = 2$ $\frac{dy}{dx} = \frac{1}{3y}$  At A, $y = p \Rightarrow 3p^2 = 2x - 1 \Rightarrow x = \frac{3p^2+1}{2}$ Equation of tangent at A: $y - p = \frac{1}{3p}\left(x - \left(\frac{3p^2+1}{2}\right)\right)$ $6py = 2x - (3p^2+1) + 6p^2$ $6py = 2x + 3p^2 - 1$ (shown)
(ii)	At B, $x = 0$, $6py = 3p^2 - 1 \Rightarrow y = \frac{3p^2-1}{6p}$ Coordinates of C is $(0, p)$ Area of $\Delta ABC = \frac{1}{2} \times AC \times BC$ $\because$ right angled at C $= \frac{1}{2}\left(\frac{3p^2+1}{2}\right)\left(p - \left(\frac{3p^2-1}{6p}\right)\right)$ $= \frac{1}{4}(3p^2+1)\left(\frac{6p^2-3p^2+1}{6p}\right)$ $= \frac{1}{24p}(3p^2+1)^2$ $= \frac{1}{24p}(9p^4+6p^2+1)$ $= \frac{1}{24}\left(9p^3+6p+\frac{1}{p}\right)$ (shown)
(iii)	Let $E = \frac{1}{24}\left(9p^3+6p+\frac{1}{p}\right)$ Let $\frac{dE}{dp} = \frac{1}{24}\left(27p^2+6-\frac{1}{p^2}\right) = 0$ $27p^2+6-\frac{1}{p^2} = 0$ $27p^4+6p^2-1 = 0$ $(9p^2-1)(3p^2+1) = 0$ $9p^2 = 1$ since $3p^2+1 \neq 0$ $p = \frac{1}{3}$ since $p > 0$

$$\frac{d^2E}{dp^2}=\frac{1}{24}\left(54p+\frac{2}{p^3}\right)>0 \text{ since } p>0$$

Thus E is minimum when $p=\frac{1}{3}$

Alternative: Using first derivative test

p	0.3	$\frac{1}{3}$	0.35
$\frac{dE}{dp}$	$-0.112<0$	0	$0.0477>0$
Shape of graph	╲	—	╱

Minimum value of E

$$=\frac{1}{24}\left(9\left(\frac{1}{3}\right)^3+6\left(\frac{1}{3}\right)+\frac{1}{\left(\frac{1}{3}\right)}\right)$$

$$=\frac{2}{9}\text{ units}^2$$

(iv) When $p=\frac{1}{3}$, $AC=\frac{3p^2+1}{2}=\frac{2}{3}$

and $BC=\frac{3p^2+1}{6p}=\frac{2}{3}$

Thus ΔABC is a right-angled isosceles triangle.

2022 TJC Preliminary Examination H2 Mathematics Paper 2 [Suggested Solutions]

Question 1 [Solution]

Let x, y and z denote the number of participants in the age group ≤ 12 years, 13 to 49 years and ≥ 50 years respectively.

$$x + y + z = 100 \quad \text{--- (1)}$$
$$6x + 9y + 4z = 700 \quad \text{--- (2)}$$
$$z \geq 35$$

From GC,

$$x = \frac{200}{3} - \frac{5}{3}z \geq 0 \quad \Rightarrow \quad z \leq 40 \quad \therefore \quad 35 \leq z \leq 40$$

$$y = \frac{100}{3} + \frac{2}{3}z$$

Method 1

Since $x = \frac{200}{3} - \frac{5}{3}z = \frac{5}{3}(40 - z)$ **must be a positive integer,**

$z = 40$ or $z = 37$

Thus the possible number of participants are $\begin{cases} x = 5 \\ y = 58 \\ z = 37 \end{cases}$ or $\begin{cases} x = 0 \\ y = 60 \\ z = 40 \end{cases}$

Method 2 Using GC

z	x	y
37	5	58
40	0	60

Question 2 [Solution]

(i) $y^2 = 1 - \frac{4}{x^2} \Rightarrow x^2 = \frac{4}{1 - y^2}$

$$\text{Volume} = \pi \int_0^{\frac{\sqrt{3}}{2}} x^2 \, dy$$
$$= 4\pi \int_0^{\frac{\sqrt{3}}{2}} \frac{1}{1 - y^2} \, dy$$
$$= 4 \times \frac{1}{2(1)} \pi \left[\ln \left| \frac{1+y}{1-y} \right| \right]_0^{\frac{\sqrt{3}}{2}}$$
$$= 2\pi \ln \left(\frac{1 + \frac{\sqrt{3}}{2}}{1 - \frac{\sqrt{3}}{2}} \right) \text{ units}^3 \quad \text{or} \quad 2\pi \ln \left(\frac{2 + \sqrt{3}}{2 - \sqrt{3}} \right) \text{ units}^3$$
$$\text{or} \quad 2\pi \ln \left(7 + 4\sqrt{3} \right) \text{ units}^3$$

(ii) Volume $= 2\times\left[\pi\left(\frac{\sqrt{3}}{2}\right)^2(4) - \pi\int_2^4 y^2\ \mathrm{d}x\right]$

$$= 2\left[3\pi - \pi\int_2^4\left(1-\frac{4}{x^2}\right)\mathrm{d}x\right]$$

$$= 2\pi\left(3 - \left[x+\frac{4}{x}\right]_2^4\right)$$

$$= 2\pi\left(3 - [5-4]\right)$$

$$= 4\pi \text{ units}^3$$

Question 2 [Solution]

(i)

$$\overrightarrow{OC} = \begin{pmatrix}0\\20\\0\end{pmatrix},\quad \overrightarrow{OA} = \begin{pmatrix}43\\0\\-1\end{pmatrix}$$

A normal is $\overrightarrow{OC}\times\overrightarrow{OA} = \begin{pmatrix}0\\20\\0\end{pmatrix}\times\begin{pmatrix}43\\0\\-1\end{pmatrix} = -20\begin{pmatrix}1\\0\\43\end{pmatrix}$

Since origin lies on the plane *OABC*,

Equation of plane *OABC* is $\mathbf{r}\cdot\begin{pmatrix}1\\0\\43\end{pmatrix} = 0$

(ii) Let *N* be the foot of perpendicular from *P* to the plane *OABC*.

Line *PN*: $\mathbf{r} = \begin{pmatrix}39.6\\8\\0.8\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\43\end{pmatrix},\quad \lambda\in\mathbb{R}$

At *N*, $\begin{pmatrix}39.6+\lambda\\8\\0.8+43\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\0\\43\end{pmatrix} = 0$

$$39.6+\lambda+34.4+1849\lambda = 0 \Rightarrow \lambda = -\frac{1}{25}$$

$$\overrightarrow{ON} = \begin{pmatrix}39.6\\8\\0.8\end{pmatrix} - \frac{1}{25}\begin{pmatrix}1\\0\\43\end{pmatrix} = \begin{pmatrix}\frac{989}{25}\\8\\-\frac{23}{25}\end{pmatrix}$$

(iii) $\overrightarrow{OM}=\begin{pmatrix}43\\10\\-1\end{pmatrix}$

$$\overrightarrow{PM}=\begin{pmatrix}43\\10\\-1\end{pmatrix}-\begin{pmatrix}39.6\\8\\0.8\end{pmatrix}=\begin{pmatrix}3.4\\2\\-1.8\end{pmatrix}$$

Method 1

Let θ be the acute angle between line *PM* and the plane *OABC*.

$$\sin\theta=\frac{\left|\begin{pmatrix}3.4\\2\\-1.8\end{pmatrix}\cdot\begin{pmatrix}1\\0\\43\end{pmatrix}\right|}{\sqrt{18.8}\sqrt{1850}}=\frac{74}{\sqrt{34780}}$$

$\theta=23.4^\circ$ or 0.408 rad

Method 2

Let ϕ be the acute angle between line *PM* and the normal of plane *OABC*.

$$\cos\phi=\frac{\left|\begin{pmatrix}3.4\\2\\-1.8\end{pmatrix}\cdot\begin{pmatrix}1\\0\\43\end{pmatrix}\right|}{\sqrt{18.8}\sqrt{1850}}=\frac{74}{\sqrt{34780}}$$

$\phi=66.6^\circ$ or 1.162 rad

Therefore, required angle $=90^\circ-66.6^\circ=23.4^\circ$ or 0.408 rad

Question 4 [Solution]

(i) $u=\frac{1}{2}\left(1+\left(2\cos^2\frac{\theta}{2}-1\right)+\mathrm{i}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\right)$

$=\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+\mathrm{i}\sin\frac{\theta}{2}\right)$

$\therefore\ |u|=\cos\left(\frac{\theta}{2}\right)$, and $\arg u=\frac{\theta}{2}$

Alternative

$u=\frac{1}{2}\left(1+\cos\theta+\mathrm{i}\sin\theta\right)$

$=\frac{1}{2}\left(1+\mathrm{e}^{i\theta}\right)$

$=\frac{1}{2}\mathrm{e}^{i\frac{\theta}{2}}\left(\mathrm{e}^{-i\frac{\theta}{2}}+\mathrm{e}^{i\frac{\theta}{2}}\right)$

$$= \frac{1}{2} e^{i\frac{\theta}{2}} \left(2\cos\frac{\theta}{2} \right)$$

$$= \cos\frac{\theta}{2} \left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2} \right)$$

$\therefore\ |u| = \cos\left(\frac{\theta}{2}\right)$, and $\arg u = \frac{\theta}{2}$

(ii) $|v| = \sec\left(\frac{\theta}{2}\right)$, and $\arg v = -\frac{\theta}{2}$

$$v = \sec\left(\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right) = 1 - i\tan\frac{\theta}{2}$$

$\therefore$ real part of v is 1 (shown)

(iii) $|w| = \sec^2\left(\frac{\theta}{2}\right)$, and $\arg w = -\theta$

$$w = \sec^2\left(\frac{\theta}{2}\right)(\cos\theta - i\sin\theta)$$

$$= \sec^2\left(\frac{\theta}{2}\right)\left(2\cos^2\frac{\theta}{2} - 1 - i\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)\right)$$

$$= 2 - \sec^2\left(\frac{\theta}{2}\right) - i\left(2\tan\frac{\theta}{2}\right)$$

$\therefore$ real part of $w = 2 - \sec^2\left(\frac{\theta}{2}\right) = 2 - |w|$ (shown)

(iv) $\arg\left(\frac{v-w}{v}\right) = \arg\left(\frac{u^{-1} - u^{-2}}{u^{-1}}\right)$

$$= \arg\left(1 - u^{-1}\right)$$

$$= \arg(1 - v)$$

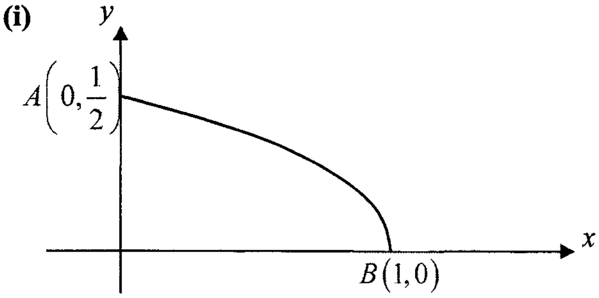
$$= \arg\left(1 - \left(1 - i\tan\frac{\theta}{2}\right)\right)$$

$$= \arg\left(i\tan\frac{\theta}{2}\right)$$

$$= \frac{\pi}{2} \text{ or } -\frac{\pi}{2}$$

Question 5 [Solution]

(i)



(ii) $\frac{dx}{d\theta} = -3\sin 3\theta$, $\frac{dy}{d\theta} = \cos\theta$

$$\frac{dy}{dx} = \frac{\cos\theta}{-3\sin 3\theta}$$

$$\frac{dy}{dt} = \frac{dy}{dx}.\frac{dx}{dt} = \frac{\cos\theta}{-3\sin 3\theta}(0.1)$$

When $x = \frac{1}{2}$, $\cos 3\theta = \frac{1}{2} \Rightarrow 3\theta = \frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{9}$

$$\frac{dy}{dt} = \frac{\cos\frac{\pi}{9}}{-3\sin\frac{\pi}{3}}(0.1) = -0.0362$$

The rate of decrease of its y-coordinate is 0.0362 units per second.

(iii) At A, when $x = 0$, $\cos 3\theta = 0 \Rightarrow 3\theta = \frac{\pi}{2}$, i.e. $\theta = \frac{\pi}{6}$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos\frac{\pi}{6}}{-3\sin\frac{\pi}{2}} = -\frac{\sqrt{3}}{6}$$

Equation of the tangent at A:

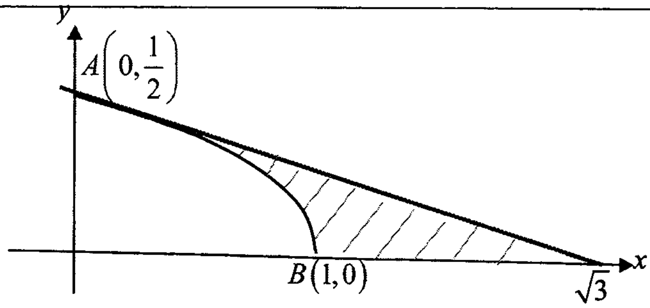
$$y - \frac{1}{2} = -\frac{\sqrt{3}}{6}(x - 0) \quad \text{i.e} \quad y = -\frac{\sqrt{3}}{6}x + \frac{1}{2}$$

(iv)

$$-\frac{\sqrt{3}}{6}x + \frac{1}{2} = 0 \Rightarrow -\frac{\sqrt{3}}{6}x = -\frac{1}{2} \Rightarrow x = \sqrt{3}$$

Area required

$$= \frac{1}{2}\left(\frac{1}{2}\right)\left(\sqrt{3}\right) - \int_0^1 y\,dx$$



$$= \frac{1}{2}\left(\frac{1}{2}\right)\left(\sqrt{3}\right) - \int_{\frac{\pi}{6}}^{0} (\sin\theta)(-3\sin 3\theta)\, d\theta$$

$$= \frac{\sqrt{3}}{4} - 3\int_0^{\frac{\pi}{6}} \sin 3\theta \sin\theta\, d\theta$$

$$= \frac{\sqrt{3}}{4} + \frac{3}{2}\int_0^{\frac{\pi}{6}} (\cos 4\theta - \cos 2\theta)\, d\theta$$

$$= \frac{\sqrt{3}}{4} + \frac{3}{2}\left[\frac{1}{4}\sin 4\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{6}}$$

$$= \frac{\sqrt{3}}{4} + \frac{3}{2}\left[\frac{1}{4}\left(\frac{\sqrt{3}}{2}\right) - \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$= \frac{\sqrt{3}}{4} - \frac{3\sqrt{3}}{16}$$

$$= \frac{\sqrt{3}}{16}\text{ units}^2$$

Question 6 [Solution]

(i) Number of arrangements = $5! \times 5! \times 2 = 28800$

(ii) Number of arrangements = $\frac{5!}{3!} \times 3! \times 4! = 2880$

(iii) Number of arrangements = $(7-1)! \times 2! \times 2! \times 2! \times 10 = 57600$

Question 7 Solution

(i)

y	0	1	2
$P(Y=y)$	α	$\beta+\frac{1}{10}$	$\frac{2}{5}$

$$E(Y) = \left(\beta + \frac{1}{10}\right) + 2\times\frac{2}{5} = \beta + \frac{9}{10}$$

$$E(Y^2) = \left(\beta + \frac{1}{10}\right) + 2^2\times\frac{2}{5} = \beta + \frac{17}{10}$$

$$\text{Var}(Y) = E(Y^2) - (E(Y))^2$$

$$= \left(\beta + \frac{17}{10}\right) - \left(\beta + \frac{9}{10}\right)^2$$

Given Var $(Y) = 0.56$, $\left(\beta+\frac{17}{10}\right)-\left(\beta+\frac{9}{10}\right)^2 = 0.56$

From GC, $\beta = \frac{3}{10}$ or $\beta = -\frac{11}{10}$ (rejected as $\because\ \beta>0$)

$\frac{1}{10}+\alpha+\beta+\frac{2}{5}=1 \ \Rightarrow \alpha = \frac{1}{5}$

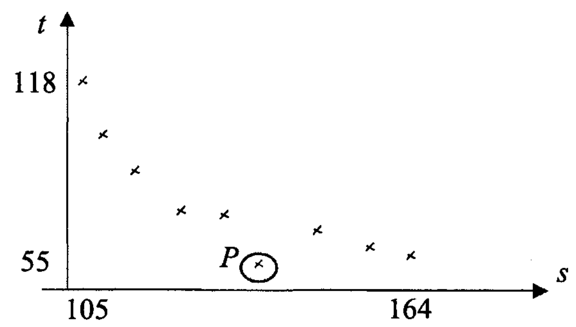
(ii) Since sample size of 50 is large, by **Central Limit Theorem,**

$\overline{Y} \sim \mathrm{N}\left(\frac{6}{5}, \frac{0.56}{50}\right)$ approximately

$\mathrm{P}\left(\overline{Y}>1.3\right)=0.172$

Question 8 Solution

(i)



(ii) When $s = 132$, the time to failure $t = 55$ days seems to be much lower than the expected value based on the trend. This could be due to a faulty car battery.

(iii) Model I : $t = a + bs^2$ where $a > 0$ and $b < 0$

In model I, as s increases, t decreases at an increasing rate but the data trend shows that as s increases, t decreases at decreasing rate. Hence model I is not suitable.

Model II: $t = a + b\sqrt{s}$ where a > 0 and b < 0

Model III: $t = a + be^{-\sqrt{s}}$ where a > 0 and b > 0

Model II, $|r| = 0.929$

Model III, $|r| = 0.988$

Model III has $|r|$-value which is closer to 1 than that in Model II. Hence model III has a stronger positive linear correlation between t and $e^{-\sqrt{s}}$ and it is a better linear model than model II.

(iv) The regression line is $t = 54.43 + 1699879.37e^{-\sqrt{s}}$

When $s = 135$, $t = 54.43 + 1699879.37e^{-\sqrt{135}}$
$t = 69.7 \approx 70$ days

The time to failure of the battery is 70 days.

The estimate is reliable because
(i) $s = 135$ is within the given data range (105, 164) and
(ii) $|r|$-value = 0.988 is very close to 1 indicating a strong positive linear relation between t and $e^{-\sqrt{s}}$.

Question 9 Solution

(i) Let X be the number of standard size packets (out of 10 in a family pack) that contains a winning coupon. $X \sim B(10, p)$

Most likely number of winning coupons in a family pack is 2

$\Rightarrow$ mode of $X = 2$

Thus $P(X=2) > P(X=1)$ **and** $P(X=2) > P(X=3)$

$\binom{10}{2} p^2 (1-p)^8 > \binom{10}{1} p (1-p)^9$ and $\binom{10}{2} p^2 (1-p)^8 > \binom{10}{3} p^3 (1-p)^7$

$45p > 10(1-p)$ and $45(1-p) > 120p$

$p > \frac{2}{9}(1-p)$ and $(1-p) > \frac{8}{3}p$

$p > \frac{2}{11}$ and $p < \frac{3}{11}$

Thus $\frac{2}{11} < p < \frac{3}{11}$

(ii) $X \sim B(10, 0.2)$

P(a family pack has at least 1 winning coupon) $= P(X \ge 1) = 1 - P(X = 0) = 0.89263$ (5 s.f.)

Let Y be the number of family packs (out of N packs) with at least 1 winning coupon.

$Y \sim B(N, 0.89263)$

Given: $P(Y \ge 30) > 0.99$

$\Rightarrow$ $1 - P(Y \le 29) > 0.99$

Using GC,

N	$1 - P(Y \le 29)$
38	$0.9829 < 0.99$
39	$0.9931 > 0.99$

Least $N = 39$

(iii)(a) Let W be the number of standard size packets (out of $12 \times 10 = 120$ packets in a carton) that contains a winning coupon.

$W \sim B(120, 0.2)$

P(a carton contains exactly 24 winning coupons)

$= P(W = 24) = 0.0907$ (3 s.f.)

(iii)(b)P(every family pack in a carton contains exactly 2 winning coupons each)

$= [P(X = 2)]^{12} = 5.75 \times 10^{-7}$ (3 s.f.)

(iii)(c) The answer for **(b)** is smaller than that for **(a)** because the case in (b) is only one of the many cases for (a).

In addition to the case in **(b)**, **(a)** includes many other cases such as 4 family packs containing 10 winning coupons, 1 family pack containing 8 winning coupons and the remaining family packs do not contain any winning coupons.

Question 10 Solution

(i) Let X be the time (in minutes) taken by a skilled craftsperson to make a wooden souvenir.

$X \sim N(72, 10^2)$

Let T_n be the time (in minutes) taken by a skilled craftsperson to make n wooden souvenirs.

$T_7 = X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7 \sim N(504, 700)$

$P(T \le 480) = 0.182$ (3 sf)

(ii) $X_1 - X_2 \sim N(0, 200)$

$P(|X_1 - X_2| < 15) = P(-15 < X_1 - X_2 < 15)$

$= 0.711$ (3sf)

(iii) P(makes at most 6 souvenirs in a workday)

$= P(T_7 > 480) = 0.81793 = 0.818$ (3sf)

P(makes at most 5 souvenirs in a workday)

$= P(T_6 > 480) = 0.02502$

Hence, P(makes exactly 6 souvenirs in a workday)

$= 0.81793 - 0.02502$

$= 0.793$ (3sf)

Let $Y = 1.5(X_1 + X_2 + X_3) - 4X$

$Y \sim N((1.5\times3-4)\times72,\ (1.5^2\times3+4^2)\times10^2)$

i.e. $Y \sim N(36, 2275)$

Required probability = $P(Y > 0) = 0.774$ (3sf)

The time taken to make each wooden souvenir is independent of each other.

Question 11 [Solution]

(i)(a) Each piece of thread has an equal chance of being selected and is selected independently of one another. All samples of 50 pieces of threads have the same chance of being selected.

(i)(b) $\bar{x} = \dfrac{441.6}{50} = 8.832$

Unbiased estimate of population variance is

$$s^2 = \frac{n}{n-1}\left(\frac{\sum(x-\bar{x})^2}{n}\right) = \frac{50}{49}\left(\frac{20.9}{50}\right) = \frac{20.9}{49} = 0.4265306$$

Let μ be the population mean tensile strength of thread

$H_0 : \mu = 9.0$

$H_1 : \mu \neq 9.0$

Test at 2% level of significance

Under H_0, the test statistics is $Z = \dfrac{\bar{X} - 9.0}{\frac{s}{\sqrt{n}}} \sim N(0,1)$ approximately.

Reject H_0 if p-value ≤ 0.02

From GC, p-value $= 0.0689 > 0.02$

Since the p-value is more than the level of significance, we do not reject H_0.

Thus there is insufficient evidence at 2% level of significance that the mean tensile strength of the cotton threads is not 9.0 N.

(ii)(a) Let Y be the tensile strength of a piece of nylon thread manufactured by Company B, and μ_n be the population mean tensile strength of nylon thread.

Unbiased estimate for population variance is

$s^2 = \dfrac{60}{59}(3.3^2) = 11.074576$

$H_0 : \mu_n = k$

$H_1 : \mu_n < k$

Test at 2% level of significance Reject H_0 if $z_{cal} \le -2.05374$ Under H_0, since the sample size $n = 60$ is large, by Central Limit Theorem, $\bar{Y} \sim N(k, \frac{11.074576}{60})$ approximately. For H_0 to be rejected, z_{cal} lies inside the critical region $z_{cal} = \dfrac{35.2-k}{\sqrt{\frac{11.07458}{60}}} \le -2.05374$ $\Rightarrow k \ge 36.1$ (3 s.f.)
(ii)(b) At 2% level of significance means that there is a probability of 0.02 that we wrongly concluded that the company has overstated the mean tensile strength of the threads when the mean tensile strength is actually k N.
(ii)(c) It is <u>not necessary</u> to make any assumption about the population distribution, as sample size of 60 is large enough, the Central Limit Theorem approximates the sample mean tensile strength of the threads, $\bar{Y}$, to a normal distribution.

BP~621

TAMPINES MERIDIAN JUNIOR COLLEGE

JC2 PRELIMINARY EXAMINATION

CANDIDATE NAME: ____________________

CIVICS GROUP: ____________________

H2 MATHEMATICS **9758/01**

Paper 1

13 SEPTEMBER 2022

3 hours

Candidates answer on the question paper.

Additional material: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and Civics Group on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

For Examiners' Use	
1	
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Total	

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

This document consists of 6 printed pages and 0 blank pages.

[Turn Over

1 **(i)** Express $\frac{3}{r}+\frac{2}{r+1}-\frac{5}{r+2}$ as a single fraction. [1]

(ii) Hence find $\sum_{r=1}^{n}\frac{4r+3}{r(r+1)(r+2)}$. [3]

(iii) Use your answer to part **(ii)** to find $\sum_{r=3}^{n}\frac{4r-5}{r(r-1)(r-2)}$. [2]

2 **(a)** Find $\int \frac{1}{\sqrt{(1-x^2)\sin^{-1}x}}\,dx$. [2]

(b) Find $\int \frac{x-3}{x^2-2x+4}\,dx$. [3]

3 The curve C is defined by the parametric equations

$$x=a^2t^2,\ \ y=e^{at}, \qquad \text{for } t\geq 0,$$

where a is a positive constant.

Find the exact area enclosed by C, the axes and the line $x=8$. [5]

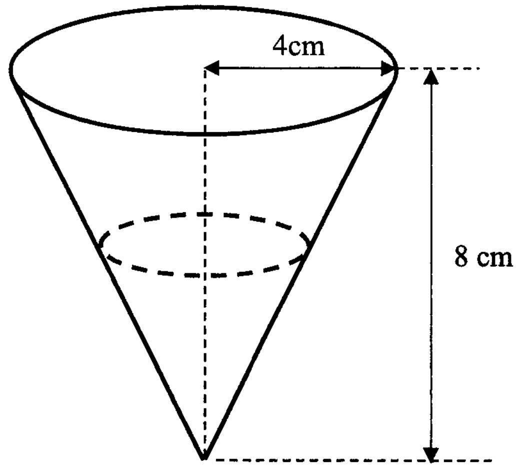
4 On the same axes, sketch the curves with equation $y=-x^2+5x-3$ and $y=|3-x|$, labelling the axial intercepts. [2]

Hence, without using a calculator, solve the inequality $-x^2+5x-3<|3-x|$. [4]

5 Given that $\ln y=\sin kx$ where k is a non-zero constant, show that $\frac{d^2y}{dx^2}+k^2y\ln y-\frac{1}{y}\left(\frac{dy}{dx}\right)^2=0$. Hence obtain the expansion of y in ascending powers of x, up to and including the term in x^2. [5]

Using the standard series given in MF26, verify that the same result is obtained and determine the coefficient of x^3. [4]

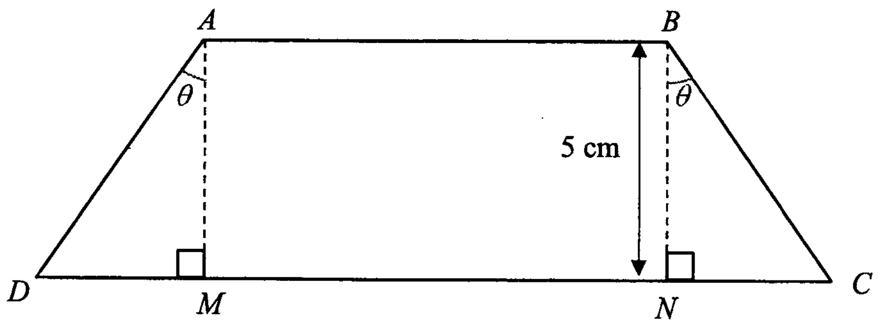
6 **(a)** Disposable cups in the shape of an inverted right cone of radius 4 cm and height 8 cm are being produced by a factory. For quality check, the factory supervisor took a cup and tested it by filling it completely with water. Water was found leaking at the vertex of the cup at a constant rate of 1.5 cm^3 per second. The diagram below shows the cup.



Find the exact rate at which the water level is decreasing when the depth of the water is 2 cm. [4]

[The volume of a cone of base radius r and height h is given by $\frac{1}{3}\pi r^2 h$.]

(b) The diagram below shows a trapezium $ABCD$ with height 5 cm such that $DA + AB + BC = 20$ cm. Points M and N are the foot of the perpendiculars from points A and B to line DC respectively, and $\angle DAM = \angle CBN = \theta$, where $0 < \theta < \frac{\pi}{2}$.



(i) Show that the area, A cm^2, of trapezium $ABCD$ is $100 - 50\sec\theta + 25\tan\theta$. [3]

(ii) Hence, by differentiation, find the maximum area of trapezium $ABCD$, giving your answer in exact form. [You do not need to verify that it is a maximum area.] [4]

[Turn Over

7 **(a)** The function f is defined as

$$f : x \mapsto ax + \frac{a}{x-1}, \; x \in \mathbb{R}, x \neq 1,$$

where a is a constant greater than 1.

(i) Sketch the graph of $y = f(x)$, indicating clearly the coordinates of the turning points and the equations of the asymptotes. [3]

(ii) Explain why f does not have an inverse. [1]

The function g is defined as

$$g : x \mapsto \frac{1}{x^2-1}, \; x \in \mathbb{R}, x \neq -1, x \neq 1.$$

(iii) Determine, with a reason, whether gf exists. [2]

(b) A curve C has equation $y = e^{\frac{x^2-4}{12}}$, where $x > 0$.

The curve C undergoes the following sequence of transformations:

A: Stretch by factor $\frac{1}{2}$ parallel to the x-axis.

B: Translate by 1 unit in the negative y-direction.

C: Reflection in the line $y = x$.

Find the equation of the new curve in the form $y = q(x)$ and state the domain of q.

[5]

8 The plane p_1 has cartesian equation $x + z = 3$. The plane p_2 is perpendicular to p_1 and contains the line l_1 with equation $\frac{x+2}{5} = \frac{y+1}{2} = \frac{3-z}{3}$.

(i) Show that the cartesian equation of the plane p_2 is $-x + 4y + z = 1$. [2]

(ii) Find a vector equation of the line l_2 given that p_1 and p_2 intersect at l_2. [2]

(iii) It is given that the point B with coordinates (0 , 4, 3) is on p_1 and the perpendicular distance from B to p_2 is k. Find the position vector of the foot of perpendicular from B to p_2 and deduce the value of k. [4]

(iv) Hence, find the vector equations of the lines in p_1 such that the perpendicular distance from each line to p_2 is k. [3]

9 **(a)** The complex number z is given by $z = x + yi$, where x and y are non-zero real numbers. Given that $|z| = 1$, find the possible values of z for which $\frac{(z^2)^*}{z}$ is real. [6]

(b) Without the use of a calculator, find the roots of the equation $z^2 = 33 + 56i$, expressing your answer in cartesian form $x + iy$ where x and y are real. [4]

Hence, find in cartesian form, the roots of the equation $w^2 = -33 + 56i$. [2]

10 A squirrel falls vertically from a tall tree. The distance, x metres, that the squirrel has fallen from the tree after t seconds is observed. It is given that $x = 0$ and $\frac{dx}{dt} = 0$ when $t = 0$.

The motion of the squirrel is modelled by the differential equation

$$\frac{d^2x}{dt^2} + 0.1\left(\frac{dx}{dt}\right)^2 = 10.$$

(i) By substituting $y = \frac{dx}{dt}$, show that the differential equation can be written as $\frac{dy}{dt} = 10 - 0.1y^2$. [1]

(ii) Find y in terms of t and hence find x in terms of t. [8]

(iii) How far has the squirrel fallen after 2 seconds? [1]

(iv) For a falling object, the terminal velocity is the value approached by the velocity after a long time. Find the terminal velocity of the falling squirrel. [2]

[Turn Over

11 Mrs Toh intends to invest \$1000 per year in a savings plan, starting from 1 January 2023. Savings plan A allows her to invest a fixed amount of \$1000 into account A on the first day of every year. The amount in account A earns an interest of 3.5% per annum at the end of each year of investment.

(i) Show that the total amount in account A at the end of n years is in the form $p\left(q^n - 1\right)$, where p and q are exact constants to be determined. [3]

(ii) On what date will the total amount in account A first exceed \$36000? [3]

Savings plan B allows Mrs Toh to invest \$1000 into account B on 1 January 2023. The amount to be invested on the first day of the subsequent years will increase by \$ k per annum. A fixed annual bonus of \$40 is added to account B at the end of each year of investment.

(iii) If $k = 36$, find the year in which the total amount in account A first exceeds the total amount in account B. [4]

(iv) Find the least value of k, giving your answer to the nearest whole number, such that the total amount in account B is more than the total amount in account A at the end of 31 December 2032. [2]

End of Paper