2024 VJC Prelim Paper 1

1 **(a)** Express $\frac{33-8x}{x^2+2x-15}+2$ as a single algebraic fraction. Hence, without using a calculator, solve exactly the inequality $\frac{33-8x}{x^2+2x-15}>-2$. [4]

(b) Using your answer to part **(a)**, find the set of values of x for which $\frac{33-8\mathrm{e}^{2x}}{\mathrm{e}^{4x}+2\mathrm{e}^{2x}-15}>-2$. [2]

2024 VJC Prelim Paper 1

2 The sum of the first n terms of a sequence, u_r is given by $\sum_{r=1}^{n} u_r = 1 - \frac{n}{(n+1)!}$.

(a) Find u_n in terms of n, for $n \geq 2$, expressing your answer as a single algebraic fraction. [2]

(b) Show that $\sum_{r=5}^{n} u_r < \frac{1}{30}$ for all $n \geq 5$. [2]

(c) Explain why $\sum_{r=1}^{\infty} u_r$ is a convergent series. [1]

2024 VJC Prelim Paper 1

3 The functions f and g are defined by

$$\mathrm{f} : x \mapsto \frac{ax-6}{x-3} \quad \text{for } x \in \mathbb{R},\ x \neq 3,$$

$$\mathrm{g} : x \mapsto \mathrm{e}^{-x} \quad \text{for } x \in \mathbb{R},\ x \geq \ln 3.$$

The function f is such that $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ for all x in the domain of f.

(a) Find the value of a. [3]

2024 VJC Prelim Paper 1

(b) State the exact value of $\mathrm{f}^6(\pi)$. [1]

(c) Find the exact range of fg. [3]

4 **(a)** Given that **a**, **b** and **c** are non-zero vectors such that $(\mathbf{a}+\mathbf{b})\times(\mathbf{a}+\mathbf{c})=\mathbf{b}\times\mathbf{c}$, and $\mathbf{b}\neq\mathbf{c}$, find the relationship between **a**, **b** and **c**. [4]

(b) It is given instead that **a**, **b** and **c** satisfy the equation $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}$ with $|\mathbf{a}|=2$, $|\mathbf{b}|=3$ and $|\mathbf{c}|=4$. Find the value of

$$\mathbf{a}\cdot\mathbf{b}\ +\ \mathbf{b}\cdot\mathbf{c}\ +\ \mathbf{c}\cdot\mathbf{a}.$$ [3]

2024 VJC Prelim Paper 1

5 It is given that $\mathrm{f}(n)=\dfrac{n}{5^{n-1}}$ where n is a positive integer.

(a) By considering $\mathrm{f}(r)-\mathrm{f}(r+1)$, find an expression for $\displaystyle\sum_{r=2}^{n}\frac{4r-1}{5^r}$. [3]

(b) Hence find an expression for $\displaystyle\sum_{r=1}^{n}\frac{4r+6}{5^{r+1}}$. [3]

2024 VJC Prelim Paper 1

6 **(a)** Find the exact value of $\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{2\sin^{-1}x}{\sqrt{1-x^2}}\,\mathrm{d}x$. [3]

(b) Find the exact value of $\int_0^{\frac{\pi}{3}} \left|\cos 2x\right| \mathrm{d}x$. [3]

2024 VJC Prelim Paper 1

(c) Find $\int \frac{1}{-x^2+2kx+3k^2}\,\mathrm{d}x$, where k is a positive constant. [4]

2024 VJC Prelim Paper 1

7 **(a)** The curve C has equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \frac{ax^2 + bx + c}{x + d},$$

and a, b, c and d are constants, and $a \neq 0$.

Given that C has asymptote $y = x + 1$, find the value of a and show that $b = d + 1$. [2]

If f is an increasing function for all $x \in \mathbb{R}$, $x > -d$, show that $c < d$. [3]

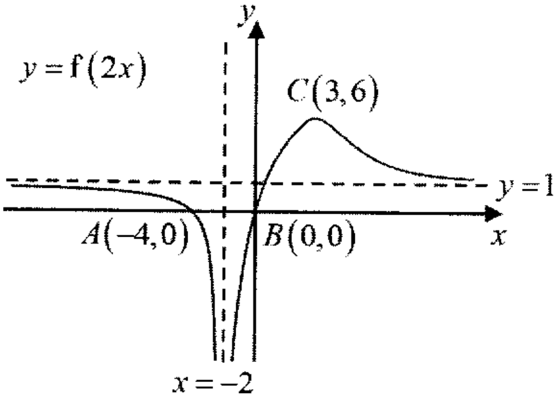
(b) It is further given that $c = 1$ and $d = 2$.

(i) Sketch C. [3]

(ii) By sketching a suitable graph in the same diagram in part **(b)(i)**, find the number of real roots to the equation $\left(\dfrac{x^2+3x+1}{x+2}\right)^2 + 4x^2 = 16$. [2]

2024 VJC Prelim Paper 1

8 **(a)** The diagram shows the graph with equation $y = \mathrm{f}(2x)$. The graph passes through the points $A(-4,0)$, $B(0,0)$ and $C(3,6)$, and has asymptotes $x = -2$ and $y = 1$.



On separate clearly labelled diagrams, deduce the graphs of

(i) $y = \mathrm{f}(2x-2)$, [2]

(ii) $y = \mathrm{f}(x)$. [2]

2024 VJC Prelim Paper 1

(b) The curve C_1 undergoes the transformations in the order given below:

1. A translation of 2 units in the negative x direction.
2. A stretch parallel to the x axis, factor 2, y axis invariant.
3. A translation of 1 unit in the positive y direction.

The resulting curve C_2 has equation

$$y=\frac{x^2+9x+22}{x+4}\ ,\ x\in\mathbb{R},\ x\neq -4\,.$$

Find, in the simplest form, the equation for C_1. [4]

2024 VJC Prelim Paper 1

9 Find the area of the region bounded by the graphs of $y=2x^2+3$ and $y=-2x+7$. [3]

State the area of the region bounded by the 2 graphs if both graphs are translated 7 units in the negative y-direction. [1]

2024 VJC Prelim Paper 1

The region R is bounded by $y=2x^2+3$, $y=-2x+7$, the x-axis and the y-axis. Find the exact volume of the solid generated when R is rotated 2π about the y-axis. [5]

2024 VJC Prelim Paper 1

10 Given that $z = 2 - \mathrm{i}$ is a root of the equation $4z^4 - 12z^3 + 17z^2 + pz + q = 0$, where p and q are real, find p and q. [4]

Using the values of p and q found, find the other roots of the equation $4z^4 - 12z^3 + 17z^2 + pz + q = 0$ in exact form. [4]

2024 VJC Prelim Paper 1

11 Dendrologists are specialised scientists who study trees and woody plants. Their work is diverse and can encompass various activities related to the identification, classification, biology, and ecology of trees. A group of dendrologists are studying the growth of 2 species of trees, codenamed Tree *Vee* and Tree *Jay*.

In the 1st year, the height of Tree *Vee* and Tree *Jay* are both H cm.

In the 2nd year, Tree *Vee*'s height increases by s cm and subsequently, the increase in height every year is 10% less than the previous year's increase. Show that the height of Tree *Vee* in the 4th year is given by $(H+2.71s)$ cm. [1]

Show that the height of Tree *Vee* in the nth year is given by $\left[H+10s\left(1-0.9^{n-1}\right)\right]$ cm. [3]

Hence, write down in terms of H and s, the theoretical maximum height (in cm) of Tree *Vee*. [1]

11 [Continued]

In the 2nd year, Tree *Jay*'s height increases by t cm and subsequently, the increase in height every year is 0.5 cm less than the previous year's increase. Show that the height of Tree *Jay* in the 10th year is given by $(H-18+9t)$ cm. [2]

It is now given that $t=20$.

After the 10th year, Tree *Jay*'s height increases at a constant rate of 7 cm per year. Express Tree Jay's height (in cm) in the n^{th} year (where $n \geq 11$) in terms of H and n. [2]

2024 VJC Prelim Paper 1

It is further given that $s = 30$, and the 1st year is the year 2024. Find the years in which the heights of Tree *Vee* and Tree *Jay* are within 7 cm of each other, **after 2034**. [3]

12 Game developers closely monitor the number of people playing their game. Understanding player numbers and behaviour can not only help in optimising in-game purchases, advertisement placements, and other revenue-generating aspects, it can also help the company manage server loads and ensure the game runs smoothly without performance issues.

Two game developers are interested in the number of players playing the mobile game "Mobile Saga". They attempt to model the number of players x, in **hundred thousands**, at time t months after the launch of the game using a differential equation. On the day of the launch, there were 55 000 players.

(a) One game developer suggests that x and t are related by the differential equation $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{3}{5}x-kt^2$, where k is a positive constant.

(i) By substituting $x=u\mathrm{e}^{\frac{3}{5}t}$, show that the differential equation can be written as $\frac{\mathrm{d}u}{\mathrm{d}t}=-kt^2\,\mathrm{e}^{-\frac{3}{5}t}$. [2]

(ii) Hence show that $x=\frac{5k}{3}t^2+\frac{50k}{9}t+\frac{250k}{27}+\left(\frac{11}{20}-\frac{250k}{27}\right)\mathrm{e}^{\frac{3}{5}t}$. [4]

12 [Continued]

(iii) Company A intends to place an advertisement in the game only if there are more than 76 000 players playing the game. Given that $k=\frac{1}{10}$, find the length of time for which Company A will place an advertisement in "Mobile Saga", giving your answer correct to the nearest month. [2]

12 [Continued]

(b) The other game developer suggests that x and t are related by the differential equation $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-\frac{10}{(1+t)^3}$. Given further that there were 180 000 players playing "Mobile Saga" after 1 month, find x in terms of t. [4]

2024 VJC Prelim Paper 2

Section A: Pure Mathematics (40 marks)

1 **(a)** Show that $\dfrac{1}{\sqrt{1+x^2}-\sqrt{1-x^2}}=\dfrac{1}{2x^2}\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)$. [1]

(b) Hence use appropriate expansions from the List of Formulae (MF26) to find the first two non-zero terms in the series expansion of $\dfrac{x^2}{\sqrt{1+x^2}-\sqrt{1-x^2}}$, $x \neq 0$ in ascending powers of x. [3]

(c) State the set of values of x for which the series expansion is valid. [1]

(d) It is given that the two terms found in part **(b)** are equal to the first two terms in the series expansion of $\cos\left(ax^b\right)$. Find the possible value(s) of the constants a and b. [2]

2024 VJC Prelim Paper 2

2 **Do not use a calculator in answering this question.**

The complex numbers z_1, z_2 and z_3 are such that $z_1 = -e^{i\frac{2\pi}{3}}$, $z_2 = -\sqrt{3} + i$ and $z_3 = \dfrac{z_1}{z_2}$.

(a) Express each of z_1, z_2 and z_3 in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. [3]

(b) Sketch an Argand diagram showing the points P_1, P_2 and P_3 where P_1, P_2 and P_3 represent the complex numbers z_1, z_2 and z_3 respectively. [2]

(c) Find the area of triangle OP_1P_2. [2]

(d) Find the smallest positive integer n for which $\left(z_2^*\right)^n$ is purely imaginary. [2]

2024 VJC Prelim Paper 2

3 The line l_1 has equation $\mathbf{r} = 3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k} + \lambda(\mathbf{i} - 2\mathbf{j} - \mathbf{k})$, where λ is a real parameter. The point A has position vector $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

(a) The plane p contains the line l_1 and the point A. Find a cartesian equation of the plane p. [3]

(b) Find the position vector of the point A', the reflection of the point A in the line l_1. [4]

(c) The plane q is such that q is parallel to p and passes through the point with position vector $-3\mathbf{j}+\mathbf{k}$. Find a cartesian equation of q and the exact shortest distance between p and q. [3]

(d) The line l_2 has the equation $\frac{y-3}{2}=\frac{z-7}{3}$, $x=2$. Given that l_2 intersects p at point S, find the area of the triangle OAS. [4]

2024 VJC Prelim Paper 2

4 The curve C is defined by the parametric equations

$$x=a\left(1+\frac{1}{t}\right) \quad \text{and} \quad y=a\left(t-\frac{1}{t^2}\right)$$

where a is a positive constant and $t \neq 0$.

(a) Show that $\frac{\mathrm{d}y}{\mathrm{d}x}=-\left(\frac{2+t^3}{t}\right)$. [3]

(b) Find, in terms of a, the coordinates of the turning point on C, and explain why it is a maximum. [4]

2024 VJC Prelim Paper 2

(c) Sketch *C*. [3]

Section B: Statistics (60 marks)

5 Two married couples, two single adults and two children formed a team of 8 to take part in a series of games.

(a) In the first game, the team sits in a circle. Find the number of arrangements that can be formed if each married couple must be seated together. [2]

(b) A group of three people are to be selected from the team for the second game. Find the number of different groups that can be formed if there must **not** be a married couple in the group. [2]

(c) In the third game, each team member selects a unique number from the set $\{1, 2, \ldots, 8\}$. Find the number of different ways this can be done if the numbers selected by the children are both greater than the numbers selected by the two single adults. [2]

2024 VJC Prelim Paper 2

6 A random variable X has the probability distribution given in the following table.

x	1	4	6	8
$\mathrm{P}(X=x)$	a	b	c	d

Given that $\mathrm{E}(X)=4$, $\mathrm{Var}(X)=\frac{19}{4}$ and $\mathrm{P}(X<4)=\mathrm{P}(X>4)$, find the values of a, b, c and d. [5]

2024 VJC Prelim Paper 2

7 For events A, B and C, it is given that $\mathrm{P}(A)=0.7$, $\mathrm{P}(B)=0.5$, $\mathrm{P}(C\mid A')=0.6$ and $\mathrm{P}(A\mid C')=0.76$.

(a) Find the greatest and least possible values of $\mathrm{P}(A\cap B)$. [2]

(b) Find $\mathrm{P}(C\cap A')$. [1]

(c) Find $\mathrm{P}(C' \cap A')$. [2]

(d) Find $\mathrm{P}(C)$. [3]

2024 VJC Prelim Paper 2

8 A small company makes wine glasses. Each day, n randomly chosen wine glasses are checked and the number of wine glasses found to be cracked is denoted by X.

(a) State, in context of the question, two assumptions needed for X to be well modelled by a binomial distribution. [2]

Assume now that X has the distribution $\mathrm{B}(n, p)$, where $n \geq 3$.

(b) Given that the mean of X and the variance of X are 1.8 and 1.773 respectively, find the value of n and the value of p. [2]

2024 VJC Prelim Paper 2

(c) Given instead that the probability of finding 2 cracked wine glasses is thrice the probability of finding 3 cracked wine glasses, find p in terms of n. [2]

2024 VJC Prelim Paper 2

9 **(a)** S and T are independent random variables with the distributions $\mathrm{N}\left(18,3^2\right)$ and $\mathrm{N}\left(\mu,\sigma^2\right)$ respectively. It is given that $\mathrm{P}(T<4)=\mathrm{P}(T>9)$ and $\mathrm{P}(S<3T)=0.65$. Calculate the values of μ and σ. [4]

2024 VJC Prelim Paper 2

(b) A fruit stall sells grapes that are packed in packets with masses in grams that follow the distribution $\mathrm{N}\left(850,30^2\right)$. The grapes are sold at \$18 per kilogram.

(i) Find the probability that a customer pays more than \$30 for two packets of grapes. [2]

(ii) The fruit stall accepts payment by cash or PayNow. The number of customers who pay by PayNow in a day is a random variable with mean 12 and variance 4.8. In a month of 30 days, find the probability that the average number of customers per day who pay by PayNow is more than 12.3. [3]

2024 VJC Prelim Paper 2

10 The yield per hectare, y kg, of a crop is believed to depend on the average rainfall, x mm, in the month of June. For 10 regions, records are kept of the values of x and y, and these are shown in the table below. The yield from the tenth region was accidentally deleted from the records after the data was analysed, and this is indicated by the value p.

Average rainfall (x mm)	149	110	188	135	156	140	168	118	122	174
Yield of crop (y kg)	13.8	6.5	15.2	12.2	14.4	12.2	14.7	9.5	9.9	p

Given that the equation of the regression line of y on x is $y = -2.5652 + 0.10168x$, show that $p = 14.4$. [2]

(a) Draw a scatter diagram for these values, labelling the axes clearly. Calculate the product moment correlation coefficient between x and y. [2]

(b) It is thought that a model of the form $y = a + b\ln x$ may also be a suitable fit to the data. Calculate least square estimates of a and b, and find the value of the product moment correlation coefficient between y and $\ln x$. [3]

(c) Use your answers to parts **(a)** and **(b)** to explain which of

$$y = -2.5652 + 0.10168x \text{ or } y = a + b\ln x$$

is the better model. [2]

(d) Using an appropriate regression line, estimate the yield for a region that experienced 200 mm of rainfall in June. Comment on the reliability of your estimate. [2]

(e) In some regions, rainfall is measured in inches instead of in mm. Given that there are 25.4 mm in an inch, show how the regression line found in part **(b)** can be re-written so that it can be used when x, the average rainfall in June, is given in inches. [1]

2024 VJC Prelim Paper 2

11 In the swimming training school AquaV, the time taken to swim a lap of the pool by the trainees is found to have a mean of 35 seconds. The school adopted a new international training programme Breakthru for 3 months and wanted to analyse if Breakthru is effective in improving the timings of the trainees.

A sample of 30 trainees is taken and the times taken, x seconds, to swim a lap of the pool by the trainees are summarised by

$$\sum(x-30)=94, \quad \sum(x-30)^2=758.$$

(a) Test, at the 5% significance level, whether there is any evidence that the mean time taken to swim a lap of the pool has improved after the trainees underwent 3 months of Breakthru, defining any parameters you use. [7]

(b) State an assumption used in carrying out the test. [1]

2024 VJC Prelim Paper 2

In another swimming training school AquaZ, the time taken to swim a lap of the pool by the trainees is normally distributed with a mean of 38 seconds. AquaZ similarly adopted Breakthru for 3 months and then also carried out a test at the 5% significance level to determine whether there is an improvement in the swimming times. A sample of 30 trainees was taken and their timings were measured. The sample standard deviation was found to be 4 seconds and the mean time was denoted by $\overline{x}$.

(c) Find the set of values of $\overline{x}$ for which the result of the test would be to reject the null hypothesis. [4]

(d) If the times taken by the 30 trainees is summarised by $\sum(x-30)=234$, determine the conclusion of the test. [2]

2024 VJC Prelim Paper 1 Solutions

1 **(a)** Express $\frac{33-8x}{x^2+2x-15}+2$ as a single algebraic fraction. Hence, without using a calculator, solve exactly the inequality $\frac{33-8x}{x^2+2x-15}>-2$. [4]

1a	$\frac{33-8x}{x^2+2x-15}+2$ $=\frac{33-8x+2\left(x^2+2x-15\right)}{x^2+2x-15}$ $=\frac{2x^2-4x+3}{x^2+2x-15}$ $=\frac{2(x-1)^2+1}{(x+5)(x-3)}$ Solve: $\frac{2(x-1)^2+1}{(x+5)(x-3)}>0$ Since $(x-1)^2\geq 0$ for all $x\in\mathbf{R}$, $2(x-1)^2+1>0$ for all $x\in\mathbb{R}$. Hence, we solve $(x+5)(x-3)>0$. $\therefore x>3$ or $x<-5$	There is a need to <u>explain</u> clearly why the numerator is always positive. The better method is to complete the square and state that $(x-1)^2\geq 0$, so $2(x-1)^2+1>0$. Some only described the discriminant to be negative or said that there are no real roots – both statements do NOT imply that the numerator is always positive. Note: 1 is NOT a critical value.

(b) Using your answer to part **(a)**, find the set of values of x for which $\frac{33-8e^{2x}}{e^{4x}+2e^{2x}-15}>-2$. [2]

1b	Replace x with e^{2x}: $e^{2x}>3$ or $e^{2x}<-5$ (rej. $\because e^{2x}>0$ for all $x\in\mathbb{R}$) $\therefore\left\{x\in\mathbb{R}:\ x>\frac{\ln 3}{2}\right\}$	<u>Explain</u> clearly why $e^{2x}<-5$ is rejected. Many showed working up to $2x<\ln(-5)$ to reject the statement which is incorrect. Firstly, $\ln(-5)$ is undefined. Secondly, IF the inequality were $e^{2x}>-5$, would you say that $2x>\ln(-5)$ and reject to conclude that there are no solutions?!

2024 VJC Prelim Paper 1 Solutions

2 The sum of the first n terms of a sequence, u_r is given by $\sum_{r=1}^{n}u_r=1-\frac{n}{(n+1)!}$.

(a) Find u_n in terms of n, for $n\geq 2$, expressing your answer as a single algebraic fraction. [2]

2a	$u_n=\sum_{r=1}^{n}u_r-\sum_{r=1}^{n-1}u_r$ $=1-\frac{n}{(n+1)!}-\left(1-\frac{n-1}{n!}\right)$ $=\frac{-n+(n-1)(n+1)}{(n+1)\times n!}$ $=\frac{n^2-n-1}{(n+1)!}$	Note that $\sum_{r=1}^{n}u_r$ is usually denoted by S_n. Here, $u_n=S_n-S_{n-1}$.

(b) Show that $\sum_{r=5}^{n}u_r<\frac{1}{30}$, for all $n\geq 5$. [2]

2b	$\sum_{r=5}^{n}u_r=\sum_{r=1}^{n}u_r-\sum_{r=1}^{4}u_r$ $=1-\frac{n}{(n+1)!}-\left(1-\frac{4}{5!}\right)$ $=\frac{1}{30}-\frac{n}{(n+1)!}$ $<\frac{1}{30}$, since $\frac{n}{(n+1)!}>0$ for all $n\geq 5$	The explanation to why $\frac{1}{30}-\frac{n}{(n+1)!}$ is less than $\frac{1}{30}$ is to correctly mention that $\frac{n}{(n+1)!}>0$. Most commonly seen answer was that $\frac{n}{(n+1)!}\to 0$. While it is true, it does not explain why $\frac{1}{30}-\frac{n}{(n+1)!}<\frac{1}{30}$.

(c) Explain why $\sum_{r=1}^{\infty}u_r$ is a convergent series. [1]

2c	$1-\frac{n}{(n+1)!}=1-\frac{n}{(n+1)\times n\times(n-1)!}=1-\frac{1}{(n+1)\times(n-1)!}$ As $n\to\infty$, $(n+1)\times(n-1)!\to\infty$, $\frac{1}{(n+1)\times(n-1)!}\to 0$, $\sum_{r=1}^{n}u_r\to 1-0=1$, which is a constant. Hence $\sum_{r=1}^{\infty}u_r$ is a convergent series.	Many gave incomplete explanations to say that as $n\to\infty$, $(n+1)!\to\infty$, so $\frac{n}{(n+1)!}\to 0$. This completely disregards the fact that the numerator, $n\to\infty$ too and did not show complete understanding of why $\frac{n}{(n+1)!}\to 0$.

BP-842

2024 VJC Prelim Paper 1 Solutions

3 The functions f and g are defined by

$$\mathrm{f}: x \mapsto \frac{ax-6}{x-3} \quad \text{for } x \in \mathbb{R},\ x \neq 3,\ b \neq 9,$$

$$\mathrm{g}: x \mapsto \mathrm{e}^{-x} \quad \text{for } x \in \mathbb{R},\ x \geq \ln 3.$$

The function f is self-inverse if $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ for all values of x in the domain of f. It is given that f is self-inverse.

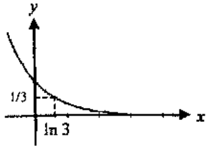
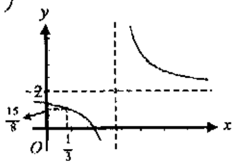
(a) Find the value of a. [3]

3a	Let $y = \dfrac{ax-6}{x-3}$ $xy - 3y = ax - 6$ $x = \dfrac{3y-6}{y-a}$ $\mathrm{f}^{-1}(x) = \dfrac{3x-6}{x-a}$ Given that $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$, $\dfrac{ax-6}{x-3} = \dfrac{3x-6}{x-a}$. $\therefore\ a = 3$	Some wrote INCORRECTLY that $\mathrm{f}(x) = \mathrm{f}^{-1}(x) \Rightarrow \mathrm{f}(x) = x$. It is important to know that $\mathrm{f}(x) = \mathrm{f}^{-1}(x) \Rightarrow \mathrm{ff}(x) = x$.

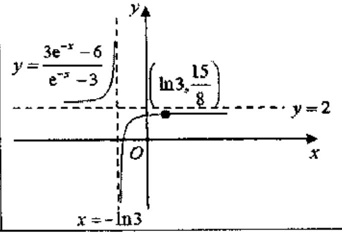
(b) Find the exact value of $\mathrm{f}^6(\pi)$. [1]

3b	$\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ $\mathrm{ff}(x) = x$ $\mathrm{f}^2(x) = x$ $\mathrm{f}^6(\pi) = \mathrm{f}^2\mathrm{f}^2\mathrm{f}^2(\pi) = \pi$	

(c) Find the exact range of fg. [3]

3c	Given $\mathrm{f}(x) = \dfrac{3x-6}{x-3} = 3\left(\dfrac{x-2}{x-3}\right)$   $\mathrm{D_g} = [\ln 3, \infty) \xrightarrow{\mathrm{g}} \mathrm{R_g} = \left(0, \dfrac{1}{3}\right] \xrightarrow{\mathrm{f}} \mathrm{R_{fg}} = \left[\dfrac{15}{8}, 2\right)$	The first method is to find $\mathrm{R_g}$ and use it as the new domain of f. To do this, it is advisable to sketch the graphs of g and of f.

2024 VJC Prelim Paper 1 Solutions

	Alternative $\mathrm{fg}(x) = \dfrac{3\mathrm{e}^{-x}-6}{\mathrm{e}^{-x}-3},\ x \geq \ln 3$  	The alternative is to draw directly the graph of $y = \mathrm{fg}(x) = \dfrac{3\mathrm{e}^{-x}-6}{\mathrm{e}^{-x}-3}$, and note that $\mathrm{D_{fg}} = \mathrm{D_g} = [\ln 3, \infty)$. Some who did this method, left out the horizontal asymptote of the graph or incorrectly stated that it to be $y = 3$. BOTH methods should be learnt and internalised.

4 (a) Given that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-zero vectors such that $(\mathbf{a}+\mathbf{b}) \times (\mathbf{a}+\mathbf{c}) = \mathbf{b} \times \mathbf{c}$, and $\mathbf{b} \neq \mathbf{c}$, find the relationship between $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. [4]

4a	$(\underset{\sim}{a}+\underset{\sim}{b}) \times (\underset{\sim}{a}+\underset{\sim}{c}) = \underset{\sim}{b} \times \underset{\sim}{c}$ $\underset{\sim}{a} \times \underset{\sim}{a} + \underset{\sim}{a} \times \underset{\sim}{c} + \underset{\sim}{b} \times \underset{\sim}{a} + \underset{\sim}{b} \times \underset{\sim}{c} = \underset{\sim}{b} \times \underset{\sim}{c}$ $\underset{\sim}{0} + \underset{\sim}{a} \times \underset{\sim}{c} + \underset{\sim}{b} \times \underset{\sim}{a} = \underset{\sim}{0} \quad (\because \underset{\sim}{a} \times \underset{\sim}{a} = \underset{\sim}{0})$ $\underset{\sim}{a} \times (\underset{\sim}{c} - \underset{\sim}{b}) = \underset{\sim}{0} \quad (\because \underset{\sim}{b} \times \underset{\sim}{a} = -\underset{\sim}{a} \times \underset{\sim}{b})$ Since $\underset{\sim}{a} \neq \underset{\sim}{0}$ and $\underset{\sim}{b} \neq \underset{\sim}{c}$, $\underset{\sim}{a}$ is parallel to $\underset{\sim}{c} - \underset{\sim}{b}$.	The notation of $\underset{\sim}{0}$ is still NOT seen in most answers. Note that 0 and $\underset{\sim}{0}$ are NOT to be used interchangeably. Some answers still show misunderstanding that $\underset{\sim}{b} \times \underset{\sim}{a}$ and $\underset{\sim}{a} \times \underset{\sim}{b}$ are equal but they are NOT. Note that $\underset{\sim}{a} \times (\underset{\sim}{c} - \underset{\sim}{b}) = \underset{\sim}{0}$ does not directly imply that $\underset{\sim}{a}$ is parallel to $\underset{\sim}{c} - \underset{\sim}{b}$. One has to check and state/explain that neither $\underset{\sim}{a}$ nor $\underset{\sim}{c} - \underset{\sim}{b}$ is $\underset{\sim}{0}$.

(b) It is given instead that a, b and c satisfy the equation $\mathbf{a}+\mathbf{b}+\mathbf{c} = \mathbf{0}$ with $\lvert\mathbf{a}\rvert = 2$, $\lvert\mathbf{b}\rvert = 3$ and $\lvert\mathbf{c}\rvert = 4$. Find the value of

$$\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a}.$$ [3]

4b	$(\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}).(\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}) = 0$ $\underset{\sim}{a}.\underset{\sim}{a} + \underset{\sim}{b}.\underset{\sim}{b} + \underset{\sim}{c}.\underset{\sim}{c} + 2\underset{\sim}{a}.\underset{\sim}{b} + 2\underset{\sim}{b}.\underset{\sim}{c} + 2\underset{\sim}{a}.\underset{\sim}{c} = 0$ $\lvert\underset{\sim}{a}\rvert^2 + \lvert\underset{\sim}{b}\rvert^2 + \lvert\underset{\sim}{c}\rvert^2 + 2(\underset{\sim}{a}.\underset{\sim}{b} + \underset{\sim}{b}.\underset{\sim}{c} + \underset{\sim}{a}.\underset{\sim}{c}) = 0$ $2^2 + 3^2 + 4^2 + 2(\underset{\sim}{a}.\underset{\sim}{b} + \underset{\sim}{b}.\underset{\sim}{c} + \underset{\sim}{a}.\underset{\sim}{c}) = 0$ $\underset{\sim}{a}.\underset{\sim}{b} + \underset{\sim}{b}.\underset{\sim}{c} + \underset{\sim}{a}.\underset{\sim}{c} = -\dfrac{29}{2}$	

5 It is given that $\mathrm{f}(n)=\frac{n}{5^{n-1}}$ where n is a positive integer.

(a) By considering $\mathrm{f}(r)-\mathrm{f}(r+1)$, find an expression for $\sum_{r=2}^{n}\frac{4r-1}{5^r}$. [3]

5 **a**	$\mathrm{f}(r)-\mathrm{f}(r+1)=\frac{r}{5^{r-1}}-\frac{r+1}{5^r}$ $=\frac{5r-r-1}{5^r}=\frac{4r-1}{5^r}$ $\sum_{r=2}^{n}\frac{4r-1}{5^r}=\sum_{r=2}^{n}\left[\mathrm{f}(r)-\mathrm{f}(r+1)\right]$ $=\mathrm{f}(2)-\cancel{\mathrm{f}(3)}$ $+\cancel{\mathrm{f}(3)}-\cancel{\mathrm{f}(4)}$ $+\ldots$ $+\cancel{\mathrm{f}(n-1)}-\cancel{\mathrm{f}(n)}$ $+\cancel{\mathrm{f}(n)}-\mathrm{f}(n+1)$ $=\mathrm{f}(2)-\mathrm{f}(n+1)$ $=\frac{2}{5}-\frac{n+1}{5^n}$	

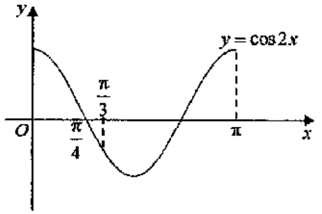
(b) Hence find an expression for $\sum_{r=1}^{n}\frac{4r+6}{5^{r+1}}$. [3]

5 **b**	$\sum_{r=1}^{n}\frac{4r+6}{5^{r+1}}$ Replace r by $r-1$ $\sum_{r-1=1}^{r-1=n}\frac{4(r-1)+6}{5^{r-1+1}}=\sum_{r=2}^{n+1}\frac{4r+2}{5^r}$ $=\sum_{r=2}^{n+1}\frac{4r-1}{5^r}+\sum_{r=2}^{n+1}\frac{3}{5^r}$ $=\frac{2}{5}-\frac{n+2}{5^{n+1}}+3\left[\frac{\frac{1}{25}\left(1-\left(\frac{1}{5}\right)^n\right)}{1-\frac{1}{5}}\right]$ $=\frac{11}{20}-\frac{n+2}{5^{n+1}}-\frac{3}{20}\left(\frac{1}{5}\right)^n$	When we use the replacement of variable method, we need to change the upper limit as well. Learn to recognise a sum of a GP, like in this case: $\sum_{r=2}^{n+1}\frac{3}{5^r}$ is a sum of a GP with first term $\frac{3}{5^2}$ and common ratio $\frac{1}{5}$.

6 (a) Find the exact value of $\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\frac{2\sin^{-1}x}{\sqrt{1-x^2}}\mathrm{d}x$. [3]

6 **a**	$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\frac{2\sin^{-1}x}{\sqrt{1-x^2}}\,\mathrm{d}x$ $=2\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\frac{1}{\sqrt{1-x^2}}\cdot\sin^{-1}x\,\mathrm{d}x$ $=\left[\left(\sin^{-1}x\right)^2\right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}$ $=\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)^2-\left(\sin^{-1}\frac{1}{2}\right)^2$ $=\left(\frac{\pi}{3}\right)^2-\left(\frac{\pi}{6}\right)^2$ $=\frac{\pi^2}{12}$	Be familiar with all the standard forms. Know how to check if the integrand is truly of the form $\frac{\mathrm{f}'(x)}{\mathrm{f}(x)}$, or of the form $\mathrm{f}'(x)\left[\mathrm{f}(x)\right]^n$. Check through all standard forms before even thinking about integration by parts. Those who did it by parts spent much more time on this question.

(b) Find the exact value of $\int_0^{\frac{\pi}{3}}\lvert\cos 2x\rvert\,\mathrm{d}x$. [3]

6 **b**	 $\int_0^{\frac{\pi}{3}}\lvert\cos 2x\rvert\mathrm{d}x=\int_0^{\frac{\pi}{4}}\cos 2x\,\mathrm{d}x+\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}-\cos 2x\,\mathrm{d}x$ $=\left[\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}}-\left[\frac{\sin 2x}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ $=\frac{1}{2}-\left(\frac{\sqrt{3}}{4}-\frac{1}{2}\right)$ $=1-\frac{\sqrt{3}}{4}$	Sketching a graph is good way to tell when the expression is positive or negative. If you cannot visualise clearly, sketch it! There is a need to split the integral into parts where the expression $\cos 2x$ is positive and negative within the interval $\left(0,\frac{\pi}{3}\right)$.

(c) Find $\int \frac{1}{-x^2+2kx+3k^2}\,\mathrm{d}x$, where k is a positive constant. [4]

6 c	$\int \frac{1}{-x^2+2kx+3k^2}\,\mathrm{d}x$ $=\int \frac{1}{-\left(x^2-2kx\right)+3k^2}\,\mathrm{d}x$ $=\int \frac{1}{-\left[\left(x-k\right)^2-k^2\right]+3k^2}\,\mathrm{d}x$ $=\int \frac{1}{-\left(x-k\right)^2+4k^2}\,\mathrm{d}x$ $=\int \frac{1}{\left(2k\right)^2-\left(x-k\right)^2}\,\mathrm{d}x$ $=\frac{1}{2\left(2k\right)}\ln\left\|\frac{2k+\left(x-k\right)}{2k-\left(x-k\right)}\right\|+C$ $=\frac{1}{4k}\ln\left\|\frac{k+x}{3k-x}\right\|+C$ **<u>Alternative</u>** $\int \frac{1}{\left(k+x\right)\left(3k-x\right)}\mathrm{d}x$ $=\int\left[\frac{1}{4k\left(k+x\right)}+\frac{1}{4k\left(3k-x\right)}\right]\mathrm{d}x$ $=\frac{1}{4k}\ln\left\|k+x\right\|-\frac{1}{4k}\ln\left\|3k-x\right\|+C$ $=\frac{1}{4k}\ln\left\|\frac{k+x}{3k-x}\right\|+C$	

7 **(a)** The curve C has equation $y=\mathrm{f}(x)$ where

$$\mathrm{f}(x)=\frac{ax^2+bx+c}{x+d},$$

and a, b, c and d are constants, and $a\neq 0$.

Given that C has asymptote $y=x+1$, find the value of a and show that $b=d+1$. [2]

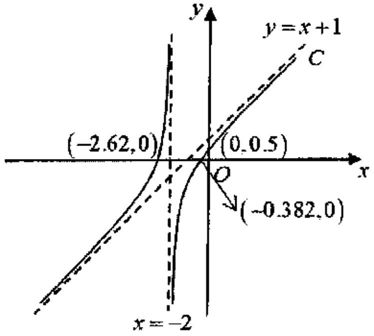
7a	$y=\mathrm{f}(x)=\frac{ax^2+bx+c}{x+d}=x+1+\frac{R}{x+d}$ From the numerator: $ax^2+bx+c=\left(x+1\right)\left(x+d\right)+R$ Comparing the coefficients, we get: $a=1,\ \ b=d+1$ **<u>Alternative</u>** $\mathrm{f}(x)=\frac{ax^2+bx+c}{x+d}=ax+\left(b-ad\right)+\frac{c-bd+ad^2}{x+d}$ $ax+\left(b-ad\right)=x+1$ $\begin{cases}a=1\\b-ad=1\end{cases}\Rightarrow a=1$ and $b=d+1$	Many candidates were able to express it in the correct form of $x+1+\frac{R}{x+d}$ and compared coefficients after expansion. A handful did not regard the remainder and made common mistakes like saying $ax^2+bx+c=\left(x+1\right)+\left(x-d\right)$.

If f is an increasing function for all $x\in\mathbb{R}$, $x>-d$, show that $c<d$. [3]

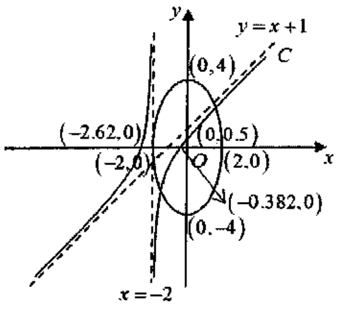
7a	Comparing coefficients from previous part, we get: $a=1,\ \ b=d+1$ and $c=d+R\Rightarrow R=c-d$ [OR alternative: $c-bd+ad^2=c-\left(d+1\right)d+d^2=c-d$] $\therefore y=\mathrm{f}(x)=x+1+\frac{c-d}{x+d}$ If f is increasing for $x>-d$, it must be the case that $\frac{\mathrm{d}y}{\mathrm{d}x}>0$ for all $x\in\mathbb{R}, x\neq-d$. $\frac{\mathrm{d}y}{\mathrm{d}x}=1-\frac{c-d}{\left(x+d\right)^2}>0$ Since $\left(x+d\right)^2>0$ for all $x\in\mathbb{R}$, $x\neq-d$, $\left(x+d\right)^2-\left(c-d\right)>0$. i.e. $x^2+2dx+d^2-c+d>0$ for all $x\in\mathbb{R}, x\neq-d$ This means that discriminant < 0 $\Rightarrow\left(2d\right)^2-4(1)\left(d^2-c+d\right)<0$ $4c-4d<0$ $\Rightarrow c<d$ (shown)	Differentiating the expression $x+1+\frac{c-d}{x+d}$ is much easier than differentiating $\frac{x^2+bx+c}{x+d}$. We need to explain why $\left(x+d\right)^2-\left(c-d\right)>0$ for all $x\in\mathbb{R}, x\neq-d$ will lead to $c<d$.

(b) It is further given that $c=1$ and $d=2$.

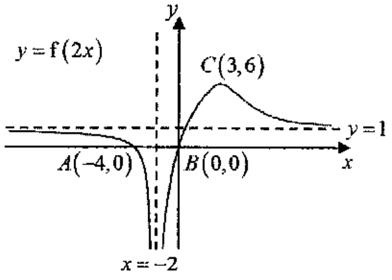
(i) Sketch C. [3]

7b (i)	Sketch $y=\mathrm{f}(x)=x+1-\frac{1}{x+2}$ 	When asked to sketch a graph, always label the coordinates of any stationary points, axial intercepts, and the equations of any asymptotes. If question did not ask for exact values, leave all answers rounded off to 3 s.f..

(ii) By sketching a suitable graph in the same diagram in part **(b)(i)**, find the number of real roots to the equation $\left(\frac{x^2+3x+1}{x+2}\right)^2+4x^2=16$. [2]

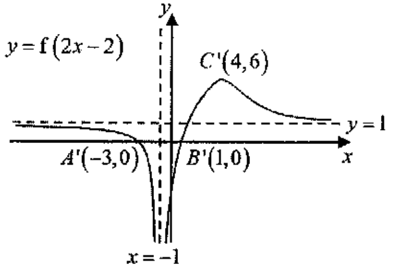
7b (ii)	Sketch $\frac{y^2}{4^2}+\frac{x^2}{2^2}=1$ on the same diagram, i.e. ellipse with centre at $(0,0)$ and passing through (–2, 0), (2, 0), (0, –4) and (0, 4).  The 2 points of intersection implies that there are 2 real roots to the equation $\left(\frac{x^2+3x+1}{x+2}\right)^2+4x^2=16$.	When sketching any conic section, like a circle or ellipse, the graph should show the correct 'shape'. In this case, the ellipse should look 'tall', passing through (–2, 0). Many did not recognise that the graph to be drawn is $\frac{y^2}{4^2}+\frac{x^2}{2^2}=1$, and went on to make y the subject unnecessarily, often resulting in only the positive square root taken ($y=\sqrt{16-4x^2}$) which is incorrect.

8 (a) The diagram shows the graph with equation $y=\mathrm{f}(2x)$. The graph passes through the points $A(-4,0)$, $B(0,0)$ and $C(3,6)$, and has asymptotes $x=-2$ and $y=1$.

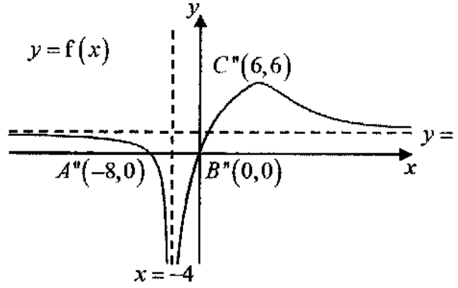


On separate clearly labelled diagrams, deduce the graphs of

(i) $y=\mathrm{f}(2x-2)$, [2]

8a (i)	$y=\mathrm{f}(2x-2)=\mathrm{f}\big(2(x-1)\big)$ 	Note that it is a transformation <u>from</u> $y=\mathrm{f}(2x)$ <u>to</u> $y=\mathrm{f}(2x-2)=\mathrm{f}\big(2(x-1)\big)$ i.e. **replace** x with $x-1$ i.e. translate 1 unit (not 2 units) in the positive x-direction

(ii) $y=\mathrm{f}(x)$. [2]

8a (ii)		Note that it is a transformation <u>from</u> $y=\mathrm{f}(2x)$ <u>to</u> $y=\mathrm{f}(x)$ i.e. **replace** x with $\frac{x}{2}$ i.e. stretch factor is 2, not ½.

(b) The curve C_1 undergoes the transformations in the order given below:

1. A translation of 2 units in the negative x direction.
2. A stretch parallel to the x axis, factor 2, y axis invariant.
3. A translation of 1 unit in the positive y direction.

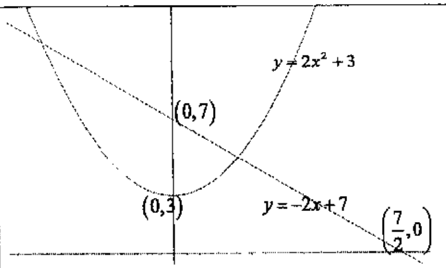
The resulting curve C_2 has equation

$$y=\frac{x^2+9x+22}{x+4}\ ,\ x\in\mathbb{R}\ ,\ x\neq -4\ .$$

Find, in the simplest form, the equation for C_1. [4]

8b	Let C_1 have equation $y=\mathrm{f}(x)$. 1. $y=\mathrm{f}(x)\to y=\mathrm{f}(x+2)$; 2. $y=\mathrm{f}(x+2)\to y=\mathrm{f}\left(\frac{1}{2}x+2\right)$; 3. $y=\mathrm{f}\left(\frac{1}{2}x+2\right)\to y=\mathrm{f}\left(\frac{1}{2}x+2\right)+1$. Now $y=\mathrm{f}\left(\frac{1}{2}x+2\right)+1=\frac{x^2+9x+22}{x+4}$. $\mathrm{f}\left(\frac{1}{2}x+2\right)=\frac{x^2+9x+22}{x+4}-1=\frac{x^2+8x+18}{x+4}$ $\mathrm{f}\left(\frac{1}{2}(x+4)\right)=\frac{(x+4)^2+2}{x+4}=x+4+\frac{2}{x+4}$ Let $w=\frac{1}{2}(x+4)\Rightarrow x+4=2w$ $\therefore \mathrm{f}(w)=2w+\frac{2}{2w}=2w+\frac{1}{w}$ $\therefore y=\mathrm{f}(x)=2x+\frac{1}{x},\quad x\neq 0.$ **Alternative**: May be easier to use the "reverse method": 3. $\frac{x^2+9x+22}{x+4}-1=\frac{x^2+8x+18}{x+4}$. 2. Replace x with $2x$: $\frac{(2x)^2+8(2x)+18}{(2x)+4}=...=\frac{2x^2+8x+9}{x+2}$. 1. Replace x with $x-2$: $\frac{2(x-2)^2+8(x-2)+9}{(x-2)+2}=...=2x+\frac{1}{x},\quad x\neq 0$.	Need to reverse the order and do the correct replacement. Note that $y=\frac{4x^2+2}{2x}$ is not in simplified form The alternative method is more challenging.

9 Find the area of the region bounded by the graphs of $y=2x^2+3$ and $y=-2x+7$. [3]

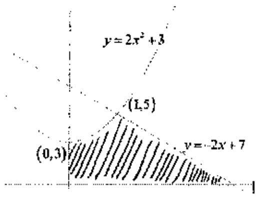
9	 Solving $y=2x^2+3$ and $y=-2x+7$ $-2x+7=2x^2+3$ $2x^2+2x-4=0$ $x^2+x-2=0$ $x=-2$ or $x=1$ Area $=\int_{-2}^{1}(-2x+7)\mathrm{d}x-\int_{-2}^{1}(2x^2+3)\mathrm{d}x$ $=\left[-x^2+4x-\frac{2x^3}{3}\right]_{-2}^{1}$ $=\left(-1+4-\frac{2}{3}\right)-\left(-4-8+\frac{2(8)}{3}\right)$ $=9$	For questions that do not prohibit the use of GC nor require exact solutions, one can use GC to evaluate numerical answers. There is NO need to split the integral to $\int_{-2}^{0}\left(-2x+7-\left(2x^2+3\right)\right)\mathrm{d}x$ $+\int_{0}^{1}\left(-2x+7-\left(2x^2+3\right)\right)\mathrm{d}x$ as the line is above the curve for $-2\le x\le 1$

State the area of the region bounded by the 2 graphs if both graphs are translated 7 units in the negative y-direction. [1]

9	Area = 9	

2024 VJC Prelim Paper 1 Solutions

The region R is bounded by $y=2x^2+3$, $y=-2x+7$, the x-axis and the y-axis. Find the exact volume of the solid generated when R is rotated 2π about the y-axis. [5]

9	Substitute $x=1$ into $y=-2x+7$ get $y=5$. Expressing x^2 as the subject $y=2x^2+3 \qquad y=-2x+7$ $x^2=\frac{y-3}{2}$ and $x^2=\left(\frac{y-7}{-2}\right)^2=\frac{1}{4}(y-7)^2$ Volume $=\pi\int_0^5\frac{1}{4}(y-7)^2\,dy-\pi\int_3^5\left(\frac{y-3}{2}\right)dy$ $=\pi\left[\frac{(y-7)^3}{12}\right]_0^5-\pi\left[\frac{(y-3)^2}{4}\right]_3^5$ $=\frac{323\pi}{12}$	It is important to identify the correct region, R.  For this question, you are required to find the volume generated when R is rotated about <u>y-axis</u>, not x-axis. For hollow volumes, always use $\int_a^b \pi x_1{}^2\,dy-\int_c^d \pi x_2{}^2\,dy$ (for rotation about y-axis)

10 Given that $z=2-i$ is a root of the equation $4z^4-12z^3+17z^2+pz+q=0$, where p and q are real, find p and q. [4]

10	$4(2-i)^4-12(2-i)^3+17(2-i)^2+p(2-i)+q=0$ $4(-7-24i)-12(2-11i)+17(3-4i)+p(2-i)+q=0$ $(-28-24+51+2p+q)+(-96+132-68-p)i=0$ Comparing real and imaginary parts, $-1+2p+q=0$ ...(1) $-32-p=0$ ...(2) Solving, $p=-32, \quad q=65$ **<u>Alternative</u>** Since all the coefficients of $4z^4-12z^3+17z^2+pz+q=0$ are real, by Conjugate Root Theorem, $z^*=2+i$ is also a root. A quadratic factor is $[z-(2-i)][z-(2+i)]=z^2-4z+5$ $4z^4-12z^3+17z^2+pz+q=(z^2-4z+5)(az^2+bz+c)$ Comparing coefficients, $a=4$ $b-4a=-12\Rightarrow b=4$ $c-4b+5a=17\Rightarrow c=13$ $-4c+5b=p\Rightarrow p=-32$ $5c=q\Rightarrow q=65$	Tip: Question did not prohibit the use of GC, hence can use GC to quickly evaluate $(2-i)^4$, $(2-i)^3$ and $(2-i)^2$

2024 VJC Prelim Paper 1 Solutions

Using the values of p and q found, find the other roots of the equation $4z^4-12z^3+17z^2+pz+q=0$ in exact form. [4]

10	Since all the coefficients of $4z^4-12z^3+17z^2-32z+65=0$ are real, by Conjugate Root Theorem, $z^*=2+i$ is also a root. A quadratic factor is $[z-(2-i)][z-(2+i)]=z^2-4z+5$ By long division (or by observation), $4z^4-12z^3+17z^2-32z+65=(z^2-4z+5)(4z^2+4z+13)$ $4z^4-12z^3+17z^2-32z+65=0$ $(z^2-4z+5)(4z^2+4z+13)=0$ $z=2-i,\ 2+i$ or $z=\frac{-4\pm\sqrt{16-4(4)(13)}}{8}$ $=\frac{-4\pm\sqrt{-192}}{8}$ $=\frac{-4\pm4\sqrt{-12}}{8}$ $=\frac{-1\pm2\sqrt{3}i}{2}$ $=-\frac{1}{2}\pm\sqrt{3}i$ The other roots are $z=2+i,\ -\frac{1}{2}+\sqrt{3}i$ or $-\frac{1}{2}-\sqrt{3}i$	Need to mention the use of Conjugate Root Theorem as all coefficients of the polynomial are real. Tip: use $(a+b)(a-b)=a^2-b^2$ to quickly expand $[z-(2-i)][z-(2+i)]$ $=[(z-2)+i][(z-2)-i]$ $=(z-2)^2-i^2$ Remember to answer the question – there are in total **3 other roots** of the equation.

BP-847

BP~848

11 Dendrologists are specialised scientists who study trees and woody plants. Their work is diverse and can encompass various activities related to the identification, classification, biology, and ecology of trees. A group of dendrologists are studying the growth of 2 species of trees, codenamed Tree *Vee* and Tree *Jay*.

In the 1st year, the height of Tree *Vee* and Tree *Jay* are both H cm.

In the 2nd year, Tree *Vee*'s height increases by s cm and subsequently, the increase in height every year is 10% less than the previous year's increase. Show that the height of Tree *Vee* in the 4th year is given by $\left(H+2.71s\right)$ cm. [1]

11

Tree *Vee*	
n	Increase in height (cm)
1	-
2	s
3	$0.9\,s$
4	$0.9^2\,s$
…	

Height of Tree *Vee* in the 4th year in cm

$= H+s+0.9s+0.9^2s$

$= H+2.71s$

Show that the height of Tree *Vee* in the n^{th} year is given by $\left[H+10s\left(1-0.9^{n-1}\right)\right]$ cm. [3]

11 Height of Tree *Vee* in the n^{th} year in cm

$= H+s+0.9s+0.9^2s+\ldots+0.9^{n-2}s$

$= H+\dfrac{s\left(1-0.9^{n-1}\right)}{1-0.9}$

$= H+10s\left(1-0.9^{n-1}\right)$

It is a "show" question.

Need to <u>write down the terms of the series</u> before using sum of GP formula to get the final result.

Hence, write down in terms of H and s, the theoretical maximum height (in cm) of Tree *Vee*. [1]

11 Theoretical maximum height (cm) of Tree *Vee*

$= H+10s$

In the 2nd year, Tree *Jay*'s height increases by t cm and subsequently, the increase in height every year is 0.5 cm less than the previous year's increase. Show that the height of Tree *Jay* in the 10th year is given by $\left(H-18+9t\right)$ cm. [2]

11

[illegible]	
[illegible]	[illegible]
1	-
2	t
3	$t-0.5$
4	$t-2(0.5)$
…	

Height of Tree *Jay* (cm) in the 10th year

$= H+t+\left(t-0.5\right)+\left(t-2(0.5)\right)+\ldots+\left(t-8(0.5)\right)$

$= H+\frac{9}{2}\left[2t+(9-1)(-0.5)\right]$

$= H+\frac{9}{2}(2t-4)$

$= H-18+9t$

Again, it is a "show" question.

Best to <u>write down the terms of the series</u> before using sum of AP formula to get the final result

It is now given that $t=20$.

After the 10th year, Tree *Jay*'s height increases at a constant rate of 7 cm per year. Express Tree Jay's height (in cm) in the n^{th} year (where $n\geq 11$) in terms of H and n. [2]

11 Height of Tree *Jay* (cm) in the 10th year

$= H-18+9(20)$

$= H+162$

[illegible]	
[illegible]	[illegible]
10	H+162
11	H+169
12	H+176
…	

Height of Tree *Jay* (cm) in the n^{th} year

$= H+162+(n-10)(7)$

$= H+7n+92$

It is further given that $s = 30$, and the 1st year is the year 2024. Find the years in which the heights of Tree *Vee* and Tree *Jay* are within 7 cm of each other, **after 2034.** [3]

11 Difference in height (cm) of Tree Vee and Tree Jay in nth year

$=[H+7n+92]-[H+10s(1-0.9^{n-1})]$

$=7n+92-300(1-0.9^{n-1})$

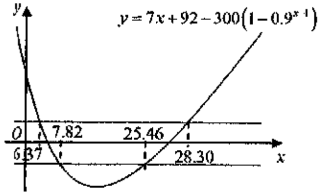
Using GC,

[illegible]	Difference in height (cm) of Tree Vee and Tree Jay= $7n+92-300(1-0.9^{n-1})$
25	−9.07
26	−4.46
27	0.383
28	5.44
29	10.7

$2024 + (26 - 1) = 2049$

The years are 2049, 2050 and 2051.

Alternative (graphical solution)



Required values are $n = 26, 27, 28$.

The years are 2049, 2050 and 2051.

As n must be an integer, use a table of values to solve the <u>inequality</u>

$|7n+92-300(1-0.9^{n-1})|<7$

Tip: to obtain the years, you can consider using this table

n	Year
$n=1$	2024
$n=2$	2025
…	
$n=26$	2024+26-1 =2049

---

12 Game developers closely monitor the number of people playing their game. Understanding player numbers and behaviour can not only help in optimising in-game purchases, advertisement placements, and other revenue-generating aspects, it can also help the company manage server loads and ensure the game runs smoothly without performance issues.

Two game developers are interested in the number of players playing the mobile game "Mobile Saga". They attempt to model the number of players x, in **hundred thousands**, at time t months after the launch of the game using a differential equation. On the day of the launch, there were 55 000 players.

(a) One game developer suggests that x and t are related by the differential equation $\frac{dx}{dt}=\frac{3}{5}x-kt^2$, where k is a positive constant.

(i) By substituting $x=ue^{\frac{3}{5}t}$, show that the differential equation can be written as $\frac{du}{dt}=-kt^2e^{-\frac{3}{5}t}$. [2]

12 a i

$x=ue^{\frac{3}{5}t}$ ---(1)

$\frac{dx}{dt}=0.6x-kt^2$ ---(2)

Differentiating (1) w.r.t t

$\frac{dx}{dt}=\frac{3}{5}ue^{\frac{3}{5}t}+e^{\frac{3}{5}t}\frac{du}{dt}$ ---(3)

Substitute (1) & (3) into (2)

$\frac{3}{5}ue^{\frac{3}{5}t}+e^{\frac{3}{5}t}\frac{du}{dt}=0.6\left(ue^{\frac{3}{5}t}\right)-kt^2$

$e^{\frac{3}{5}t}\frac{du}{dt}=-kt^2$

$\frac{du}{dt}=-kt^2e^{-\frac{3}{5}t}$

For such DE questions involving substitution, **identify the variables** to be substituted before performing any differentiation

Given DE: $\frac{dx}{dt}=\frac{3}{5}x-kt^2$ in terms of $\frac{dx}{dt},t,x$

$\downarrow$

Result: $\frac{du}{dt}=-kt^2e^{-\frac{3}{5}t}$ - in terms of $\frac{du}{dt},t$

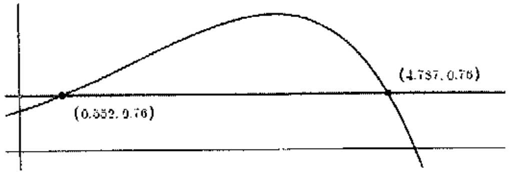
Looking at the above, one has to use $x=ue^{\frac{3}{5}t}$ substitute $\frac{dx}{dt}$ with $\frac{du}{dt}$.

Hence, differentiate $x=ue^{\frac{3}{5}t}$ wrt t, bearing in mind both u and t are variables, not constants.

(ii) Hence show that $x=\frac{5k}{3}t^2+\frac{50k}{9}t+\frac{250k}{27}+\left(\frac{11}{20}-\frac{250k}{27}\right)e^{\frac{3}{5}t}$. [4]

12 a ii	$\frac{\mathrm{d}u}{\mathrm{d}t}=-kt^2e^{-\frac{3}{5}t}$ Integrating w.r.t. t: $u=-k\int t^2e^{-\frac{3}{5}t}\,\mathrm{d}t$ $=-k\left[-\frac{5}{3}t^2e^{-\frac{3}{5}t}-\int(2t)\left(-\frac{5}{3}e^{-\frac{3}{5}t}\right)\mathrm{d}t\right]$ $=\frac{5k}{3}t^2e^{-\frac{3}{5}t}-\frac{10k}{3}\int te^{-\frac{3}{5}t}\,\mathrm{d}t$ $=\frac{5k}{3}t^2e^{-\frac{3}{5}t}-\frac{10k}{3}\left[\left(-\frac{5}{3}te^{-\frac{3}{5}t}\right)-\int-\frac{5}{3}e^{-\frac{3}{5}t}\,\mathrm{d}t\right]$ $=\frac{5k}{3}t^2e^{-\frac{3}{5}t}+\frac{50k}{9}te^{-\frac{3}{5}t}+\frac{250k}{27}e^{-\frac{3}{5}t}+C$ $xe^{-\frac{3}{5}t}=\frac{5k}{3}t^2e^{-\frac{3}{5}t}+\frac{50k}{9}te^{-\frac{3}{5}t}+\frac{250k}{27}e^{-\frac{3}{5}t}+C$ $x=\frac{5k}{3}t^2+\frac{50k}{9}t+\frac{250k}{27}+Ce^{\frac{3}{5}t}$ When $t=0$, $x=0.55$ $0.55=\frac{250k}{27}+C \qquad \Rightarrow C=\frac{11}{20}-\frac{250k}{27}$ Hence $x=\frac{5k}{3}t^2+\frac{50k}{9}t+\frac{250k}{27}+\left(\frac{11}{20}-\frac{250k}{27}\right)e^{\frac{3}{5}t}$	

(iii) Company A intends to place an advertisement in the game only if there are more than 76 000 players playing the game. Given that $k=\frac{1}{10}$, find the length of time for which Company A will place an advertisement in "Mobile Saga", giving your answer correct to the nearest month. [2]

12 a iii	$x=\frac{1}{10}\left(\frac{5}{3}t^2+\frac{50}{9}t+\frac{250}{27}\right)+\left(-\frac{203}{540}\right)e^{\frac{3}{5}t}$  $\frac{1}{10}\left(\frac{5}{3}t^2+\frac{50}{9}t+\frac{250}{27}\right)+\left(-\frac{203}{540}\right)e^{\frac{3}{5}t}=0.76$ From the GC, longest duration = $4.787-0.552=4.235$ ≈ 4 months	Note that t is not discrete, hence **cannot use a table** to solve the inequality. The question asked for the length of time, i.e. **duration**, when there are more than 76 000 players playing the game. It is not sufficient to solve for 1 value of t.

(b) The other game developer suggests that x and t are related by the differential equation $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-\frac{10}{(1+t)^3}$. Given further that there were 180 000 players playing "Mobile Saga" after 1 month, find x in terms of t. [4]

12 b	$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-\frac{10}{(1+t)^3}$ Integrating w.r.t t: $\frac{\mathrm{d}x}{\mathrm{d}t}=-10\int\frac{1}{(1+t)^3}\,\mathrm{d}t=\frac{5}{(1+t)^2}+C$ Integrating w.r.t. t: $x=5\int\frac{1}{(1+t)^2}\,\mathrm{d}t=-\frac{5}{(1+t)}+Ct+D$ When $t=0$, $x=0.55$ $0.55=-5+D \quad \Rightarrow D=\frac{111}{20}$ When $t=1$, $x=1.8$ $1.8=-\frac{5}{(1+1)}+C+\frac{111}{20} \quad \Rightarrow C=-\frac{5}{4}$ $\therefore x=-\frac{5}{1+t}-\frac{5}{4}t+\frac{111}{20}$	After obtaining $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{5}{(1+t)^2}+C$, one has to integrate **wrt t** once again to obtain $x=-\frac{5}{(1+t)}+Ct+D$, and NOT $x=-\frac{5}{(1+t)}+Cx+D$.

General Comments • The rubric of the paper states that non-exact numerical answers should be given correct to 3 significant figures. • The use of graphing calculators is encouraged, but in questions where a calculator is prohibited, you need to show sufficient working in answering that question. • Need to be aware that every step shown in a **<u>given answer question</u>** needs to maintain <u>an appropriate level of accuracy.</u> • Read the question carefully.

Section A: Pure Mathematics (40 marks)

1 **(a)** Show that $\dfrac{1}{\sqrt{1+x^2}-\sqrt{1-x^2}}=\dfrac{1}{2x^2}\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)$. [1]

1a	$\dfrac{1}{\sqrt{1+x^2}-\sqrt{1-x^2}}\times\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}$ $=\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{(1+x^2)-(1-x^2)}$ $=\dfrac{1}{2x^2}\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)$ (shown)	Mostly well done Remember that: $(a+b)(a-b)=a^2-b^2$.

(b) Hence use appropriate expansions from the List of Formulae (MF26) to find the first two non-zero terms in the series expansion of $\dfrac{x^2}{\sqrt{1+x^2}-\sqrt{1-x^2}}$ in ascending powers of x for $x\neq 0$. [3]

1b	$\dfrac{x^2}{\sqrt{1+x^2}-\sqrt{1-x^2}}$ $=\dfrac{1}{2}\left[\left(1+x^2\right)^{\frac{1}{2}}+\left(1-x^2\right)^{\frac{1}{2}}\right]$ $=\dfrac{1}{2}\left[1+\dfrac{1}{2}x^2+\dfrac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\left(x^2\right)^2+...\right]+\dfrac{1}{2}\left[1-\dfrac{1}{2}x^2+\dfrac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\left(-x^2\right)^2+...\right]$ $=\dfrac{1}{2}\left[2-\dfrac{1}{4}x^4+...\right]$ $\approx 1-\dfrac{1}{8}x^4$	Mostly well done Do note that repeated differentiation is <u>not</u> allowed here as question says "use appropriate expansions from MF26".

(c) State the set of values of x for which the series expansion is valid. [1]

1c	$\left\|x^2\right\|<1,\ x\neq 0$ $x^2<1$ (or draw the graph) $x^2-1<0$ $(x+1)(x-1)<0$ $\therefore -1<x<1$ Set of values of x: $\{\ x\in\mathbb{R}: -1<x<1,\ x\neq 0\ \}$	Learn how to solve inequalities properly and not "jump" the step.

(d) It is given that the two terms found in part **(b)** are equal to the first two terms in the series expansion of $\cos\left(ax^b\right)$. Find the possible value(s) of the constants a and b. [2]

1d	$\cos\left(ax^b\right)\approx 1-\dfrac{\left(ax^b\right)^2}{2}\equiv 1-\dfrac{1}{8}x^4$ $\dfrac{a^2}{2}=\dfrac{1}{8}$ and $2b=4$ $a=\pm\dfrac{1}{2}$ and $b=2$	Quite a few arithmetic errors spotted here, e.g. $\dfrac{\left(ax^b\right)^2}{2}\neq\dfrac{ax^{2b}}{2}$. Do practise well to gain fluency

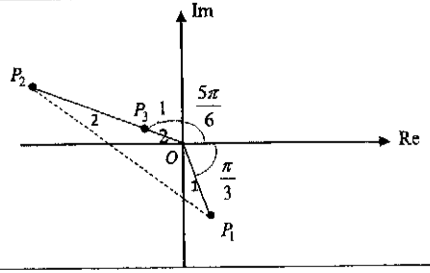
2 **Do not use a calculator in answering this question.**

The complex numbers z_1, z_2 and z_3 are such that $z_1 = -\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$, $z_2 = -\sqrt{3} + \mathrm{i}$ and $z_3 = \frac{z_1}{z_2}$.

(a) Express each of z_1, z_2 and z_3 in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. [3]

2a	$z_1 = -\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$ $= \mathrm{e}^{\mathrm{i}\pi}.\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$ $= \mathrm{e}^{\mathrm{i}\frac{5\pi}{3}}$ $= \mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{3}\right)}$ $z_2 = -\sqrt{3} + \mathrm{i}$ $= 2\mathrm{e}^{\mathrm{i}\left(\pi - \tan^{-1}\frac{1}{\sqrt{3}}\right)}$ $= 2\mathrm{e}^{\mathrm{i}\left(\pi - \frac{\pi}{6}\right)}$ $= 2\mathrm{e}^{\mathrm{i}\frac{5\pi}{6}}$ $z_3 = \frac{z_1}{z_2}$ $= \frac{\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{3}\right)}}{2\mathrm{e}^{\mathrm{i}\frac{5\pi}{6}}}$ $= \frac{1}{2}\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{3} - \frac{5\pi}{6}\right)}$ $= \frac{1}{2}\mathrm{e}^{\mathrm{i}\left(-\frac{7\pi}{6}\right)}$ $= \frac{1}{2}\mathrm{e}^{\mathrm{i}\frac{5\pi}{6}}$	Polar form for complex numbers was poorly performed in general. Do note that $-1 = \mathrm{e}^{\mathrm{i}\pi}$ (i.e. modulus = 1, argument = π. When in doubt, please plot the complex number as a point on an argand diagram – this will help you to better determine the correct modulus and argument.

(b) Sketch an Argand diagram showing the points P_1, P_2 and P_3 where P_1, P_2 and P_3 represent the complex numbers z_1, z_2 and z_3 respectively. [2]

2b		This follows from (a). Do note that you are required to indicate the modulus in addition to the argument i.e. distance from the origin. Also, ensure that the relative positioning of the 3 points is correct.

(c) Find the area of triangle OP_1P_2. [2]

2c	Area of triangle OP_1P_2 $= \frac{1}{2}(1)(2)\sin\left(\frac{\pi}{6} + \frac{\pi}{2} + \frac{\pi}{6}\right)$ $= \sin\frac{5\pi}{6}$ $= 0.5$	For those who have done (a) and (b) correctly, this was well done and students were able to apply the correct formula $\frac{1}{2}ab(\sin c)$.

(d) Find the smallest positive integer n for which $\left(z_2{}^*\right)^n$ is purely imaginary. [2]

2d	$\left(z_2{}^*\right)^n = \left(2\mathrm{e}^{\mathrm{i}\left(-\frac{5\pi}{6}\right)}\right)^n$ $= 2^n\mathrm{e}^{\mathrm{i}\left(-\frac{5n\pi}{6}\right)}$ For $\left(z_2{}^*\right)^n$ to be purely imaginary, $-\frac{5n\pi}{6} = \frac{\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, -\frac{3\pi}{2}, \frac{5\pi}{2}, -\frac{5\pi}{2}, \ldots$ Smallest positive integer $n = 3$	Most students were able to translate "purely imaginary" to the argument being an odd multiple of $\frac{\pi}{2}$. So, either write it in a general form e.g. $(2k+1)\frac{\pi}{2}$ or $(2k-1)\frac{\pi}{2}$, or since the question is asking for the smallest n, you can also list out the first few (negative) odd multiples of $\frac{\pi}{2}$ and check which value of n works.

3 The line l_1 has equation $\mathbf{r}=3\mathbf{i}-4\mathbf{j}-5\mathbf{k}+\lambda(\mathbf{i}-2\mathbf{j}-\mathbf{k})$, where λ is a real parameter. The point A has position vector $\mathbf{i}-2\mathbf{j}+\mathbf{k}$.

(a) The plane p contains the line l_1 and the point A. Find a cartesian equation of the plane p. [3]

3a	$\underset{\sim}{r}=\begin{pmatrix}3\\-4\\-5\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\-1\end{pmatrix},\lambda\in\mathbb{R}\quad ; \overrightarrow{OA}=\begin{pmatrix}1\\-2\\1\end{pmatrix}$ Vector parallel to $p=\begin{pmatrix}1\\-2\\1\end{pmatrix}-\begin{pmatrix}3\\-4\\-5\end{pmatrix}=\begin{pmatrix}-2\\2\\6\end{pmatrix}=2\begin{pmatrix}-1\\1\\3\end{pmatrix}$ Normal vector $=\begin{pmatrix}-1\\1\\3\end{pmatrix}\times\begin{pmatrix}1\\-2\\-1\end{pmatrix}=\begin{pmatrix}5\\2\\1\end{pmatrix}$ Equation of p $\underset{\sim}{r}\cdot\begin{pmatrix}5\\2\\1\end{pmatrix}=\begin{pmatrix}1\\-2\\1\end{pmatrix}\cdot\begin{pmatrix}5\\2\\1\end{pmatrix}\Rightarrow\underset{\sim}{r}\cdot\begin{pmatrix}5\\2\\1\end{pmatrix}=2$ Cartesian equation of p: $5x+2y+z=2$	Make sure you copy the vector correctly and not make any silly mistakes at the start The origin O may not be in plane p, hence you cannot assume that $\overrightarrow{OA}$ is a vector in the plane (as it turns out, O is not in the plane). Also, always check that the normal of the plane you have obtained after cross product is correct. A quick way to check is via dot product: $\begin{pmatrix}5\\2\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\3\end{pmatrix}=\ldots=0$ and $\begin{pmatrix}5\\2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\-1\end{pmatrix}=\ldots=0$. (since the normal must be perpendicular to the 2 vectors used in the cross product.) Finally, if your normal is $\begin{pmatrix}10\\4\\2\end{pmatrix}$, it is a good idea to reduce it to $\begin{pmatrix}5\\2\\1\end{pmatrix}$ first before finding the cartesian equation of the plane (so that the equation of the plane can be in the "reduced" form).

(b) Find the position vector of the point A', the reflection of the point A in the line l_1. [4]

3b	Let F be the foot of the perpendicular from A to the line l_1, and B be the point on l_1 with position vector $3\mathbf{i}-4\mathbf{j}-5\mathbf{k}$ Let $\overrightarrow{BA}=\begin{pmatrix}1\\-2\\1\end{pmatrix}-\begin{pmatrix}3\\-4\\-5\end{pmatrix}=\begin{pmatrix}-2\\2\\6\end{pmatrix}$ $\overrightarrow{BF}=\left[\begin{pmatrix}-2\\2\\6\end{pmatrix}\cdot\frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\-1\end{pmatrix}\right]\frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\-1\end{pmatrix}=-2\begin{pmatrix}1\\-2\\-1\end{pmatrix}$ $\overrightarrow{OF}=\overrightarrow{OB}+\overrightarrow{BF}=\begin{pmatrix}3\\-4\\-5\end{pmatrix}-2\begin{pmatrix}1\\-2\\-1\end{pmatrix}=\begin{pmatrix}1\\0\\-3\end{pmatrix}$ $\overrightarrow{OF}=\frac{\overrightarrow{OA}+\overrightarrow{OA'}}{2}\Rightarrow\overrightarrow{OA'}=2\overrightarrow{OF}-\overrightarrow{OA}$ $\overrightarrow{OA'}=2\begin{pmatrix}1\\0\\-3\end{pmatrix}-\begin{pmatrix}1\\-2\\1\end{pmatrix}=\begin{pmatrix}1\\2\\-7\end{pmatrix}$	Most students were able to apply the method of finding the projection vector or finding the foot of the perpendicular but made several errors, e.g. For $\begin{pmatrix}-2\\2\\6\end{pmatrix}$, you will have to use $\begin{pmatrix}-2\\2\\6\end{pmatrix}$ and not $\begin{pmatrix}-1\\1\\3\end{pmatrix}$ which is only half of $\overrightarrow{BA}$ (contrast this with the idea of "reducing" the normal). Almost all students were familiar with applying the midpoint theorem which is good

(c) The plane q is such that q is parallel to p and passes through the point with position vector $-3\mathbf{j}+\mathbf{k}$. Find a cartesian equation of q and the exact shortest distance between p and q. [3]

3c	$5x+2y+z=k$ Sub $-3\mathbf{j}+\mathbf{k}$ into the equation: $5(0)+2(-3)+1=k\Rightarrow k=-5$ Cartesian equation of q: $5x+2y+z=-5$ Exact shortest distance between p and q $=\left\lvert\left[\begin{pmatrix}0\\-3\\1\end{pmatrix}-\begin{pmatrix}1\\-2\\1\end{pmatrix}\right]\cdot\begin{pmatrix}5\\2\\1\end{pmatrix}\right\rvert\frac{1}{\sqrt{25+4+1}}$ $=\left\lvert\begin{pmatrix}-1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}5\\2\\1\end{pmatrix}\right\rvert\frac{1}{\sqrt{30}}$ $=\frac{7}{\sqrt{30}}$ **Alternative** Exact shortest distance between p and q $=\frac{5+2}{\sqrt{5^2+2^2+1^2}}=\frac{7}{\sqrt{30}}$	Always draw a simple diagram (if you need) to help you to determine the points to be used in the planes and whether dot or cross product should be used to find the required distance.

(d) The line l_2 has the equation $\frac{y-3}{2}=\frac{z-7}{3}$, $x=2$. Given that l_2 intersects p at point S, find the area of the triangle OAS. [4]

3d

Substitute $\underset{\sim}{r}=\begin{pmatrix}2\\3\\7\end{pmatrix}+\mu\begin{pmatrix}0\\2\\3\end{pmatrix}$ into $\underset{\sim}{r}\cdot\begin{pmatrix}5\\2\\1\end{pmatrix}=2$

$$\left[\begin{pmatrix}2\\3\\7\end{pmatrix}+\mu\begin{pmatrix}0\\2\\3\end{pmatrix}\right]\cdot\begin{pmatrix}5\\2\\1\end{pmatrix}=2$$

$$(10+6+7)+(4+3)\mu=2$$

$$\mu=-3$$

$$\overrightarrow{OS}=\begin{pmatrix}2\\3\\7\end{pmatrix}+(-3)\begin{pmatrix}0\\2\\3\end{pmatrix}=\begin{pmatrix}2\\-3\\-2\end{pmatrix}$$

Area of triangle OAS

$$=\frac{1}{2}\left|\begin{pmatrix}2\\-3\\-2\end{pmatrix}\times\begin{pmatrix}1\\-2\\1\end{pmatrix}\right|$$

$$=\frac{1}{2}\left|\begin{pmatrix}-7\\-4\\-1\end{pmatrix}\right|=\frac{\sqrt{66}}{2}$$

Do note that it is **<u>wrong</u>** to write:

$l_2=\begin{pmatrix}2\\3\\7\end{pmatrix}+\mu\begin{pmatrix}0\\2\\3\end{pmatrix}$ (Why?)

Many students were also unable to convert the cartesian equation of l_2 to vector equation correctly, resulting in the wrong $\overrightarrow{OS}$ that was found. Please learn well from now

4 The curve C is defined by the parametric equations

$$x=a\left(1+\frac{1}{t}\right)\quad\text{and}\quad y=a\left(t-\frac{1}{t^2}\right)$$

where a is a positive constant and $t\neq 0$.

(a) Show that $\frac{dy}{dx}=-\left(\frac{2+t^3}{t}\right)$. [3]

4a

$$\frac{dx}{dt}=-a\left(\frac{1}{t^2}\right)$$

$$\frac{dy}{dt}=a\left(1+\frac{2}{t^3}\right)$$

$$\frac{dy}{dx}=a\left(1+\frac{2}{t^3}\right)\div\left(-a\left(\frac{1}{t^2}\right)\right)$$

$$=\left(\frac{t^3+2}{t^3}\right)\times\left(-t^2\right)$$

$$=-\left(\frac{2+t^3}{t}\right)$$

Mostly well done

(b) Find, in terms of a, the coordinates of the turning point on C, and explain why it is a maximum. [4]

4b

At stationary point, $\frac{dy}{dx}=0\Rightarrow\frac{2+t^3}{t}=0$

$$t^3=-2\Rightarrow t=-\sqrt[3]{2}$$

$$x=a\left(1-\frac{1}{\sqrt[3]{2}}\right)=0.2062995a$$

$$y=a\left(-\sqrt[3]{2}-\frac{1}{2^{\frac{2}{3}}}\right)=-1.88988a$$

Coordinates of turning point are $(0.206a,\ -1.89a)$.

<u>To determine that it is a max turning point, use first derivative sign test.</u>

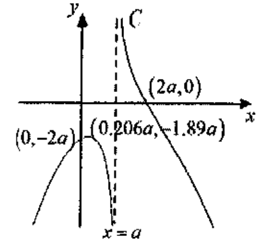
t	$t=-1.25$	$t=-\sqrt[3]{2}=-1.25992$	$t=-1.27$
x	$x=0.2a$ (left)	$x=a\left(1-\frac{1}{\sqrt[3]{2}}\right)$ $=0.206a$	$x=0.213a$ (right)
$\frac{dy}{dx}$	0.0375	0	−0.0381
Sign of $\frac{dy}{dx}$	Positive	Zero	Negative

Therefore, turning point is a maximum.

Since there is no requirement to give the coordinates in exact form, please give the answer as $(0.206a,\ -1.89a)$ as it will help you to position the turning point when sketching the graph in (c). (Note a is positive.)

Also, do note that finding the second derivative for parametric equations is <u>out of the syllabus</u> – i.e. it is not simply to differentiate $\frac{dy}{dx}$ once more with respect to x (i.e. chain rule is required). Hence, for explaining why the turning point is a maximum, you will have to perform the first derivative test, showing <u>a table of values of x and t</u> that were used and the corresponding values of $\frac{dy}{dx}$ that were found.

(c) Sketch C. [3]

4c	 Asymptote at $x=a$: As $t\to+\infty$, $x\to a^+$, $y\to+\infty$ As $t\to-\infty$, $x\to a^-$, $y\to-\infty$ Axial intercepts: At $x=0$, $t=-1$, $y=-2a$ At $y=0$, $t=1$, $x=2a$	Do not forget to adjust t to include negative values. If you forget, you should realise something is not right, as you would be missing the maximum turning point that was mentioned in (b). Please learn how to find the asymptote and axial intercepts and not just rely on the GC. The GC may give you the general shape, but not the details that you may need especially when an unknown a is present in the parametric equations.

Section B: Statistics (60 marks)

5 Two married couples, two single adults and two children form a team of 8 to take part in a series of games.

(a) In the first game, the team sits in a circle. Find the number of arrangements that can be formed if each married couple must be seated together. [2]

5a	N (team seated in a circle & each couple together) $=(6-1)!\times 2!\times 2!$ $=480$ 	2 couples – hence do not forget it is 2! × 2!

(b) A group of three people are to be selected from the team for the second game. Find the number of different groups that can be formed if there must **not** be a married couple in the group. [2]

5b	N (select group of 3, husband & wife cannot both be selected) = N (no restriction) – N (1 couple & 1 other person) $=\binom{8}{3}-\binom{2}{1}\times\binom{6}{1}$ $=44$ **Alternative** Case 1: N(no married person) = $\binom{4}{3}=4$ Case 2: N(1 married person) = $\binom{4}{1}\times\binom{4}{2}=24$ Case 3: N(1 married person from each couple) $=2\times\binom{2}{1}\times\binom{4}{1}=16$ N (group of 3, husband & wife cannot both be selected) $=4+24+16=44$	A common mistake for those who did by the complement method was: $\binom{8}{3}-\binom{2}{1}\times\binom{4}{1}$ resulting in the wrong answer.

(c) In the third game, each team member selects a unique number from the set $\{1, 2, \ldots, 8\}$. Find the number of different ways this can be done if the numbers selected by the children are both greater than the numbers selected by the two single adults. [2]

5c	Number of ways $={}^8C_4\times 2!\times 2!\times 4!$ $=6720$	This was meant to be the differentiating question so kudos to the few who got it! As for the majority of you, fret not, there are still 98 marks in the paper to be earned 😊 So stay calm and focused in the A levels and not be thrown off.

6 A random variable X has the probability distribution given in the following table.

x	1	4	6	8
$\mathrm{P}(X=x)$	a	b	c	d

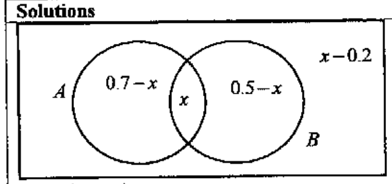
Given that $\mathrm{E}(X)=4$, $\mathrm{Var}(X)=\frac{19}{4}$ and $\mathrm{P}(X<4)=\mathrm{P}(X>4)$, find the values of a, b, c and d.

[5]

6	$\sum_{\text{all } r}\mathrm{P}(X=r)=1$ $a+b+c+d=1$ ---(1) $\mathrm{P}(X<4)=\mathrm{P}(X>4)$ $a=c+d$ $a-c-d=0$ ---(2) $\mathrm{E}(X)=4$ $a+4b+6c+8d=4$ ---(3) $\mathrm{Var}(X)=\frac{19}{4}$ $a+16b+36c+64d-4^2=\frac{19}{4}$ $a+16b+36c+64d=\frac{83}{4}$ ---(4) Solving, $a=\frac{1}{4}$, $b=\frac{1}{2}$, $c=\frac{1}{8}$ and $d=\frac{1}{8}$	Mostly well done ☺ Do note that X is a discrete random variable here, i.e. it only takes on possible values of 1, 4, 6 and 8. Some students mistook or assumed X to be normal which is not true. Also, always remember that if you have 4 variables to solve completely (i.e. a, b, c and d), you will need (at least) 4 equations. So don't forget the equation that comes from the sum of the probabilities is 1.

7 For events A, B and C, it is given that $\mathrm{P}(A)=0.7$, $\mathrm{P}(B)=0.5$, $\mathrm{P}(C\mid A')=0.6$ and $\mathrm{P}(A\mid C')=0.76$.

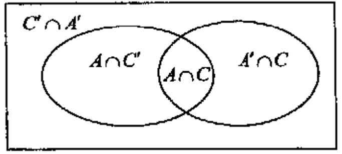
(a) Find the greatest and least possible values of $\mathrm{P}(A\cap B)$. [2]

Solutions	Comments
 Let $\mathrm{P}(A\cap B)=x$. $0\le x-0.2\le 1 \Rightarrow 0.2\le x\le 1$ $0\le 0.7-x\le 1 \Rightarrow 0\le x\le 0.7$ $0\le 0.5-x\le 1 \Rightarrow 0\le x\le 0.5$ Hence, greatest and least values of $\mathrm{P}(A\cap B)$ are 0.5 and 0.2 respectively.	Since set C is not involved in this part, Venn diagram drawn includes only sets A and B.

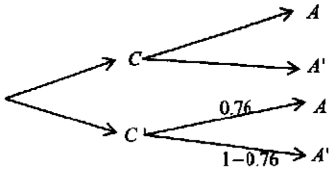
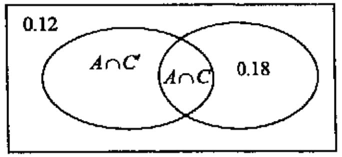
(b) Find $\mathrm{P}(C\cap A')$. [1]

Solutions	Comments
$\mathrm{P}(C\cap A')=\mathrm{P}(C\mid A')\times\mathrm{P}(A')$ $=(0.6)(1-0.7)=0.18$	Mostly well done ☺ Apply conditional probability $\mathrm{P}(C\mid A')=\frac{\mathrm{P}(C\cap A')}{\mathrm{P}(A')}$

(c) Find $\mathrm{P}(C'\cap A')$. [2]

Solutions	Comments
$\mathrm{P}(C'\cap A')=1-\mathrm{P}(A\cup C)$ $=1-[\mathrm{P}(A)+\mathrm{P}(C\cap A')]$ $=1-(0.7+0.18)$ $=0.12$	 No need to include set B in the Venn diagram

(d) Find $\mathrm{P}(C)$. [3]

Solutions	Comments
$\mathrm{P}(A\mid C')=0.76$ $\mathrm{P}(A'\mid C')=1-0.76=0.24$ $\frac{\mathrm{P}(A'\cap C')}{\mathrm{P}(C')}=0.24$ $\mathrm{P}(C')=\frac{0.24}{\mathrm{P}(A'\cap C')}=\frac{0.24}{0.12}=0.5$ $\mathrm{P}(C)=1-0.5=0.5$ **Alternative** $\mathrm{P}(A\mid C')=0.76$ $\frac{\mathrm{P}(A\cap C')}{\mathrm{P}(C')}=0.76$ $\mathrm{P}(A\cap C')=0.76[1-\mathrm{P}(C)]$ $\mathrm{P}(A\cap C')+0.76\mathrm{P}(C)=0.76$ ---(1) $\mathrm{P}(A\cup C)=0.88$ $\mathrm{P}(A\cap C')+\mathrm{P}(C)=0.88$ ---(2) Solving (1) & (2): $\mathrm{P}(C)=0.5$	 **Alternative** 

8 A small company makes wine glasses. Each day, n randomly cosen wine glasses are checked and the number of wine glasses found to be cracked is denoted by X.

(a) State, in context of the question, two assumptions needed for X to be well modelled by a binomial distribution. [2]

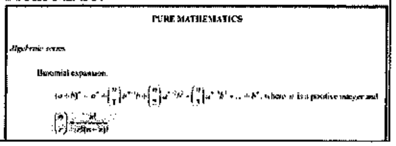
Solutions	**Comments**
The assumptions are 1. Cracked wine glasses occur independently of one 2. The probability that a wine glass is cracked remains constant.	These are **<u>INCORRECT</u>** statements • wine glasses are independent of each other • probability (number) of cracked wine glass is independent of each other • probability that a wine glass is cracked is constant for **all n wine glasses (OR each day)** • selecting / choosing / finding/ getting a cracked wine glass is independent.

Assume now that X has the distribution $\mathrm{B}(n, p)$, where $n \geq 3$

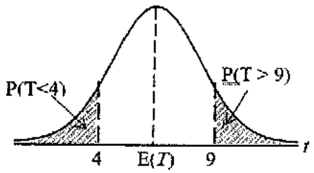
(b) Given that the mean of X and the variance of X are 1.8 and 1.773 respectively, find the value of n and the value of p. [2]

Solutions	**Comments**
$X \sim \mathrm{B}(n,\ p)$ $\mathrm{E}(X) = 1.8 \qquad \mathrm{Var}(X) = 1.773$ $np = 1.8 \quad ---(1) \quad np(1-p) = 1.773 \quad ---(2)$ $\frac{(2)}{(1)}: \frac{np(1-p)}{np} = \frac{1.773}{1.8} = 0.985$ $1 - p = 0.985$ $p = 0.015$ $n = \frac{1.8}{0.015} = 120$	Mostly well done $p = 0.015$ is exact and should be left as such.

(c) Given instead that the probability of finding 2 cracked wine glasses is thrice the probability of finding 3 cracked wine glasses, find p in terms of n. [2]

Solutions	**Comments**
$X \sim \mathrm{B}(n,\ p)$ $\mathrm{P}(X=2) = 3\mathrm{P}(X=3)$ $\binom{n}{2} p^2 (1-p)^{n-2} = 3\binom{n}{3} p^3 (1-p)^{n-3}$ $\frac{n(n-1)}{2} p^2 (1-p)^{n-2} = \frac{3n(n-1)(n-2)}{3!} p^3 (1-p)^{n-3}$ Since $p > 0$, $1 - p > 0$, $n > 0$ and $n - 1 > 0$ $\frac{1-p}{2} = \frac{3(n-2)p}{6}$ $(n-2+1)p = 1$ $p = \frac{1}{n-1}$	From MF26:  PURE MATHEMATICS Algebraic series Binomial expansion.

9 **(a)** S and T are independent random variables with the distributions $\mathrm{N}(18, 3^2)$ and $\mathrm{N}(\mu, \sigma^2)$ respectively. It is given that $\mathrm{P}(T<4) = \mathrm{P}(T>9)$ and $\mathrm{P}(S<3T) = 0.65$. Calculate the values of μ and σ. [4]

Solutions	**Comments**
Since $\mathrm{P}(T<4) = \mathrm{P}(T>9)$, by symmetry, $\mu = \frac{4+9}{2} = 6.5$ $\mathrm{E}(S-3T) = \mathrm{E}(S) - 3\mathrm{E}(T) = 18 - 3\mu = -1.5$ $\mathrm{Var}(S-3T) = \mathrm{Var}(S) + 3^2\mathrm{Var}(T) = 3^2 + 3^2\sigma^2 = 9 + 9\sigma^2$ $S - 3T \sim \mathrm{N}(-1.5, 9+9\sigma^2)$ $\mathrm{P}(S<3T) = 0.65$ $\mathrm{P}(S-3T<0) = 0.65$ $\mathrm{P}\left(Z < \frac{0-(-1.5)}{\sqrt{9+9\sigma^2}}\right) = 0.65$ From GC, $\mathrm{P}(Z < 0.38532) = 0.65$ $\frac{1.5}{\sqrt{9+9\sigma^2}} = 0.38532$ Solving, $\sigma = 0.82694 \approx 0.827$	

BP-858

(b) A fruit stall sells grapes that are packed in packets with masses in grams that follow the distribution $\mathrm{N}\left(850,30^2\right)$. The grapes are sold at \$18 per kilogram.

(i) Find the probability that a customer pays more than \$30 for two packets of grapes. [2]

Solutions	**Comments**
Let X g be the mass of a packet of grapes. $X \sim \mathrm{N}\left(850,30^2\right)$ $\therefore X_1+X_2 \sim \mathrm{N}(1700,1800)$ $\mathrm{P}\left(\frac{18}{1000}(X_1+X_2)>30\right)=\mathrm{P}\left(X_1+X_2>\frac{30000}{18}\right)$ $=0.78397$ ≈ 0.784 **Alternative** Let Y be the total cost of 2 packets of grapes $\therefore Y=\frac{18}{1000}(X_1+X_2)$ $\mathrm{E}(Y)=\frac{18}{1000}\left[2\mathrm{E}(X)\right]=30.6$ $\mathrm{Var}(Y)=\left(\frac{18}{1000}\right)^2\left[2\mathrm{Var}(X)\right]=0.5832$ $\therefore Y \sim \mathrm{N}(30.6,0.5832)$ $\mathrm{P}(Y>30)=0.78397\approx 0.784$	Selling price: \$18 per kg → $\$\frac{18}{1000}$ per g

(ii) The fruit stall accepts payment by cash or PayNow. The number of customers who pay by PayNow in a day is a random variable with mean 12 and variance 4.8. In a month of 30 days, find the probability that the average number of customers per day who pay by PayNow is more than 12.3. [3]

Solutions	**Comments**
Let Y be the number of customers who pay by PayNow in a day. $\mathrm{E}(Y)=12, \quad \mathrm{Var}(Y)=4.8$ Then $\bar{Y}=\frac{Y_1+Y_2+\ldots+Y_{30}}{30}$ is the average number of customers per day who pay by PayNow. By Central Limit Theorem, $\bar{Y} \sim \mathrm{N}\left(12,\frac{4.8}{30}\right)$ approx. $\mathrm{P}\left(\bar{Y}>12.3\right)=0.22663$ ≈ 0.227	**Incorrect to assume** $Y \sim \mathrm{N}(12,4.8)$ OR $Y \sim \mathrm{B}(n,p)$ It is necessary to state "by Central Limit Theorem, $\bar{Y} \sim \mathrm{N}\left(12,\frac{4.8}{30}\right)$ approx"

---

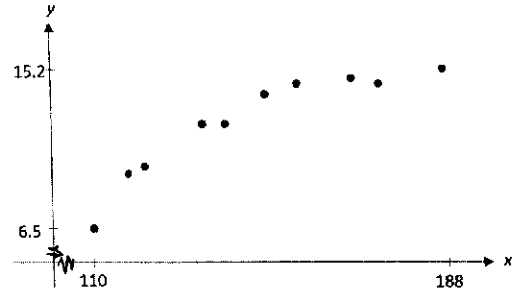
10 The yield per hectare, y kg, of a crop is believed to depend on the average rainfall, x mm, in the month of June. For 10 regions, records are kept of the values of x and y, and these are shown in the table below. The yield from the tenth region was accidentally deleted from the records after the data was analysed, and this is indicated by the value p.

Average rainfall (x mm)	149	110	188	135	156	140	168	118	122	174
Yield of crop (y kg)	13.8	6.5	15.2	12.2	14.4	12.2	14.7	9.5	9.9	p

Given that the equation of the regression line of y on x is $y=-2.5652+0.10168x$, show that $p=14.4$. [2]

Solutions	**Comments**
$\sum x=1460$, $\sum y=108.4+p$, $n=10$ Since $\left(\bar{x},\bar{y}\right)$ lies on the regression line of y on x, $\bar{y}=-2.5652+0.10168\bar{x}$ $\frac{108.4+p}{10}=-2.5652+0.10168\frac{1460}{10}$ $=12.28008$ $p=14.4008$ ≈ 14.4 (shown)	$(174,p)$ does not lie on the regression line, hence you will get an **approximated** value for p when you substitute $x=174$ into the regression line. **"Show"** → you need to give at least the 5 s.f value of $p=14.401$ before concluding that $p=14.4$.

(a) Draw a scatter diagram for these values, labelling the axes clearly. Calculate the product moment correlation coefficient between x and y. [2]

Solutions	**Comments**
 From GC, product moment correlation coefficient, $r=0.92147\approx 0.921$	☐ Label minimum and maximum x and y values. ☐ Check that the 4th and 5th points are level as they have the same y-value. ☐ Make sure you have **10** points sketched in your scatter diagram.

(b) It is thought that a model of the form $y = a + b\ln x$ may also be a suitable fit to the data. Calculate least square estimates of a and b, and find the value of the product moment correlation coefficient between y and $\ln x$. [3]

Solutions	Comments
From GC, the regression line of y on $\ln x$ is $y = -63.028 + 15.154\ln x$ $a = -63.028 \approx -63.0$ $b = 15.154 \approx 15.2$ $r = 0.94660 \approx 0.947$	The rubric of the paper states that non-exact numerical answers should be given correct to 3 significant figures.

(c) [redacted] to explain which of

[redacted]

$$y = -2.5652 + 0.10168x \text{ or } y = a + b\ln x$$

is the better model. [2]

Solutions	Comments
1) From the scatter diagram, it is observed that as x increases, y increases by decreasing amounts, and 2) product moment correlation coefficient between y and $\ln x$ is 0.947, which is closer to 1 than that of x and y, which is 0.921 . Hence $y = a + b\ln x$ is the better model.	☐ Describe how your y values change as x increases as seen in the scatter diagram in (a) ☐ You should compare which of the two r values is **<u>closer to 1</u>**.

(d) Using an appropriate regression line, estimate the yield for a region that experienced 200 mm of rainfall in June. Comment on the reliability of your estimate. [2]

Solutions	Comments
$y = -63.028 + 15.154\ln 200$ $= 17.263$ ≈ 17.3 Since $x = 200$ lies outside the given range of x values, $110 \le x \le 188$, the estimated yield may not be reliable.	Remember to note down the specific range of x i.e. state $110 \le x \le 188$

(e) In some regions, rainfall is measured in inches instead of in mm. Given that there are 25.4 mm in an inch, show how the regression line found in part **(b)** can be re-written so that it can be used when x, the average rainfall in June, is given in inches. [1]

Solutions	Comments
Replace x by $25.4x$, $y = -63.028 + 15.154\ln(25.4x)$ $= -63.028 + 15.154[\ln 25.4 + \ln x]$ $= -14.009 + 15.154\ln x$ $= -14.0 + 15.2\ln x$	All x values (in mm) need to be multiply by $\frac{1}{25.4}$ to convert to inches $\Rightarrow$ i.e. stretch // x-axis by factor $\frac{1}{25.4}$ $\Rightarrow$ replace x by $25.4x$. Answer has to be simplified.

---

11 In the swimming training school AquaV, the time taken to swim a lap of the pool by the trainees is found to have a mean of 35 seconds. The school adopted a new international training programme Breakthru for 3 months and wanted to analyse if Breakthru is effective in improving the timings of the trainees.

A sample of 30 trainees is taken and the times taken, x seconds, to swim a lap of the pool by the trainees are summarised by

$$\sum(x-30) = 94, \quad \sum(x-30)^2 = 758.$$

(a) Test, at the 5% significance level, whether there is any evidence that the [redacted] taken to swim a lap of the pool [redacted] after the trainees underwent 3 months of Breakthru, defining any parameters you use. [7]

[redacted]

Need to define μ

Solutions	Comments
Let μ be the population mean time taken by the trainees to swim a lap. $H_0: \mu = 35$ $H_1: \mu < 35$ Level of significance: 5 % Test Statistic: Since $n = 30$ is large, by Central Limit Theorem, $\overline{X}$ is approximately normally distributed. When H_0 is true, $Z = \dfrac{\overline{X} - 35}{\frac{S}{\sqrt{n}}} \sim N(0,1)$ approximately Computation: $n = 30,\ \bar{x} = \dfrac{\sum(x-30)}{30} + 30 = \dfrac{94}{30} + 30 = 33\dfrac{2}{15} \approx 33.133$ $s^2 = \dfrac{1}{29}\left[\sum(x-30)^2 - \dfrac{\left(\sum(x-30)\right)^2}{30}\right]$ $= \dfrac{1}{29}\left[758 - \dfrac{94^2}{30}\right] = 15.982$ From GC, p-value = 0.00526 Since p-value = $0.00526 < 0.05$, H_0 is rejected at 5% significance level. There is sufficient evidence that the trainees' mean time taken to swim a lap has improved.	When carrying out a hypothesis test, need to write down <u>Step 1:</u> Definition of μ, <u>Step 2:</u> Correct Hypotheses statements <u>Step 3:</u> Test Statistics <u>Step 4:</u> computation of $\bar{x}$, s^2 (because population variance is not known) and p-value. <u>Step 5:</u> Conclusion 1. Compare p-value to sig. level → H_0 rejected (or not rejected) at the level of significance. 2. There is sufficient (or insufficient) evidence that mean time has improved.

BP-859

(b) State an assumption used in carrying out the test. [1]

Solutions	Comments
Assumption: The sample of 30 trainees taken is a random sample. OR Assumption: The sample of 30 trainees taken is such that every trainee has an equal probability of being selected for the sample **and** each trainee is selected independently.	Need a **random** sample to carry out a hypothesis test.

In another swimming training school AquaZ, the time taken to swim a lap of the pool by the trainees is normally distributed with a mean of 38 seconds. AquaZ similarly adopted Breakthru for 3 months and then also carried out a test at the 5% significance level to determine whether there is an improvement in the swimming times. A sample of 30 trainees was taken and their timings were measured. The sample standard deviation was found to be 4 seconds and the mean time was denoted by $\bar{x}$

$\boxed{\text{sample variance} = 4^2}$

(c) Find the set of values of $\bar{x}$ for which the result of the test would be to reject the null hypothesis. [4]

Solutions	Comments
$H_0 : \mu = 38$ $H_1 : \mu < 38$ Level of significance: 5 % Test Statistic: When H_0 is true, $Z = \dfrac{\bar{X} - 38}{\dfrac{S}{\sqrt{n}}} \sim N(0,1)$ approximately Computation: $s^2 = \dfrac{n}{n-1}[\text{sample variance}] = \dfrac{30}{29}\left[4^2\right] = 16.552$ Rejection region: $z \le -1.64485$ Since H_0 is rejected $\Rightarrow z - \text{calculated} \le -1.64485$ $\dfrac{\bar{x} - 38}{\dfrac{\sqrt{16.552}}{\sqrt{30}}} \le -1.64485$ $\bar{x} \le 38 + \dfrac{\sqrt{16.552}}{\sqrt{30}}(-1.64485)$ $\bar{x} \le 36.778$ Set of values of $\bar{x}$: $\{\bar{x} \in \mathbb{R} : \bar{x} \le 36.8\}$	Note the difference between - population variance $\left(\sigma^2\right)$ - unbiased estimate of population variance $\left(s^2\right)$ - sample variance

(d) If the times taken by the 30 trainees is summarised by $\sum(x-30) = 234$, determine the conclusion of the test. [2]

Solutions	Comments
$\bar{x} = \dfrac{234}{30} + 30 = 37.8$ Since $37.8 > 36.778$, from result in **(c)**, H_0 is not rejected at 5% significance level. There is insufficient evidence that the trainees' mean time taken to swim a lap has improved.	