ANDERSON SERANGOON JUNIOR COLLEGE

H2 MATHEMATICS **9758**

JC2 Prelim Paper 1 (100 marks) **9 Sept 2024**

3 hours

Additional Material(s): List of Formulae (MF 26)

CANDIDATE NAME

CLASS | | | / | | |

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Question number	Marks
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This document consists of 23 printed pages and 5 blank pages.

[Turn Over

1 **(a)** Sketch the graphs of $y = 3e^x$ and $y = x + 3$ on the same diagram. Indicate clearly the coordinates of the points of intersection between the 2 graphs.

Solve the inequality $3e^x > x + 3$. [3]

(b) Hence find $\int_{-2}^{2} \left|3e^x - x - 3\right| \, dx$, giving your answer in an exact form. [2]

2 **(i)** Find $\dfrac{d}{dx}\left(e^{\sin^{-1} x}\sqrt{1 - x^2}\right)$. [1]

(ii) Hence using integration by parts, find $\displaystyle\int x \frac{e^{\sin^{-1} x}}{\sqrt{1 - x^2}} \, dx$. [3]

3 The curve C has parametric equations

$$x = 2t - \frac{1}{t^2}, \quad y = 2t + \frac{1}{t}, \qquad t \in \square, t \neq 0.$$

The point P on the curve has parameter $t = 1$.

(i) Find the equation of tangent and normal to C at the point P. [4]

(ii) The tangent at P meets the y-axis at B. The normal at P meets the x-axis at A. If O is the origin, find the area of the quadrilateral $OAPB$. [2]

4 A sequence is such that $u_0 = 2$ and $u_n = u_{n-1} + n^3 + \left(\frac{1}{2}\right)^n$ for $n \geq 1$.

(a) It is given that $\displaystyle\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$. By considering $\displaystyle\sum_{r=1}^{n} (u_r - u_{r-1})$, find a formula for u_n in terms of n. [4]

(b) Hence, using the formula of u_n found in **(a)**, find $\displaystyle\sum_{r=9}^{n}\left((r+2)^3 + \left(\frac{1}{2}\right)^{r+2}\right)$ exactly. [3]

5 **(a)** It is given that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-zero vectors.

If $|\mathbf{a}+\mathbf{b}|=|\mathbf{a}-\mathbf{b}|$, show that the two vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular to each other. [4]

(b) (i) Explain why the result of

$$\mathbf{a}\times(\mathbf{b}+\mathbf{c})+\mathbf{b}\times(\mathbf{c}+\mathbf{a})+\mathbf{c}\times(\mathbf{a}+\mathbf{b})$$

is a vector. [1]

(ii) Simplify $\mathbf{a}\times(\mathbf{b}+\mathbf{c})+\mathbf{b}\times(\mathbf{c}+\mathbf{a})+\mathbf{c}\times(\mathbf{a}+\mathbf{b})$. Show your workings clearly. [3]

6 **(i)** The variables x and y are related by

$$(x+y)\frac{\mathrm{d}y}{\mathrm{d}x}+ky=2 \text{ and } y=1 \text{ at } x=0,$$

where k is a constant. Show that $(x+y)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(1+k)\frac{\mathrm{d}y}{\mathrm{d}x}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=0$. [1]

(ii) Given that x is small, find the series expansion of $\mathrm{g}(x)=\dfrac{1}{\sin^2\left(2x+\frac{\pi}{2}\right)}$ in ascending powers of x, up to and including the term in x^2.

If the coefficient of x^2 in the expansion of $\mathrm{g}(x)$ is equal to twice the coefficient of x^2 in the Maclaurin series for y in **(i)**, find the value of k. [5]

(iii) By further differentiation of the result found in **(i)**, and taking $k=1$, find the Maclaurin series for y, up to and including the term in x^3. [3]

7 **(a)** State a sequence of transformations that will transform the curve with equation $y^2-x^2=1$ on to the curve with equation

$9y^2-54y-x^2-2x+79=0$. [4]

(b) A curve C has equation

$$9y^2-54y-x^2-2x+79=0.$$

(i) For real values x, use a non-graphical method to determine that y cannot lie between a and b, where a and b are exact real constants to be determined. [3]

(ii) Sketch the curve C, indicating clearly the equations of all asymptotes and the coordinates of the turning points. [3]

(iii) By adding a suitable curve, determine the number of real roots of the equation,

$$9\left[(x+1)^2+3\right]^2-54\left[(x+1)^2+3\right]-x^2-2x+79=0.$$ [2]

[Turn Over

8 The functions f and g are defined by

$$f : x \mapsto \left|4+2x-x^2\right|,\ x \in \square,\ x \geq 3.5,$$

$$g : x \mapsto 4+e^{ax},\ x \in \square,\ x \geq -1,$$

where $a > 0$.

(a) Find $f^{-1}(x)$ and state its domain. [3]

(b) Find the value of x for which $f^{-1}(x) = f(x)$. [2]

(c) Show that the composite function fg exists and express the exact range of fg in the form of $A + Be^{-a} + Ce^{-2a}$, where A, B and C are real constants. [4]

(d) Without the use of a graphing calculator, solve the inequality $\dfrac{g(x)}{x^2-2x-2} \geq 0$.

Leave your answer in exact form. [3]

9 **(a)** The complex numbers z_1 and z_2 are given by

$$z_1 = \frac{1+i}{1-i} \text{ and } z_2 = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right).$$

(i) Find $z_1 + z_2$ in the form $re^{i\theta}$, where r is an exact real constant in trigonometric form such that $r > 0$, and θ is in the form $k\pi$ where k is an exact real constant such that $-1 < k \leq 1$. [3]

(ii) Find also $z_1 + z_2$ in the form $x + iy$, where x and y are exact real constant.

Hence show that $\tan\left(\dfrac{3\pi}{8}\right) = 1 + \sqrt{2}$. [2]

(b) The complex number w is given by $w = \cos\theta + i\sin\theta$, where $0 < \theta < \dfrac{\pi}{2}$.

(i) Show that $1 - w^2 = -2iw\sin\theta$. [2]

(ii) Hence find the modulus and argument of $1 - w^2$ in terms of θ. [2]

(iii) Given that $\left(\dfrac{1-w^2}{iw^*}\right)^n$ is real and negative and that $\theta = \dfrac{\pi}{5}$, find the three smallest positive integer values of n. [3]

10 A rice retailer pledges to donate a bowl of rice for every kilometre run by participants in a service-learning project. Donations will be made in complete bowls, based on the cumulative distance each individual ran by the end of the 28-day period. Distances ran by multiple individuals will not be combined. For example, if person A runs 18.8 km and person B runs 11.2 km, the retailer will donate a total of 29 bowls. Two such participants, athlete A and B, will each accumulate the distance they run for a total of 28 days via a plan each devised.

- Athlete A plans to run 5 km on the first day and then increase the distance by a fixed 0.65 km more than the previous day.
- Athlete B plans to run 7 km on the first day and then increase the distance by 4% more than the previous day.

(a) Determine the least number of days required for the cumulative distance of athlete A to exceed that of athlete B. [3]

(b) How many bowls of rice will both athletes contribute, in total, at the end of the 28-day period? [3]

(c) Suppose athlete A plans to cover at least 400 km by the end of the 28-day period, what is the minimum distance he should run in day 1 if the plan to increase by 0.65 km more than the previous day remains the same. Give your answer to the nearest metres. [3]

(d) On days where the distance athlete B is supposed to run exceeds 10 km based on his own plan, he will limit it to exactly 10 km instead. Given this change, how many bowls of rice will he contribute at the end of the 28-day period? [3]

11 In a large town, the number of people infected by a particular virus t days after the virus was first discovered is x. It is assumed that the rate of infection is proportional to x. Initially there are 5 people who are infected by the virus, and there are 5120 people who are infected by the virus 30 days after the virus was first discovered.

(i) Show that $x = 5(2)^{\frac{t}{3}}$. [5]

A cure and vaccine for the virus were discovered and administered to the population 30 days after the virus was discovered. Individuals who were cured are not at risk of reinfection. The number of people infected by the virus p days after the cure and vaccine were administered is represented by y. It is believed that the new rate of infection from then on is proportional to $6400y - y^2$.

It is given that 30 days after the cure or vaccine was administered, 3200 people remain infected with the virus.

(ii) Show that $y = \dfrac{6400}{1+2^{\left(\frac{p}{15}+H\right)}}$, where H is a constant to be determined. [6]

(iii) By finding the number of people in the town that will be infected by the virus in the long term, comment on the effectiveness of the cure and vaccine administered. [2]

End of Paper

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H2 MATHEMATICS **9758/2**

JC2 Prelim Paper 2 (100 marks) **16 Sept 2024**

3 hours

Additional Material(s): List of Formulae (MF26)

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Section A: Pure Mathematics [40 marks]

1 By using the substitution $x = \cot\theta$, for $0 < \theta < \frac{\pi}{4}$, find $\int \frac{1}{x^2\sqrt{1+x^2}}\,dx$. [4]

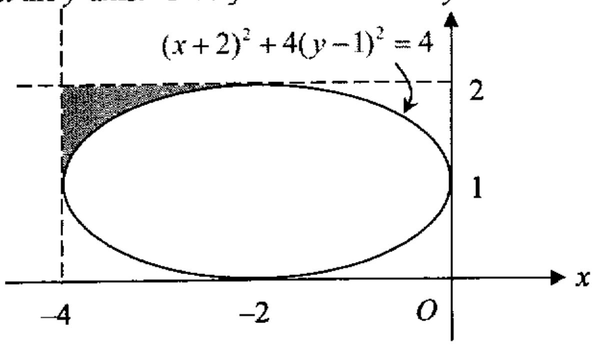
2 **(a)(i)** Find $\int \frac{9u-8}{4+9u^2}\,du$. [3]

(ii) The curve C is given by the parametric equations

$$x = u^2 + u + 1,\ y = \frac{9u}{4+9u^2}, \text{where } u \geq 0.$$

Find the exact area bounded by C, the x-axis and the line $x = 3$. [4]

(b) Find the volume of the solid formed when the shaded region bounded by the lines $x = -4$, $y = 2$ and the ellipse $(x+2)^2 + 4(y-1)^2 = 4$ is rotated through 2π radians about the y-axis. Give your answer correct to 1 decimal place.



[3]

3 With respect to the origin O, the points A, B, C, D and E have coordinates $A(2,3,4)$, $B(6,5,7)$, $C(8,9,6)$, $D(4,7,3)$ and $E(5,6,10)$.

(a) Show that the cartesian equation of the surface containing the points A, B and E is $x - 5y + 2z = -5$. [2]

A line passes through the point D and the midpoint M of the line segment EC.

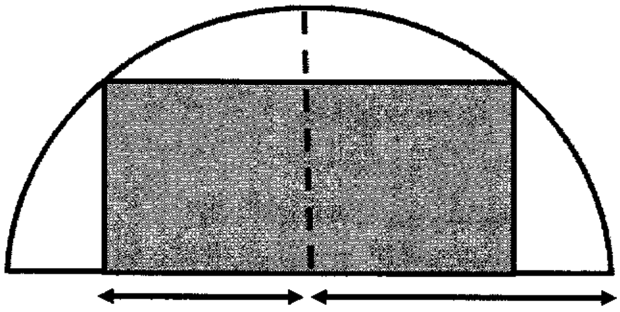
(b) Find the vector equation of the line DM. [3]

(c) Find the exact coordinates of the foot of the perpendicular from the point M to the surface found in part **(a)**. [3]

(d) Hence find the exact shortest distance from the point M to the surface found in part **(a)**. [2]

(e) Verify the line DM intersects the surface found in part **(a)** at the point P with coordinates $(9,8,13)$. Hence find the vector equation of the reflection of the line DM about this slant surface. [4]

4 A popular toy company is designing a new water play feature for children. The toy consists of a cylindrical water container that will hold water for various playful activities. This cylindrical container is designed to be inscribed within a fixed, rigid hemispherical shell made of durable plastic of negligible thickness.



The shaded region in the diagram above shows the cross-sectional view of the upright cylindrical container that is inscribed in a hemisphere with fixed radius $2a$ cm.

(a) If the radius of the cylindrical water container is r cm, show that the volume V cm^3 of the water container is given by $V = \pi r^2\sqrt{4a^2 - r^2}$. [1]

The unique feature of this toy is that the height of the cylindrical container is adjustable, allowing it to expand or contract while always touching the inner surface of the hemisphere.

(b) Water is pumped into the container at a rate of 100π cm^{3}s^{-1} while the adjustment is taking place. If $a = 50$, find the exact rate of change of the radius of the container at the instant when the height of the water container is 80 cm. [5]

(c) Using differentiation, find in terms of a, the value of r which gives a maximum value of V. Justify that this value indeed gives a maximum V. Hence write down the exact maximum volume of the cylinder in terms of a. [4]

(d) Sketch the graph showing the volume of the cylinder as the radius of the cylinder varies. [2]

Section B: Probability and Statistics [60 marks]

5 An amateur music composer is arranging a sequence of four musical notes followed by three beats. There are 7 possible notes (labeled A to G) and 5 possible beats (labeled 1 to 5). The order of the notes and beats is important in the composition. Find the probability that a randomly chosen sequence has

(i) the third beat being a higher number than the second beat, [1]

(ii) exactly two notes the same or exactly two beats the same, but not both. [3]

6 Tetrahedral dice have four faces. Two fair tetrahedral dice, one red and one blue, have faces numbered 0, 1, 2, and 3 respectively. The dice are rolled, and the numbers faced down on the two dice are recorded. The random variable T is defined as the score on the red die multiplied by the score on the blue die.

(i) Find the probability distribution of T. [3]

(ii) Find $\mathrm{E}(T)$ and show that $\mathrm{Var}(T) = \frac{115}{16}$. Show your workings clearly. [2]

(iii) Evaluate $\mathrm{P}(|T-2\mu| > \sigma)$, where $\mu = \mathrm{E}(T)$ and $\sigma^2 = \mathrm{Var}(T)$. [2]

7 The masses, in grams, of the packets of semolina flour follow the distribution $\mathrm{N}(225, 25^2)$ and the masses, in grams, of the packets of millet flour follow the distribution $\mathrm{N}(\mu, \sigma^2)$.

(a) Find the probability that 4 times the mass of a packet of semolina flour is between 0.85 kilograms and 1.05 kilograms. [2]

(b) Let M be the mean mass of 3 packets of semolina flour and 2 packets of millet flour. Given that $\mathrm{P}(M < 125) = \mathrm{P}(M > 265) = 0.02$, show that the value of μ is 150. Hence, by finding an equation involving σ, find the value of σ. [5]

8 An office team of 10 people includes 7 men and 3 women named Anne, Beth, and Cathie. For an upcoming fire drill exercise, 5 individuals will be chosen, each assigned a unique role, to carry out the drill. Determine the number of possible ways to select 5 people from this group of 10

(i) to conduct the fire drill, [1]

(ii) such that at most 1 woman is selected to conduct the fire drill. [2]

(iii) After the fire drill exercise, the 10 people are to hold a discussion at a round table with 10 identical seats. Determine the number of ways in which Beth is seated between Anne and Cathie. [1]

(iv) A group photo of the 10 people, arranged in two rows of five, was taken after the discussion. Determine the number of ways in which Beth is not standing beside Anne or Cathie. [4]

9 A bakery produces batches of cookies. On average, the proportion of flawed cookies produced is p, where $0 < p < 1$. The cookies are packed in boxes of 20. The number of flawed cookies in a box of cookies is denoted by C.

(a) State, in context, one assumption needed for the number of flawed cookies in a box to be well-modelled by a binomial distribution. [1]

(b) Given that $\mathrm{P}(C = 0 \text{ or } 1) = 0.15$, write down an equation for the value of p, and find this value numerically. [2]

For **(c)** and **(d)**, take $p = 0.08$.

(c) Ten boxes of cookies are randomly chosen. As part of the bakery's quality control process, a box of cookies will be accepted if it contains fewer than 4 flawed cookies, otherwise it will be rejected. Find the probability there are at least 2 but no more than 5 rejected boxes. [4]

(d) A random sample of 15 boxes of cookies is taken and 3 of the boxes are found to be rejected. Find the probability that the third rejected box occurs on the fifteenth box. [3]

10 **(a)** Observations of 8 pairs of values (u, g), representing the hours of internet usage per week (u) and academic performance (g) in terms of Grade Point Average (GPA), are shown in the table below.

Internet usage (u)	4.0	6.0	8.0	a	12.0	16.0	18.0	20.0
GPA (g)	3.7	3.5	3.4	3.2	3.0	2.7	2.6	2.5

It is known that the equation of the linear regression line of g on u is $g = -0.0765u + 3.99$, find the value of a correct to 1 decimal place. [2]

(b) A researcher is studying the relationship between the battery life (y, in hours) of a new smartphone model and the screen brightness setting (x, in %). The following data was collected from the tests conducted at different brightness levels.

Screen Brightness (x)	10	20	30	40	50	60	70
Battery life (y)	48.2	47.4	45.5	37.3	35.6	31.1	24.3

(i) Draw a scatter diagram for these values. [2]

(ii) One of the values of y appears to be incorrect. Circle this point on your diagram and label it P. [1]

(iii) Explain why a linear model $y = a + bx$ is not a suitable model. [1]

(iv) It is thought that the battery life (y) can be modelled by one of the formulae after removing the point P.

(A) $y = a + bx^2$,

(B) $y = a + b\ln x$,

where a and b are non-zero constants.

Find, correct to 4 decimal places, the product moment correlation coefficient between y and x^2 as well as y and $\ln x$.

Explain clearly which model is a better model for this set of data.

For the case identified, find the equation of a suitable regression line. [3]

(v) Using the regression line found in **(iv)**, estimate the battery life when the screen brightness is set to 80%. [1]

(vi) Comment on the reliability of your answer in part **(v)**. [1]

11 **(a)** The leaves of a particular plant species have an average length of 12 cm with a standard deviation of 3.5 cm. If a random sample of 100 leaves is selected, estimate the probability that their total length is at least 1138 cm. [2]

(b) An operator of a public workspace at location A claims that users of its one-seater pods spend an average of 131 minutes using the facilities. To test this claim, a random sample of 64 users was observed, revealing a mean usage time of 127 minutes with a standard deviation of 16.4 minutes.

(i) Test at 3% level of significance whether the workspace operator's claim is overstated. You should state the hypotheses and define any symbols you use. [5]

(ii) Explain the meaning of '3% level of significance' in the context of the question. [1]

(iii) The workspace operator at location B claims that the mean time spent by users of its one-seater pods is 140 minutes, with a known population standard deviation of 20.1 minutes. A new sample of 15 pod users is taken, and the sample mean usage time, $\bar{w}$, is reported. A hypothesis test is conducted at a 5% significance level, and the operator's claim is not rejected.

State two necessary assumptions for the test and determine the range of values that $\bar{w}$ can take. Give your answer correct to one decimal place. [5]

End of paper

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H2 MATHEMATICS **9758**

JC2 Prelim Paper 1 (100 marks) **9 Sept 2024**

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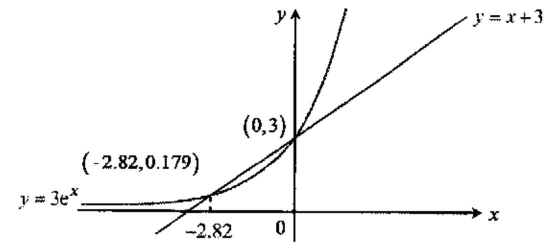
1 (a)



From the graphs, $x<-2.82$ or $x>0$.

$\int_{-2}^{2}\left|3e^x-x-3\right|\,dx$

$=\int_{-2}^{0}-(3e^x-x-3)dx+\int_{0}^{2}(3e^x-x-3)dx$

$=-\left[3e^x-\frac{x^2}{2}-3x\right]_{-2}^{0}+\left[3e^x-\frac{x^2}{2}-3x\right]_{0}^{2}$

$=-[3-(3e^{-2}-2+6)]+[(3e^2-2-6)-3]$

$=3e^{-2}+3e^2-10$

2 (i) $\frac{d}{dx}\left(e^{\sin^{-1}x}\sqrt{1-x^2}\right)=e^{\sin^{-1}x}-\frac{xe^{\sin^{-1}x}}{\sqrt{1-x^2}}\cdots\cdots(1)$

(ii) $\int x\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}dx=xe^{\sin^{-1}x}-\int e^{\sin^{-1}x}dx$

From (1), $e^{\sin^{-1}x}\sqrt{1-x^2}+\int\frac{xe^{\sin^{-1}x}}{\sqrt{1-x^2}}dx+c=\int e^{\sin^{-1}x}dx$

So $\int x\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}dx=xe^{\sin^{-1}x}-e^{\sin^{-1}x}\sqrt{1-x^2}-\int\frac{xe^{\sin^{-1}x}}{\sqrt{1-x^2}}dx-c$

$\therefore\int x\frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}dx=\frac{1}{2}\left(xe^{\sin^{-1}x}-e^{\sin^{-1}x}\sqrt{1-x^2}\right)+D$

3 (i)

$\frac{dx}{dt}=2+\frac{2}{t^3}$

$\frac{dy}{dt}=2-\frac{1}{t^2}$

$\frac{dy}{dx}=\frac{2-\frac{1}{t^2}}{2+\frac{2}{t^3}}=\frac{t(2t^2-1)}{2t^3+2}$

When $t=1$,

BP-55

$\frac{dy}{dx}=\frac{1}{4}$, $x=1$ and $y=3$.

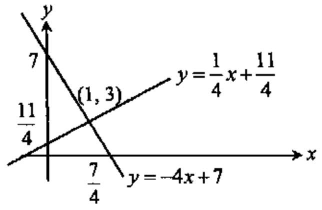
Eqn of tangent at P: $y-3=\frac{1}{4}(x-1)$

$y=\frac{1}{4}x+\frac{11}{4}$

Eqn of normal at P: $y-3=-4(x-1)$

$y=-4x+7$

(ii)



Area of $OAPB=\frac{1}{2}\left(\frac{7}{4}\right)(7)-\frac{1}{2}\left(7-\frac{11}{4}\right)(1)=4\text{ units}^2$

4 **(a)** $u_r-u_{r-1}=r^3+\left(\frac{1}{2}\right)^r$

$$\sum_{r=1}^{n}(u_r-u_{r-1})=\sum_{r=1}^{n}\left[r^3+\left(\frac{1}{2}\right)^r\right]$$

$$\begin{pmatrix} u_1-u_0 \\ +u_2-u_1 \\ +u_3-u_2 \\ + \\ \vdots \\ +u_{n-1}-u_{n-2} \\ +u_n-u_{n-1} \end{pmatrix}=\sum_{r=1}^{n}r^3+\sum_{r=1}^{n}\left(\frac{1}{2}\right)^r$$

$$u_n-u_0=\frac{n^2(n+1)^2}{4}+\left(1-\left(\frac{1}{2}\right)^n\right)$$

$$\therefore u_n=\frac{n^2(n+1)^2}{4}+\left(1-\left(\frac{1}{2}\right)^n\right)+2$$

$$=\frac{n^2(n+1)^2}{4}-\left(\frac{1}{2}\right)^n+3$$

(b) $\sum_{r=9}^{n}\left((r+2)^3+\left(\frac{1}{2}\right)^{r+2}\right)$

$$=\sum_{r=11}^{r=n+2}\left(r^3+\left(\frac{1}{2}\right)^r\right)\ (\because \text{Replace } r \text{ by } r-2)$$

$$=u_{n+2}-u_{10}$$

$$=\left[\frac{(n+2)^2(n+3)^2}{4}-\left(\frac{1}{2}\right)^{n+2}+3\right]-\left[\frac{(10)^2(11)^2}{4}-\left(\frac{1}{2}\right)^{10}+3\right]$$

$$=\frac{(n+2)^2(n+3)^2}{4}-\left(\frac{1}{2}\right)^{n+2}-3025+\frac{1}{1024}$$

5 **(a)** $|\mathbf{a}+\mathbf{b}|=|\mathbf{a}-\mathbf{b}|$

$\Rightarrow|\mathbf{a}+\mathbf{b}|^2=|\mathbf{a}-\mathbf{b}|^2$

$\Rightarrow(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})=(\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})$

$\Rightarrow|\mathbf{a}|^2+2\mathbf{a}\cdot\mathbf{b}+|\mathbf{b}|^2=|\mathbf{a}|^2-2\mathbf{a}\cdot\mathbf{b}+|\mathbf{b}|^2$

$\Rightarrow 4\mathbf{a}\cdot\mathbf{b}=0$

$\Rightarrow 4|\mathbf{a}||\mathbf{b}|\cos\theta=0$

Since $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors, then $\theta=90^\circ$, thus $\mathbf{a}\perp\mathbf{b}$

(b)(i) Since $\mathbf{a}\times(\mathbf{b}+\mathbf{c})$, $\mathbf{b}\times(\mathbf{c}+\mathbf{a})$ and $\mathbf{c}\times(\mathbf{a}+\mathbf{b})$ are all vectors, the addition of these vectors will lead to a resultant vector.

(ii) $\mathbf{a}\times(\mathbf{b}+\mathbf{c})+\mathbf{b}\times(\mathbf{c}+\mathbf{a})+\mathbf{c}\times(\mathbf{a}+\mathbf{b})$

$=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}+\mathbf{b}\times\mathbf{a}+\mathbf{c}\times\mathbf{a}+\mathbf{c}\times\mathbf{b}$

$=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}-\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{c}-\mathbf{b}\times\mathbf{c}$ ($\because \mathbf{b}\times\mathbf{a}=-\mathbf{a}\times\mathbf{b}$ and $\mathbf{c}\times\mathbf{a}=-\mathbf{a}\times\mathbf{c}$)

$=\mathbf{0}$

will not be given if the student wrote it as a scalar quantity

6 **(i)**

$(x+y)\frac{dy}{dx}+ky=2 \quad \cdots(1)$

Differentiating (1) w.r.t. x:

$$(x+y)\frac{d^2y}{dx^2}+\left(1+\frac{dy}{dx}\right)\frac{dy}{dx}+k\frac{dy}{dx}=0$$

$$(x+y)\frac{d^2y}{dx^2}+(1+k)\frac{dy}{dx}+\left(\frac{dy}{dx}\right)^2=0 \quad \cdots(2)$$

(ii) $\sin\left(2x+\frac{\pi}{2}\right)=\sin 2x\cos\frac{\pi}{2}+\cos 2x\sin\frac{\pi}{2}=\cos 2x$

$$\frac{1}{\sin^2\left(2x+\frac{\pi}{2}\right)}=\frac{1}{\cos^2 2x}$$

$$\approx\left(1-\frac{(2x)^2}{2}\right)^{-2}$$

$=\left(1-2x^2\right)^{-2}$

$=1+4x^2+\ldots$

$x=0,\ \ y=1:\ \dfrac{dy}{dx}=2-k$

$\dfrac{d^2y}{dx^2}=3k-6$

$4=2\left(\dfrac{3k-6}{2}\right)$

$k=\dfrac{10}{3}$

(iii)
Differentiating (2) w.r.t. x:

$(x+y)\dfrac{d^3y}{dx^3}+\left(1+\dfrac{dy}{dx}\right)\dfrac{d^2y}{dx^2}+2\dfrac{d^2y}{dx^2}+2\left(\dfrac{dy}{dx}\right)\left(\dfrac{d^2y}{dx^2}\right)=0$

$(x+y)\dfrac{d^3y}{dx^3}+\left(3+3\dfrac{dy}{dx}\right)\dfrac{d^2y}{dx^2}=0$

When $x=0$, $\ y=1,\ \dfrac{dy}{dx}=1,\ \dfrac{d^2y}{dx^2}=-3,\ \dfrac{d^3y}{dx^3}=18$

$\therefore y=1+x-\dfrac{3}{2}x^2+3x^3+\ldots$

7 **(a)** $9y^2-54y-x^2-2x+79=0$

$\Rightarrow 9\left(y^2-6y\right)-\left(x^2+2x\right)+79=0$

$\Rightarrow 9\left[(y-3)^2-9\right]-\left[(x+1)^2-1\right]+79=0$

$\Rightarrow 9(y-3)^2-(x+1)^2=1$

$\Rightarrow (3y-9)^2-(x+1)^2=1$

$y^2-x^2=1$

Replace x by $x+1$ ↓ Translation of 1 unit in the negative x direction

$y^2-(x+1)^2=1$

Replace y by $y-9$ ↓ Translation of 9 units in the positive y direction

$(y-9)^2-(x+1)^2=1$

Replace y by $3y$ ↓ Scaling parallel to the y – axis by a factor of $\dfrac{1}{3}$

$(3y-9)^2-(x+1)^2=1$

(b)(i) Consider the line $y=k$. To find the range of y where curve C cannot lie,

$\Rightarrow (3k-9)^2-(x+1)^2=1$ has no real roots

$\Rightarrow x^2+2x+2-(3k-9)^2=0$ has no real roots.

$\Rightarrow 4-4\left[2-(3k-9)^2\right]<0$

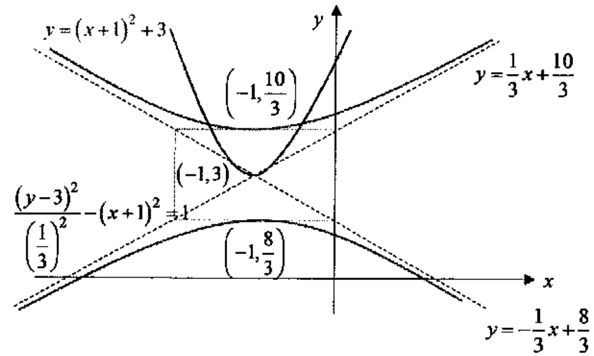
$(3k-9)^2-1<0$

$(3k-9+1)(3k-9-1)<0$

$\dfrac{8}{3}<k<\dfrac{10}{3}$

Thus y cannot lie between $\dfrac{8}{3}$ and $\dfrac{10}{3}$.

(ii)



(iii) By adding the graph of $y=(x+1)^2+3$, since there are 2 intersections between the 2 curves, so the equation

$9\left[(x+1)^2+3\right]^2-54\left[(x+1)^2+3\right]-x^2-2x+79=0$ will have 2 real roots.

8 **(a)** $y=\left|4+2x-x^2\right|,\ x\in\square,\ x\geq 3.5$

$\therefore y=-\left(4+2x-x^2\right)\quad (\because 4+2x-x^2<0 \text{ for } x\geq 3.5)$

$y=x^2-2x-4$

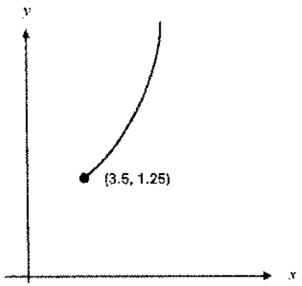
$y=(x-1)^2-1-4$

$y=(x-1)^2-5$

$x=1+\sqrt{y+5}$ or $x=1-\sqrt{y+5}$ (rej $\because x\geq 3.5$)

$x=1+\sqrt{y+5}$

$f^{-1}(x)=1+\sqrt{x+5}$



$R_f = [1.25, \infty)$

$\therefore D_{f^{-1}} = [1.25, \infty)$

(b) $f^{-1}(x) = f(x),\ x \in D_f \cap D_{f^{-1}}$

$x = f(x),\ x \in [3.5,\ \infty)$

$x = x^2 - 2x - 4$

If equate f^{-1} found in **(a)** to f, can also give the

$x^2 - 3x - 4 = 0$

$(x-4)(x+1) = 0$

$x = -1$ (rej $\because x \geq 3.5$) or $x = 4$

(c) $R_g = \left[4 + e^{-a}, \infty\right)$

$D_f = [3.5, \infty)$

Since $e^{-a} > 0$, $\therefore 4 + e^{-a} > 3.5$

$\therefore R_g \subset D_f$, fg exists.

$R_g = \left[4 + e^{-a}, \infty\right)$

$f(4 + e^{-a}) = (4 + e^{-a})^2 - 2(4 + e^{-a}) - 4$

$= 16 + 8e^{-a} + e^{-2a} - 8 - 2e^{-a} - 4$

$= 4 + 6e^{-a} + e^{-2a}$

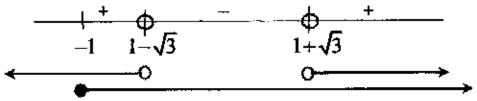
$R_{fg} = \left[4 + 6e^{-a} + e^{-2a}, \infty\right)$

(d) $\dfrac{g(x)}{x^2 - 2x - 2} \geq 0$ AND $x \in D_g$

$\dfrac{4 + e^{ax}}{(x-1)^2 - 3} \geq 0$ AND $x \geq -1$

Since $4 + e^{ax} > 0,\ \forall x \in \square$

$\therefore \dfrac{1}{(x - 1 + \sqrt{3})(x - 1 - \sqrt{3})} \geq 0$ AND $x \geq -1$



$\left(x < 1 - \sqrt{3} \text{ or } x > 1 + \sqrt{3}\right)$ AND $x \geq -1$

$-$ for $x < 1 - \sqrt{3}$ or $x > 1 + \sqrt{3}$

$\therefore -1 \leq x < 1 - \sqrt{3}$ **OR** $x > 1 + \sqrt{3}$

9 **(ai)** $z_1 = \left(\dfrac{1+i}{1-i}\right) \times \left(\dfrac{1+i}{1+i}\right) = \dfrac{1}{2}(1 + 2i - 1) = i = e^{i\left(\frac{\pi}{2}\right)}$

$z_2 = \cos\left(\dfrac{\pi}{4}\right) + i\sin\left(\dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i$

$z_1 + z_2 = e^{\frac{\pi}{2}i} + e^{\frac{\pi}{4}i} = e^{\frac{1}{2}\left(\frac{\pi}{2}i + \frac{\pi}{4}i\right)}\left(e^{\frac{\pi}{8}i} + e^{-\frac{\pi}{8}i}\right)$

$= 2\cos\dfrac{\pi}{8} \times e^{\frac{3\pi}{8}i}$

(ii) $z_1 + z_2 = \dfrac{1}{\sqrt{2}} + \left(1 + \dfrac{1}{\sqrt{2}}\right)i$

$\tan\dfrac{3\pi}{8} = \dfrac{1 + \frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}$

$= \dfrac{\frac{\sqrt{2}+1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \sqrt{2} + 1$ (shown)

(bi) Method 1

$w = \cos\theta + i\sin\theta = e^{\theta i}$

$1 - w^2 = 1 - e^{2\theta i} = e^{0i} - e^{2\theta i}$

$= e^{\theta i}\left(e^{-\theta i} - e^{\theta i}\right)$

$= w(\cos\theta - i\sin\theta - i\sin\theta - \cos\theta)$

$= w(-2i\sin\theta)$

$= -2iw\sin\theta$

Method 2

$1 - w^2 = 1 - (\cos\theta + i\sin\theta)^2$

$= 1 - \cos^2\theta + \sin^2\theta - 2i\sin\theta\cos\theta$

$= 2\sin^2\theta - 2i\sin\theta\cos\theta$

$= 2\sin\theta(\sin\theta - 2i\cos\theta)$

$= -2i\sin\theta(\cos\theta + i\sin\theta)$

$= -2iw\sin\theta$

(ii)

$\left|1 - w^2\right| = \left|-2iw\sin\theta\right| = \left|-2\sin\theta\right|\left|\,i\,\right|\left|w\right|$

$= 2\sin\theta$

$\arg(1-w^2) = \arg(-2\text{i}w\sin\theta)$

$= \arg[(-2\sin\theta)\text{i}] + \arg(w)$

$= -\frac{\pi}{2} + \theta$

(iii) Method 1

$$\left(\frac{1-w^2}{\text{i}w^*}\right)^n = \left(\frac{2\sin\theta \text{e}^{\left(-\frac{\pi}{2}+\theta\right)\text{i}}}{\text{e}^{\left(\frac{\pi}{2}-\theta\right)\text{i}}}\right)^n = (2\sin\theta)^n \text{e}^{n(-\pi+2\theta)\text{i}}$$

$\because \sin\theta > 0$ when $\theta = \frac{\pi}{5}$,

$\therefore \arg\left(\frac{1-w^2}{\text{i}w^*}\right)^n = n(-\pi+2\theta)$

Method 2

$\arg\left(\frac{1-w^2}{\text{i}w^*}\right)^n = n[\arg(1-w^2) - \arg(\text{i}) - \arg w^*]$

$= n\left(\theta - \frac{\pi}{2} - \frac{\pi}{2} - (-\theta)\right)$

$= n(2\theta - \pi)$

Since $\left(\frac{1-w^2}{\text{i}w^*}\right)^n$ is real and negative, and sub in $\theta = \frac{\pi}{5}$

$\therefore n\left(2\left(\frac{\pi}{5}\right) - \pi\right) = \pi + 2k\pi, k \in \mathbb{Z}$

$-\frac{3}{5}n = (2k+1), k \in \mathbb{Z}$

$n = -\frac{5}{3}(2k+1), k \in \mathbb{Z}$

From GC tables, when $k = -2, -5, -8$,

Smallest $n = 5, 15, 25$

10 **(a)** $\frac{n}{2}[10 + (n-1)0.65] > \frac{7(1.04^n - 1)}{1.04 - 1}$

$\frac{n(9.35 + 0.65n)}{2} - 175(1.04^n - 1) > 0$

$n > 13.396$

Least number of days = 14

(b) Total dist covered by A $= \frac{28}{2}[10 + 27(0.65)] = 385.7$

Total dist covered by B $= \frac{7(1.04^{28} - 1)}{1.04 - 1} = 349.77$

Total number of bowls = 385 + 349 = 734

(c) Let a be the distance athlete A will need to cover on the 1st day

$\frac{28}{2}[2a + 27(0.65)] \geq 400$

$a \geq 5.5107$

Hence athlete A will need to run at least 5511 m

(d) Let n be the number of days where the distance covered is at most 10 km.

$7(1.04)^{n-1} \leq 10$

$n \leq 10.094$

Total dist covered by B $= \frac{7(1.04^{10} - 1)}{1.04 - 1} + 10(18) = 264.04$

Number of bowls contributed by A from the run = 264

11 **(i)** $\frac{\text{d}x}{\text{d}t} = kx$

$\int \frac{1}{x}\,\text{d}x = \int k\,\text{d}t$

$\ln|x| = kt + C$

$x = A\text{e}^{kt}, A = \pm\text{e}^C$

When $t = 0$, $x = 5$

$\Rightarrow A = 5$

When $t = 30$, $x = 5120$

$\Rightarrow 5120 = 5\text{e}^{30k}$

$k = \frac{1}{30}\ln 1024 = \frac{1}{3}\ln 2$

$x = 5\text{e}^{\frac{t}{3}\ln 2} = 5(2)^{\frac{t}{3}}$

(ii) $\frac{\text{d}y}{\text{d}p} = a(6400y - y^2)$

$\int \frac{1}{6400y - y^2}\,\text{d}y = \int a\,\text{d}p$

$\int \frac{1}{3200^2 - (y-3200)^2}\,\text{d}y = \int a\,\text{d}p$

$\frac{1}{6400}\ln\left|\frac{3200 + (y-3200)}{3200 - (y-3200)}\right| = ap + C$

$\ln\left|\frac{y}{6400 - y}\right| = 6400ap + 6400C$

$\frac{y}{6400 - y} = B\text{e}^{6400ap}$

$y = \frac{6400B\text{e}^{6400ap}}{1 + B\text{e}^{6400ap}} = \frac{6400}{1 + D\text{e}^{-6400ap}}$

When $p = 0, y = 5120$

$5120 = \frac{6400}{1 + D}$

BP-59

$D=\frac{1}{4}\cdots\cdots(1)$

When $p=30, y=3200$

$$3200=\frac{6400}{1+\frac{1}{4}e^{-192000a}}$$

$$a=\frac{-1}{192000}\ln 4$$

$$\therefore y=\frac{6400}{1+\frac{1}{4}e^{\frac{p}{30}\ln 4}}=\frac{6400}{1+2^{-2}\left(2^{\frac{p}{15}}\right)}$$

$$y=\frac{6400}{1+2^{\left(\frac{p}{15}-2\right)}}, \text{ where } H=-2$$

(iii) As $p\to\infty, 2^{\left(\frac{p}{15}-2\right)}\to\infty$ and $y\to 0$

Since eventually no one in the town will be infected by the virus, the vaccine produced is therefore effective.

ANDERSON SERANGOON JUNIOR COLLEGE

H2 MATHEMATICS **9758**

JC2 Prelim Paper 2 (100 marks) **16 Sept 2024**

3 hours

Additional Material(s): List of Formulae (MF26)

CANDIDATE NAME

CLASS

READ THESE INSTRUCTIONS FIRST

Write your name and class in the boxes above.
Please write clearly and use capital letters.
Write in dark blue or black pen. HB pencil may be used for graphs and diagrams only.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions and write your answers in this booklet.
Do not tear out any part of this booklet.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

All work must be handed in at the end of the examination. If you have used any additional paper, please insert them inside this booklet.
The number of marks is given in brackets [ ] at the end of each question or part question.

Question number	Marks
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
Total	

This document consists of 13 printed pages and 3 blank pages.

[Turn Over

Section A: Pure Mathematics [40 marks]

1 **(a)**

Let $x = \cot\theta$

$$\frac{dx}{d\theta} = -\operatorname{cosec}^2\theta$$

$$\int \frac{1}{x^2\sqrt{1+x^2}}\,dx$$

$$= \int \frac{1}{\cot^2\theta\sqrt{1+\cot^2\theta}} \times \left(-\operatorname{cosec}^2\theta\right)d\theta$$

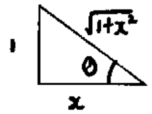
$$= \int \frac{1}{\cot^2\theta\sqrt{\operatorname{cosec}^2\theta}} \times \left(-\operatorname{cosec}^2\theta\right)d\theta$$

$$= -\int \frac{\sin^2\theta}{\cos^2\theta} \times \frac{1}{\sin\theta}\,d\theta$$

$$= \int (-\sin\theta)(\cos\theta)^{-2}\,d\theta \quad \textbf{OR} \quad = -\int \tan\theta\sec\theta\,d\theta$$

$$= -\frac{1}{\cos\theta} + C$$

$$= -\frac{\sqrt{x^2+1}}{x} + C$$



2 **(ai)**

$$\int \frac{9u-8}{4+9u^2}\,du$$

$$= \int \frac{\frac{1}{2}(18u)}{4+9u^2} - \frac{8}{2^2+(3u)^2}\,du$$

$$= \left[\frac{1}{2}\ln(4+9u^2) - \frac{8}{(2)(3)}\tan^{-1}\left(\frac{3u}{2}\right)\right] + C$$

$$= \frac{1}{2}\ln(4+9u^2) - \frac{4}{3}\tan^{-1}\left(\frac{3u}{2}\right) + C$$

(ii) Area required $= \int_1^3 y\,dx$

$$= \int_0^1 \frac{9u}{4+9u^2} \times (2u+1)\,du$$

$$= \int_0^1 \frac{18u^2+9u}{4+9u^2}\,du$$

$$= 2\int_0^1 1\,du + \int_0^1 \frac{(9u-8)}{4+9u^2}\,du$$

$$= \left[2u + \frac{1}{2}\ln(4+9u^2) - \frac{4}{3}\tan^{-1}\left(\frac{3u}{2}\right)\right]_0^1 \quad \text{(from part (i))}$$

BP-61

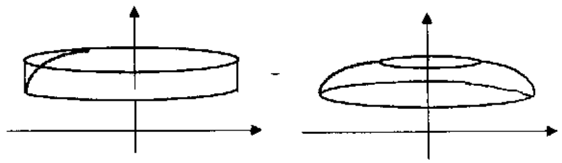
$$= \left(2+\frac{1}{2}\ln 13-\frac{4}{3}\tan^{-1}\left(\frac{3}{2}\right)\right)-\left(0+\frac{1}{2}\ln 4-\frac{4}{3}\tan^{-1}(0)\right)$$

$$= 2+\frac{1}{2}\ln\frac{13}{4}-\frac{4}{3}\tan^{-1}\left(\frac{3}{2}\right)\text{units}^2$$

(b) Given $(x+2)^2+4(y-1)^2=4$

$(x+2)^2=4[1-(y-1)^2] \Rightarrow x+2=\pm 2\sqrt{1-(y-1)^2}$

The shaded region is bounded by the section of the ellipse where $x\le -2$. Hence $x=-2-2\sqrt{1-(y-1)^2}$.



Volume of solid formed

$= \pi 4^2(2-1) - \pi\int_1^2 x^2\,dy$

$= 16\pi - \pi\int_1^2\left(-2-2\sqrt{1-(y-1)^2}\right)^2\,dy$

$= 9.6$ units^3 (From GC)

3 **(a)** $\vec{AB}=\begin{pmatrix}6\\5\\7\end{pmatrix}-\begin{pmatrix}2\\3\\4\end{pmatrix}=\begin{pmatrix}4\\2\\3\end{pmatrix}$ and $\vec{BE}=\begin{pmatrix}5\\6\\10\end{pmatrix}-\begin{pmatrix}6\\5\\7\end{pmatrix}=\begin{pmatrix}-1\\1\\3\end{pmatrix}$

Consider $\begin{pmatrix}4\\2\\3\end{pmatrix}\times\begin{pmatrix}-1\\1\\3\end{pmatrix}=\begin{pmatrix}3\\-15\\6\end{pmatrix}=3\begin{pmatrix}1\\-5\\2\end{pmatrix}$

Then $\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}1\\-5\\2\end{pmatrix}=\begin{pmatrix}2\\3\\4\end{pmatrix}\cdot\begin{pmatrix}1\\-5\\2\end{pmatrix}=2-15+5=-5$

$\Rightarrow x(1)+y(-5)+z(2)=-5$

Thus equation of surface ABE is $x-5y+2z=-5$

(b) $\vec{OM}=\frac{1}{2}\left[\begin{pmatrix}5\\6\\10\end{pmatrix}+\begin{pmatrix}8\\9\\6\end{pmatrix}\right]=\frac{1}{2}\begin{pmatrix}13\\15\\16\end{pmatrix}$

$\vec{DM}=\frac{1}{2}\begin{pmatrix}13\\15\\16\end{pmatrix}-\begin{pmatrix}4\\7\\3\end{pmatrix}=\frac{1}{2}\begin{pmatrix}5\\1\\10\end{pmatrix}$

Line DM has equation: $\mathbf{r}=\begin{pmatrix}4\\7\\3\end{pmatrix}+\lambda\begin{pmatrix}5\\1\\10\end{pmatrix}, \lambda\in\mathbb{R}$

(c) Let foot of perpendicular from M to surface be N.

Thus $\vec{ON}=\frac{1}{2}\begin{pmatrix}13\\15\\16\end{pmatrix}+t\begin{pmatrix}1\\-5\\2\end{pmatrix}$ for some $t\in\mathbb{R}$

$\Rightarrow\begin{pmatrix}\frac{13}{2}+t\\\frac{15}{2}-5t\\8+2t\end{pmatrix}\cdot\begin{pmatrix}1\\-5\\2\end{pmatrix}=-5$

$\Rightarrow\frac{13}{2}+t-\frac{75}{2}+25t+16+4t=-5$

$\Rightarrow t=\frac{1}{3}$

Thus $\vec{ON}=\begin{pmatrix}\frac{13}{2}+\frac{1}{3}\\\frac{15}{2}-5\left(\frac{1}{3}\right)\\8+2\left(\frac{1}{3}\right)\end{pmatrix}=\frac{1}{6}\begin{pmatrix}41\\35\\52\end{pmatrix}$

Thus we have $N\left(\frac{41}{6},\frac{35}{6},\frac{26}{3}\right)$.

(d) $\left|\vec{MN}\right|=\left|\frac{1}{6}\begin{pmatrix}41\\35\\52\end{pmatrix}-\frac{1}{2}\begin{pmatrix}13\\15\\16\end{pmatrix}\right|=\left|\frac{1}{3}\begin{pmatrix}1\\-5\\2\end{pmatrix}\right|$

$=\frac{1}{3}\sqrt{1+25+4}=\frac{\sqrt{30}}{3}$

(e) Consider $\begin{pmatrix}4+5\lambda\\7+\lambda\\3+10\lambda\end{pmatrix}=\begin{pmatrix}9\\8\\13\end{pmatrix}\Rightarrow\begin{cases}\lambda=1\\\lambda=1\\\lambda=1\end{cases}$

Since the value of λ is consistent, P lines on the line DM.

Also, $9-5(8)+2(13)=-5$

Hence P also lies on the plane ABE.

Thus P is the point of intersection between the line DM and the plane ABE.

Let the reflection of point M about the surface ABE be the point M'.

By ratio theorem (mid-point theorem),

$\vec{ON}=\frac{\vec{OM}+\vec{OM'}}{2}$

BP-62

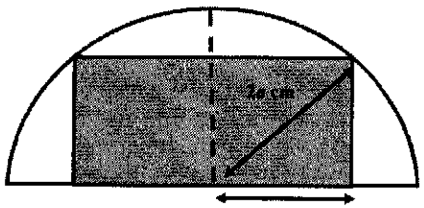
$\vec{OM}' = 2\vec{ON} - \vec{OM} = \frac{1}{3}\begin{pmatrix}41\\35\\52\end{pmatrix} - \frac{1}{2}\begin{pmatrix}13\\15\\16\end{pmatrix} = \frac{1}{6}\begin{pmatrix}43\\25\\56\end{pmatrix}$

$\vec{PM}' = \frac{1}{6}\begin{pmatrix}43\\25\\56\end{pmatrix} - \begin{pmatrix}9\\8\\13\end{pmatrix} = -\frac{1}{6}\begin{pmatrix}11\\23\\22\end{pmatrix}$

Thus equation of the reflection of DM about the surface ABE is

$\mathbf{r} = \begin{pmatrix}9\\8\\13\end{pmatrix} + \mu\begin{pmatrix}11\\23\\22\end{pmatrix}, \mu \in \square$

4 (a)



Height of the cylindrical container $= \sqrt{(2a)^2 - r^2} = \sqrt{4a^2 - r^2}$

Volume of cylindrical container $V = \pi r^2 \sqrt{4a^2 - r^2}$

Need to see that $\sqrt{(2a)^2 - r^2}$ gives the height of the container

(b) $V = \pi r^2 \sqrt{4a^2 - r^2}$

$\frac{dV}{dr} = \frac{d}{dr}\left(\pi r^2 \sqrt{4a^2 - r^2}\right)$

$\frac{dV}{dr} = 2\pi r\sqrt{4a^2 - r^2} + \pi r^2\left(\frac{1}{2}\right)\left(4a^2 - r^2\right)^{-\frac{1}{2}}(-2r)$

$\frac{dV}{dr} = 2\pi r\sqrt{4a^2 - r^2} - \frac{\pi r^3}{\sqrt{4a^2 - r^2}}$

$\frac{dV}{dr} = \frac{2\pi r\left(4a^2 - r^2\right) - \pi r^3}{\sqrt{4a^2 - r^2}}$

$\frac{dV}{dr} = \frac{\pi r\left(8a^2 - 3r^2\right)}{\sqrt{4a^2 - r^2}}$

$80 = \sqrt{4(50)^2 - r^2}$

$r^2 = 3600$

$r = 60 \ (\because r > 0)$

$\frac{dV}{dr} = \frac{\pi(60)\left[8(50^2) - 3(60^2)\right]}{\sqrt{4(50^2) - 60^2}} = 6900\pi$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$100\pi = 6900\pi \times \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{1}{69}$ cm s^{-1}

(c) $\frac{dV}{dr} = \frac{\pi r\left(2\sqrt{2}a + \sqrt{3}r\right)\left(2\sqrt{2}a - \sqrt{3}r\right)}{\sqrt{4a^2 - r^2}}$

$\frac{dV}{dr} = 0$

$\frac{\pi r\left(2\sqrt{2}a + \sqrt{3}r\right)\left(2\sqrt{2}a - \sqrt{3}r\right)}{\sqrt{4a^2 - r^2}} = 0$

$r = 0$ (rej $\because r > 0$) or $r = -\frac{2\sqrt{2}a}{\sqrt{3}}$ (rej $\because r > 0$) or $r = \frac{2\sqrt{2}a}{\sqrt{3}}$

When $0 < r < \frac{2\sqrt{2}}{\sqrt{3}}a$,

$\frac{\pi r\left(2\sqrt{2}a + \sqrt{3}r\right)}{\sqrt{4a^2 - r^2}} > 0$ & $2\sqrt{2}a - \sqrt{3}r > 0 \Rightarrow \frac{dV}{dr} > 0$.

When $r > \frac{2\sqrt{2}}{\sqrt{3}}a$,

$\frac{\pi r\left(2\sqrt{2}a + \sqrt{3}r\right)}{\sqrt{4a^2 - r^2}} > 0$ & $2\sqrt{2}a - \sqrt{3}r < 0 \Rightarrow \frac{dV}{dr} < 0$.

	$\left(\frac{2\sqrt{2}a}{3}\right)^-$	$\left(\frac{2\sqrt{2}a}{3}\right)$	$\left(\frac{2\sqrt{2}a}{3}\right)^+$
$\frac{dV}{dr}$	+ve	0	−ve
Slope	/	—	\

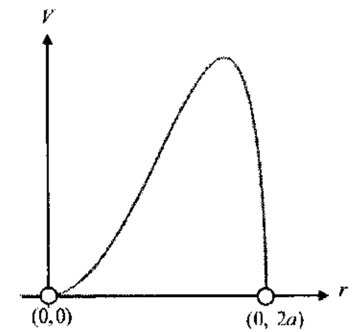
V is maximum when $r = \frac{2\sqrt{2}a}{\sqrt{3}} = \frac{2\sqrt{6}a}{3}$

When $r = \frac{2\sqrt{6}a}{3}$,

$V = \pi\left(\frac{2\sqrt{6}a}{3}\right)^2 \sqrt{4a^2 - \left(\frac{2\sqrt{6}a}{3}\right)^2}$

$V = \frac{16\pi a^3}{3\sqrt{3}} = \frac{16\sqrt{3}\pi a^3}{9}$ cm^3

(d)



Section B: Probability and Statistics [60 marks]

5 (i) Required probability $= \frac{4+3+2+1}{5^2} = \frac{2}{5}$

(ii) Required probability $= \frac{{}^7C_3\,{}^3C_1\frac{4!}{2!}}{7^4} + \frac{{}^5C_2\,{}^2C_1\frac{3!}{2!}}{5^3} - 2\left(\frac{{}^7C_3\,{}^3C_1\frac{4!}{2!}}{7^4}\right)\left(\frac{{}^5C_2\,{}^2C_1\frac{3!}{2!}}{5^3}\right)$

$= 0.50099$

$= 0.501$ (3 s.f)

6 (i) Table of outcomes:

	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	4	6
3	0	3	6	9

The probability distribution of T is given by:

T	0	1	2	3	4	6	9
$P(T=t)$	$\frac{7}{16}$	$\frac{1}{16}$	$\frac{2}{16} = \frac{1}{8}$	$\frac{2}{16} = \frac{1}{8}$	$\frac{1}{16}$	$\frac{2}{16} = \frac{1}{8}$	$\frac{1}{16}$

(ii)

$E(T) = 0\left(\frac{7}{16}\right)+1\left(\frac{1}{16}\right)+2\left(\frac{1}{8}\right)+3\left(\frac{1}{8}\right)+4\left(\frac{1}{16}\right)+6\left(\frac{1}{8}\right)+9\left(\frac{1}{16}\right) = \frac{9}{4}$

$E(T^2) = 0^2\left(\frac{7}{16}\right)+1^2\left(\frac{1}{16}\right)+2^2\left(\frac{1}{8}\right)+3^2\left(\frac{1}{8}\right)+4^2\left(\frac{1}{16}\right)+6^2\left(\frac{1}{8}\right)+9^2\left(\frac{1}{16}\right) = \frac{49}{4}$

$Var(T) = E(T^2) - [E(T)]^2 = \frac{49}{4} - \left(\frac{9}{4}\right)^2 = \frac{115}{16}$

(iii) $P(|T-2\mu| > \sigma) = P(T-2\mu > \sigma) + P(T-2\mu < -\sigma)$

$= P(T > 2\mu+\sigma) + P(T < 2\mu-\sigma)$

$= P(T > 7.18095) + P(T < 1.81905)$

$= P(T=9) + P(T=0) + P(T=1)$

$= \frac{1}{16} + \frac{7}{16} + \frac{1}{16} = \frac{9}{16}$ or 0.5625

7 (a) Let X be the mass, in grams, of a randomly chosen packet of semolina.

$X \sim N(225, 25^2)$

$4X \sim N(4\times 225,\ 4^2\times 25^2)$

$4X \sim N(900,\ 100^2)$

$P(850 \le 4X \le 1050) = 0.62466$ (5 sf)

$= 0.624$ (3 sf)

(b) Let Y be the mass, in grams, of a randomly chosen packet of millet flour.

$Y \sim N(\mu, \sigma^2)$

Let $M = \frac{X_1+X_2+X_3+Y_1+Y_2}{5}$

$M \sim N\left(\frac{675+2\mu}{5},\ \frac{3(25^2)+2(\sigma^2)}{25}\right)$

$P(M < 125) = P(M > 265) = 0.02$

$\Rightarrow \frac{675+2\mu}{5} = \frac{125+265}{2}$

$\Rightarrow \frac{675+2\mu}{5} = 195$

$\Rightarrow \mu = 150$

$P(M < 125) = 0.02$

$P\left(Z < \frac{125-195}{\sqrt{\frac{1875+2\sigma^2}{25}}}\right) = 0.02$ where $Z \sim N(0,1)$

$-\frac{70}{\sqrt{\frac{1875+2\sigma^2}{25}}} = -2.0537$

$1875 + 2\sigma^2 = 29042.99478$

$\sigma = 116.55$ (5 s.f.) $= 116$ (3 s.f.)

8 (i) Number of ways $=\binom{10}{5}\times 5! = 30240$

(ii) Number of ways $=\left[\binom{7}{5}\binom{3}{0}+\binom{7}{4}\binom{3}{1}\right]5!$

$=(21\times1+35\times3)5! = 15120$

(iii) Number of ways $=(8-1)!2! = 10080$

(iv) Case 1 – Beth in 1 row while Anne and Cathie are in another row

$\binom{7}{4}\times5!\times5!\times2 = 1008000$

Case 2 – Beth and one of them in 1 row

$\binom{7}{3}\times3!\times\binom{2}{1}\times\binom{4}{2}\times2!\times5!\times2 = 1209600$

Case 3 – Anne, Beth and Cathie are in the same row

$\underbrace{\binom{7}{2}\times2!\times\binom{3}{2}\times2!\times2!\times5!\times2}_{\text{A and C together}}+\underbrace{\binom{7}{2}\times2!\times3!\times5!\times2}_{\text{A and C separated}} = 181440$

Thus number of ways = $1008000+1209600+181440 = 2399040$

9 (a) The probability of a cookie is flawed is constant at p for each cookie.
OR
The event that a cookie is flawed is independent of another cookie being flawed.

(b) $C \sim B(20, p)$

$P(C=0) + P(C=1) = 0.15$

$(1-p)^{20} + \binom{20}{1}p^1(1-p)^{19} = 0.15$

$(1-p)^{19}(1+19p) = 0.15$

Using G.C, $p = 0.15891 = 0.159$

(c) Let X denote the number of flawed cookies in a box of 20 cookies.

$X \sim B(20,\ 0.08)$

$P(X<4) = P(X\le3) = 0.92938$

Let Y be the number of rejected boxes out of 10 boxes.

$Y \sim B(10,\ 1-0.92938)$

$Y \sim B(10,\ 0.070615)$

$P(2\le Y\le5) = P(Y\le5) - P(Y\le1)$

$=0.15388$

$=0.154$ (to 3 sig fig)

(d) Let W be the number of rejected boxes in the first 14 boxes.

$W \sim B(14,\ 0.070615)$

Let V be the number of rejected boxes out of 15 boxes.

$V \sim B(15,\ 0.070615)$

Required probability = P(3[rd] rejected box is the 15[th] box | $V=3$)

$=\dfrac{P(W=2)\times0.070615}{P(V=3)}$

$=0.2$

10 (a) Using G.C, $\bar{g} = 3.075$

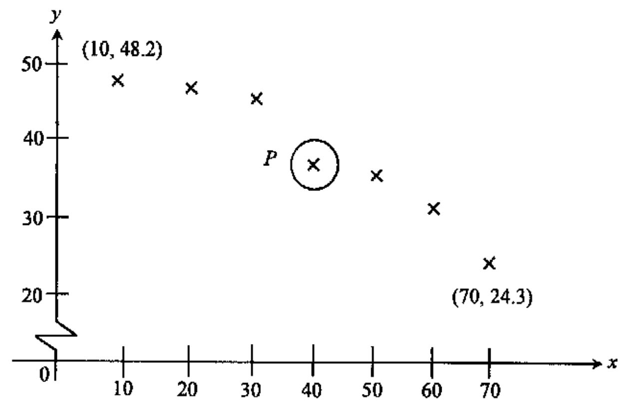
$\bar{u} = \dfrac{84+a}{8}$

Since $(\bar{u},\bar{g})$ lies on the regression line,

$3.075 = -0.0765\left(\dfrac{84+a}{8}\right)+3.99$

$a = 11.686 \approx 11.7$ (correct to 1 decimal place)

(bi) and (ii)



(iii) From the scatter diagram, as x increases, y decreases at an increasing rate. Hence a linear model is not a suitable model.

(iv) Using G.C,

$r_A = -0.9981$

$r_B = -0.8970$

Since r_A is closer to -1 than r_B, so model (A) is a better model than model (B).

From the G.C,

$y = -0.00511019x^2 + 49.2444$

$y = -0.00511x^2 + 49.2$ (3.s.f)

(v) When $x = 80$,

$y = -0.00511019(80)^2 + 49.2444 = 16.53916$

$y = 16.5$ (3.s.f)

(vi) The estimate is unreliable because the data substituted is outside the data range ($10\le x\le70$) and so the linear relationship between y and x^2 may not hold true.

11 (a) Let X denote the length of a randomly chosen green leaf, in centimetres.

Let L be the total lengths of 100 green leaves.

$L = X_1 + X_2 + X_3 ... + X_{100}$

Since 100 > 30 (n is considered large), by Central Limit Theorem,

$L \sim N(12\times100, 3.5^2\times100)$ approx.

$P(L\geq1138)=0.96175\approx0.962$ (3 sf)

(bi) Unbiased estimate of the population variance

$s^2=\frac{64}{63}(16.4^2)=273.229\approx273.23$ (2 dp) (2 decimal places)

Let Y denote the time spent in minutes using the one-seater pod facilities by a randomly chosen user at location A and μ denote the population mean time spent in minutes using the one-seater pod facilities at location A.

To test H_0: $\mu=131$

Against H_1: $\mu<131$ (Workspace operator overstating the claim)

Conduct a one-tail test at 3% level of significance, i.e., $\alpha=0.03$

Under H_0,

Since $n=64$ (> 30) is large, by Central Limit Theorem,

$\bar{Y}\sim N\left(131,\frac{273.229}{64}\right)$ approximately.

$\bar{t}=127$

Using GC, p-value = 0.026438 ≈ 0.0264 (3 sf)

Since p-value = 0.0264 < 0.03, we reject H_0. There is sufficient evidence at 3% level of significance to conclude that that the centre manager was overstating his claim.

(ii) There is a probability of 0.03 of concluding that the population average time spent using the one-seater pod facilities at location A is less than 131 minutes when in fact the population average time spent using the one-seater pod facilities in location A is 131 minutes.

(iii) Assume that the time spent by the users of the one-seater pods facilities in location B follows a Normal Distribution.

Assume also that the time spent, on the one-seater pod facilities in location B by users, are independent of each other.

Let W denote the time spent in minutes using the one-seater pod facilities by a randomly chosen user at location B.

To test H_0: $\mu=140$

Against H_1: $\mu\neq140$

at 5% level of significance

Under H_0, $\bar{W}\sim N\left(140,\frac{20.1^2}{15}\right)$

Since H_0 is not rejected,

$$-1.95996<\frac{\bar{w}-140}{\left(\frac{20.1}{\sqrt{15}}\right)}<1.95996$$

$129.828<\bar{w}<150.1718$

$129.8<\bar{w}<150.2$ (1 d.p)

[Turn Over

BP-66