Name:		Index Number:		Class:	

PreliminaryExamination
Year 6

MATHEMATICS (Higher 2) — 9740/01

Paper 1

14 September 2016

3 hours

Additional Materials: Answer Paper
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your Name, Index Number and Class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

For teachers' use:

Qn	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Total
Score												
Max Score	**6**	**6**	**7**	**7**	**8**	**9**	**10**	**10**	**11**	**13**	**13**	**100**

1 (i) Given that $\int f'(x)[f(x)]^n \, dx = \frac{[f(x)]^{n+1}}{n+1} + c$ where c is an arbitrary constant and $n \neq -1$, find $\int x\sqrt{(4-x^2)}\, dx$. [2]

(ii) Hencefind the exact volume of revolution when the region bounded by the curve $y = x^{\frac{3}{2}}(4-x^2)^{\frac{1}{4}}$, the lines $x=0$, $x=2$ and $y=3$, is rotated completely about the x-axis. [4]

2 The complex number w is such that $kw^2 + kww^* + iw - iw^* - 1 = 0$, where w^* is the complex conjugate of w and k is a real and non-zero constant.

(i) For $w = a + bi$ where a and b are real numbers, obtain an expression for b in terms of a and k. Explain why w is either purely real or purely imaginary. [4]

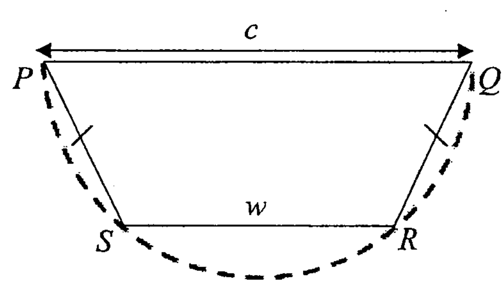
(ii) Using your result in part (i), or otherwise, find the real roots of the equation $2w^2 + 2ww^* + iw - iw^* - 1 = 0$. [2]

3 (i) Without using a calculator, find the exact solution of the inequality

$$4 - x \geq \frac{4}{x+2}.$$ [4]

(ii) Hence solve $5 - |x| \geq \frac{4}{|x|+1}$. [3]

4



To travelalong the River Nile, an adventurer decides to use a log with a semi-circular cross-section of constant diameter $c$metresto build a boat. The log istrimmed such that the uniform cross-section of the boat is an isosceles trapezium with base width$w$metres and $PS = QR$, as shown in the diagram above.

(i) Show that the cross-sectional area of the boat A metres^{2}is given by

$$A = \tfrac{1}{4}(c+w)^{\frac{3}{2}}(c-w)^{\frac{1}{2}}.$$ [2]

(ii) Find the value of w,in terms of c, that gives the stationary value of A. Hence determine whether this stationary value is a maximum or a minimum. [5]

5 Given that $y = \ln(1 + \tan x)$,

(i) show that $\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 + 2(1 - e^y)\dfrac{dy}{dx} = 0$, [3]

(ii) find the Maclaurin series for y up to and including the term in x^3, given that the value of $\dfrac{d^3y}{dx^3}$ when $x = 0$ is 4. [2]

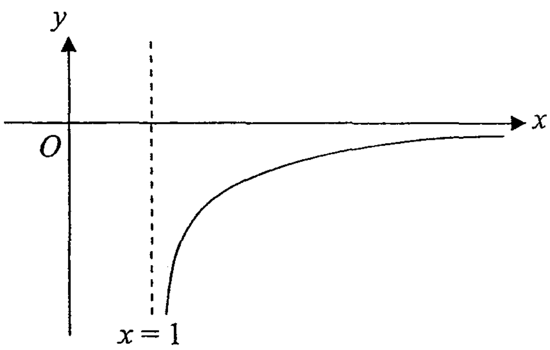
Hence find the first three terms in the series expansion of $\dfrac{\sec^2 x}{1 - \tan x}$. [3]

6 **(a)** Use the substitution $x = e^t$ to find $\displaystyle\int \frac{1}{2e^t + e^{-t}}\, dt$. [3]

(b) **(i)** Express $\dfrac{4 + x}{(1 - x)(4 + x^2)}$ in partial fractions. [2]

(ii) Evaluate $\displaystyle\int_2^n \frac{4 + x}{(1 - x)(4 + x^2)}\, dx$, giving your answer in the form $\frac{1}{2}\ln\left[\dfrac{f(n)}{8(n-1)^2}\right]$, where $f(n)$ is a function of n. [2]

The curve C has equation $y = \dfrac{4 + x}{(1 - x)(4 + x^2)}$. The diagram below shows the part of C for which $x > 1$.



Find the exact value of the area of the region between C and the positive x-axis for $x \geq 2$.

[2]

7 A curve C has parametric equations

$$x=\frac{\theta}{\sqrt{(1-\theta^2)}},\quad y=\sin^{-1}\theta,\quad \text{for } -1<\theta<1.$$

(i) Show that $\frac{dy}{dx}=1-\theta^2$. What can be said about the tangents to C as $\theta\to\pm1$? [4]

(ii) Sketch C, showing clearly its axial intercept and asymptotes. [2]

(iii) Find the equation of the tangent at the point where C has maximum gradient. By considering the intersection between C and an appropriate graph, find the set of positive values of k for which the equation $\sin^{-1}x-\frac{kx}{\sqrt{(1-x^2)}}=0$ has at most one real root. [4]

8 A sequence of real numbers $u_0, u_1, u_2, \ldots$ satisfy the recurrence relation

$$u_n=u_{n-1}+\ln\left(\frac{n}{n+1}\right)$$

for $n\geq1$ and $u_0=2$.

(i) Use the method of mathematical induction to prove that $u_n=2-\ln(n+1)$ for $n\geq0$. [4]

(ii) By considering u_r-u_{r-1}, show how the result for u_n in part (i) can be obtained using the method of differences. [4]

(iii) Show that $\sum_{n=0}^{N}u_n>(N+1)(2-\ln(N+1))$. [2]

9 Joseph started a marathon race. After a while, his trainer, Sarah, starts to collect data on Joseph's speed and she realises that the rate of change of Joseph's speedis proportional to the difference between his speedand a constant a. If the speed of Joseph at time t hours after the start of collection of data is u kilometres per hour, it is found that $\frac{\mathrm{d}u}{\mathrm{d}t}=1$ when $u=14.5$ and $\frac{\mathrm{d}u}{\mathrm{d}t}=2$ when $u=14$.

(i) Show that $\frac{\mathrm{d}u}{\mathrm{d}t}=-2(u-15)$. [3]

(ii) Find the general solution of the equation in part **(i)**,expressing$u$ in terms of t. [3]

(iii) Deduce the steady speed of Joseph eventually. [1]

The distance covered by Joseph, s kilometres, at time t hours after the start of collection of data can be modelled by

$$\frac{\mathrm{d}s}{\mathrm{d}t}=u.$$

(iv) Find$s$ in terms of t. [2]

(v) The result in part **(iv)** can be represented by a family of solution curves. Sketch an appropriate non-linear member of the family of curves that has a linear asymptote that passes through the origin. [2]

10 A curve C has equation $y=\frac{x^2-5}{(x+1)^2-12}$.

(i) Determine the equations of the three asymptotes of$C$, giving each answer in an exact form. [2]

(ii) Prove algebraically that there are no values of x for which $\frac{1}{2}<y<\frac{5}{6}$. [3]

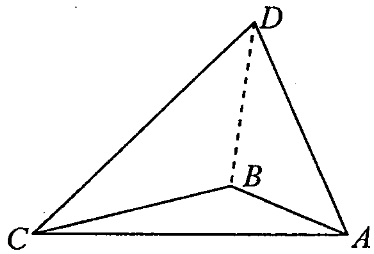
For parts **(iii)** and **(iv)**, you do not need to label the point where the graph cuts the y-axis.

(iii) Sketch C. [3]

(iv) Sketch the graph of $y=\frac{(x+1)^2-12}{x^2-5}$. [3]

(v) Describe a sequence of two transformations which transform C to the graph of $y=\frac{(x-1)^2-5}{(x-2)^2-12}$. [2]

11 The diagram below shows a tetrahedron $ABCD$. The equation of the plane ABD is $4x+y+2z=16$.



(i) Given that A is on the x-axis, find the coordinates of A. [1]

The equation of the plane CBD is $7x-11y-5z=-23$.

(ii) Find a vector equation of the line that passes through B and D. [2]

(iii) Given that B is on the xy-plane, find the coordinates of B. [2]

The cartesian equation of the line that passes through A and D is $\frac{4-x}{2}=\frac{y}{2}=\frac{z}{3}$.

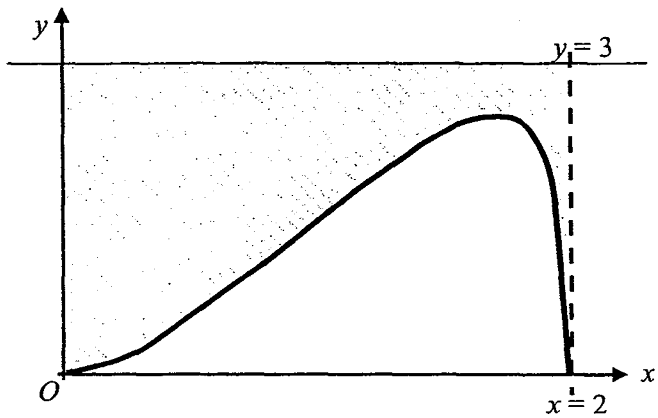
(iv) Find the coordinates of D. [3]

The coordinates of C are $(-1, 1, 1)$.

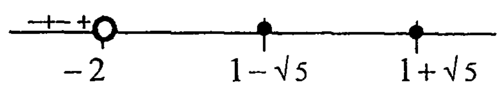
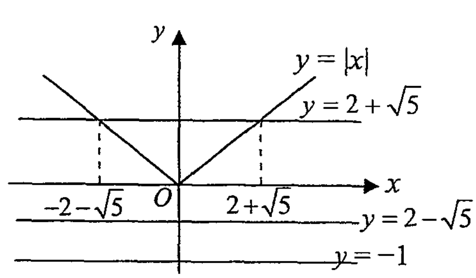
(v) By considering the area of triangle ABC, find the exact volume of the tetrahedron $ABCD$. [5]

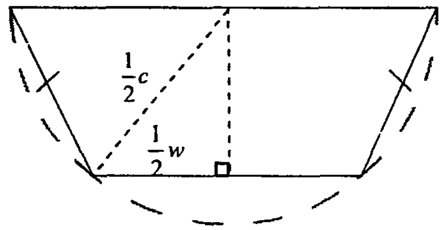
[Volume of tetrahedron = $\frac{1}{3}\times$ area of base $\times$ perpendicular height]

2016Year 6PreliminaryExamination Mark Scheme

Qn	Suggested Solution	Mark Scheme
1(i)	$\int x\sqrt{(4-x^2)}\,\mathrm{d}x = -\frac{1}{2}\int (-2x)\sqrt{(4-x^2)}\,\mathrm{d}x$ $= -\frac{1}{2}\left[\frac{2}{3}(4-x^2)^{\frac{3}{2}}\right]+C$ $= -\frac{1}{3}(4-x^2)^{\frac{3}{2}}+C$	**M1** – attempt to rewrite integrand in the form $\int [\mathrm{f}(x)]^n \mathrm{f}'(x)\mathrm{d}x$ (must see "2x") **A1** SR: Award 2 marks if answer is correct
	 Volume required $= \pi(3^2)(2) - \pi\int_0^2 \left(x^{\frac{3}{2}}(4-x^2)^{\frac{1}{4}}\right)^2 \mathrm{d}x$ $= 18\pi - \pi\int_0^2 x^3(4-x^2)^{\frac{1}{2}}\mathrm{d}x$ $= 18\pi - \pi\int_0^2 x^2(x)(4-x^2)^{\frac{1}{2}}\mathrm{d}x$ $= 18\pi - \pi\left\{\left[-x^2\frac{1}{3}(4-x^2)^{\frac{3}{2}}\right]_0^2 - \int_0^2 -\frac{1}{3}(4-x^2)^{\frac{3}{2}}(2x)\mathrm{d}x\right\}$ $= 18\pi + \frac{\pi}{3}\left[\frac{2}{5}(4-x^2)^{\frac{5}{2}}\right]_0^2$ $= 18\pi + \frac{\pi}{3}\left[-\frac{2}{5}(4)^{\frac{5}{2}}\right]$ $= 18\pi - \frac{64}{15}\pi$ $= \frac{206}{15}\pi$	**B1** – $V_1 =$ $\pi\int_0^2 \left(x^{\frac{3}{2}}(4-x^2)^{\frac{1}{4}}\right)^2 \mathrm{d}x$ **M1** – $\pi(3^2)(2) - V_1$ √**M1** – by parts using $\frac{\mathrm{d}v}{\mathrm{d}x} = x(4-x^2)^{\frac{1}{2}}$ to obtain $v = -\frac{1}{3}(4-x^2)^{\frac{3}{2}}$ from **(ii)** **A1**– exact form only
		Total Marks: 6

Qn	Suggested Solution	Mark Scheme
2(i)	$kw^2 + kww^* + iw - iw^* - 1 = 0$ $kw(w + w^*) + i(w - w^*) - 1 = 0$ $k(a + bi)(2a) + i(2bi) - 1 = 0$ $\left(2ka^2 - 2b\right) + 2abki = 1$ <u>Real part</u> $2ka^2 - 2b = 1 \Rightarrow b = \dfrac{2ka^2 - 1}{2}$ ---(1) <u>Im part</u> $ab = 0 \quad \because k \neq 0$ $\Rightarrow b = 0$ **or** $a = 0$ ie, w is either purely real or imaginary.	**M1**- Simplify with real and Im relations or equivalent **M1** – Obtain cartesian expressions <u>and</u> attempt comparison (real or imaginary) **A1** **AG1** – Obtain $ab = 0$ <u>and show interpretion for *w*</u> Do not accept $a = 0$ **and** $b = 0$ as this is not possible from previous part
(ii)	<u>Hence</u> Since w is real, $b = 0$. Using $k = 2$ and $b = 0$ From part (i): $\dfrac{2(2)a^2 - 1}{2} = 0$ $4a^2 = 1 \Rightarrow a = \pm\sqrt{\dfrac{1}{4}}$ ie, $w = -\dfrac{1}{2}$ or $w = \dfrac{1}{2}$ <u>Otherwise</u> Since w is real, $b = 0$, ie, $w = a$ Using $k = 2$ and $w = a$ eqn becomes: $2a^2 + 2a^2 + ia - ia - 1 = 0$ $4a^2 = 1 \Rightarrow a = \pm\sqrt{\dfrac{1}{4}}$ ie, $w = -\dfrac{1}{2}$ or $w = \dfrac{1}{2}$	**M1** – **"Hence"** to obtain a **A1** <u>ALT</u> **M1** – **"otherwise"** to obtain a **A1**
		Total : 6 marks

Qn	Suggested Solution	Mark Scheme
3 (i)	$4-x \geq \dfrac{4}{x+2}$ $\dfrac{4}{x+2}+x-4 \leq 0$ $\dfrac{4+(x+2)(x-4)}{x+2} \leq 0$ $\dfrac{x^2-2x-4}{x+2} \leq 0$ $\dfrac{(x-1)^2-5}{(x+2)} \leq 0$ $\dfrac{\left(x-[1-\sqrt{5}]\right)\left(x-[1+\sqrt{5}]\right)}{x+2} \leq 0$  $\therefore x<-2$ or $1-\sqrt{5} \leq x \leq 1+\sqrt{5}$	**M1** – Make RHS 0 and combine into single fraction **M1**– Expressed as factorised factors in inequality **A1** – $x<-2$ **A1** – $1-\sqrt{5} \leq x \leq 1+\sqrt{5}$
(ii)	$5-\lvert x\rvert \geq \dfrac{4}{\lvert x\rvert+1}$ $\Rightarrow \quad 4-(\lvert x\rvert-1) \geq \dfrac{4}{(\lvert x\rvert-1)+2}$ Hence, replace x with $(\lvert x\rvert-1)$ in earlier sol: $\therefore \lvert x\rvert-1<-2$ or $1-\sqrt{5} \leq \lvert x\rvert-1 \leq 1+\sqrt{5}$ $\Rightarrow \quad \lvert x\rvert<-1$ or $2-\sqrt{5} \leq \lvert x\rvert \leq 2+\sqrt{5}$ Since $\lvert x\rvert \geq 0$, $\lvert x\rvert<-1$ has no solution Note $2-\sqrt{5}<0$ $\quad \therefore 0 \leq \lvert x\rvert \leq 2+\sqrt{5}$ i.e. $-(2+\sqrt{5}) \leq x \leq 2+\sqrt{5}$  	**M1** – Correct replacement $(\lvert x\rvert-1)$ **B1** – $\lvert x\rvert<-1$ has no solution **A1** – Correct final sol
		Total Marks: 7

Qn	Suggested Solution	Mark Scheme
4(i)	 Height of cross-section of boat = $\sqrt{\frac{1}{4}c^2-\frac{1}{4}w^2}$ $A=\frac{1}{2}\sqrt{\frac{1}{4}c^2-\frac{1}{4}w^2}\,(c+w)$ $=\frac{1}{4}\sqrt{c^2-w^2}\,(c+w)$ $=\frac{1}{4}\sqrt{(c+w)(c-w)}\,(c+w)$ $=\frac{1}{4}(c+w)^{\frac{1}{2}}(c-w)^{\frac{1}{2}}(c+w)$ $=\frac{1}{4}(c+w)^{\frac{3}{2}}(c-w)^{\frac{1}{2}}$ (shown)	**B1** – See $\sqrt{\frac{1}{4}c^2-\frac{1}{4}w^2}$ **AG1** – $A=\frac{1}{4}(c+w)^{\frac{3}{2}}(c-w)^{\frac{1}{2}}$.
(ii)	$\frac{\mathrm{d}A}{\mathrm{d}w}=\frac{1}{4}\left(\frac{3}{2}\right)(c+w)^{\frac{1}{2}}(c-w)^{\frac{1}{2}}-\frac{1}{4}\left(\frac{1}{2}\right)(c+w)^{\frac{3}{2}}(c-w)^{-\frac{1}{2}}$ $=\frac{3(c+w)^{\frac{1}{2}}(c-w)-(c+w)^{\frac{3}{2}}}{8(c-w)^{\frac{1}{2}}}$ $=\frac{(c+w)^{\frac{1}{2}}[3(c-w)-(c+w)]}{8(c-w)^{\frac{1}{2}}}$ $=\frac{(c+w)^{\frac{1}{2}}[c-2w]}{4(c-w)^{\frac{1}{2}}}$ For stationary $A\Rightarrow\frac{\mathrm{d}A}{\mathrm{d}w}=0$ $\frac{1}{8}(c+w)^{\frac{1}{2}}(c-2w)=0$ $w=-c$ (reject as $w>0$) or $w=\frac{1}{2}c$ **<u>Alternative</u>** $A^2=\frac{1}{16}(c+w)^3(c-w)$ $2A\frac{\mathrm{d}A}{\mathrm{d}w}=\frac{1}{16}\left[3(c+w)^2(c-w)-(c+w)^3\right]$ $=\frac{1}{16}(c+w)^2(3c-3w-c-w)$ $=\frac{1}{16}(c+w)^2(2c-4w)$ $A\frac{\mathrm{d}A}{\mathrm{d}w}=\frac{1}{16}(c+w)^2(c-2w)$	**M1** – Correct product rule **A1** –Correct $\frac{\mathrm{d}A}{\mathrm{d}w}$ **M1** – Set $\frac{\mathrm{d}A}{\mathrm{d}w}=0$ and solve for w **A1** – $w=\frac{1}{2}c$, mustreject $w=-c$ **M1** – Correct product rule **A1** –Correct $A\frac{\mathrm{d}A}{\mathrm{d}w}$

<table>
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For stationary $A \Rightarrow \frac{\mathrm{d}A}{\mathrm{d}w} = 0$
$w = -c$ (reject as $w > 0$) or $w = \frac{1}{2}c$
<table>
<tr><td>w</td><td>$\frac{1}{2}c^{-}$</td><td>$\frac{1}{2}c$</td><td>$\frac{1}{2}c^{+}$</td></tr>
<tr><td>$\frac{\mathrm{d}A}{\mathrm{d}w}$</td><td>+ve</td><td>0</td><td>+ve</td></tr>
<tr><td>Tangent</td><td>/</td><td>–</td><td>\</td></tr>
</table>
$\therefore$ When $w = \frac{c}{2}$, A is a maximum
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<td>
M1 – Set $\frac{\mathrm{d}A}{\mathrm{d}w} = 0$ and solve for w
A1 – $w = \frac{1}{2}c$, mustreject $w = -c$
B1 – 1st derivative test correctly shown
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<td>Total Marks: 7</td>
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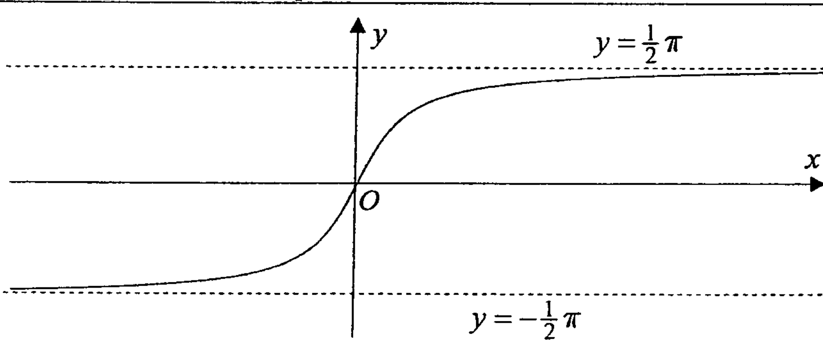
Qn	Suggested Solution	Mark Scheme
5	**Method 1** $y=\ln(1+\tan x)$ $e^y=1+\tan x \quad ------(1)$ Differentiate wrt x, $e^y\frac{dy}{dx}=\sec^2 x \quad ------(2)$ Differentiate wrt x, $e^y\frac{d^2y}{dx^2}+e^y\left(\frac{dy}{dx}\right)^2=2\sec x(\sec x\tan x)$ [from (1), $\tan x=e^y-1$] $e^y\frac{d^2y}{dx^2}+e^y\left(\frac{dy}{dx}\right)^2=2e^y\frac{dy}{dx}(e^y-1)$ [from (2), $\sec^2 x=e^y\frac{dy}{dx}$] $\Rightarrow\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+2\frac{dy}{dx}(1-e^y)=0 \quad (\because e^y\neq 0)$ (shown)	**M1** – $\frac{d}{dx}(1+\tan x)=\sec^2 x$ **M1** – Use product rule and see either $e^y\frac{d^2y}{dx^2}$ or $e^y\left(\frac{dy}{dx}\right)^2$ **AG1** – Show given result.
	Method 2 (Discouraged) $y=\ln(1+\tan x)$ $\frac{dy}{dx}=\frac{\sec^2 x}{1+\tan x}$ $\frac{d^2y}{dx^2}=\frac{(1+\tan x)2\sec^2 x\tan x-(\sec^2 x)(\sec^2 x)}{(1+\tan x)^2}$ $=\frac{2\sec^2 x\tan x}{1+\tan x}-\left(\frac{\sec^2 x}{1+\tan x}\right)^2$ $=2\left(\frac{dy}{dx}\right)\tan x-\left(\frac{dy}{dx}\right)^2$ $\Rightarrow\frac{d^2y}{dx^2}=2\left(\frac{dy}{dx}\right)(e^y-1)-\left(\frac{dy}{dx}\right)^2 \quad [\because e^y=1+\tan x]$ $\Rightarrow\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+2\left(\frac{dy}{dx}\right)(1-e^y)=0$ (shown)	**M1** – $\frac{dy}{dx}=\frac{\sec^2 x}{1+\tan x}$ **M1** – See correct use of quotient rule (or chain rule) **AG1**
(ii)	When $x=0$, $y=\ln(1+\tan 0)=0$ $e^0\frac{dy}{dx}=\sec^2 0\Rightarrow\frac{dy}{dx}=1$ $\frac{d^2y}{dx^2}+(1)^2+2(1)(1-e^0)=0\Rightarrow\frac{d^2y}{dx^2}=-1$ Given that $\frac{d^3y}{dx^3}=4$, $y=x+(-1)\frac{x^2}{2!}+(4)\frac{x^3}{3!}+...=x-\frac{1}{2}x^2+\frac{2}{3}x^3+...$	**M1** – Sub in $x=0$ into y and higher derivatives and first 2 correct **A1** – All terms up to x^3 correct [*do not award if stop at x^2 term*]

	$\frac{d}{dx}\ln(1+\tan x)=\frac{d}{dx}(x-\frac{1}{2}x^2+\frac{2}{3}x^3+...)$ $\Rightarrow \frac{\sec^2 x}{1+\tan x}=1-x+2x^2+...$ Replace x with $(-x)$, $\frac{\sec^2(-x)}{1+\tan(-x)}=1-(-x)+2(-x)^2+...$ $\Rightarrow \frac{\sec^2(x)}{1-\tan(x)}=1+x+2x^2+...$ $(\because \tan(-x)=\tan x$ and $\cos(-x)=\cos(x))$ **<u>Alternative</u>** Replace x with $(-x)$, $\ln(1-\tan x)=\ln(1+\tan(-x))$ $=-x-\frac{1}{2}(-x)^2+\frac{2}{3}(-x)^3+...$ $=-x-\frac{1}{2}x^2-\frac{2}{3}x^3+...$ $\frac{d}{dx}\ln(1-\tan x)=\frac{d}{dx}(-x-\frac{1}{2}x^2-\frac{2}{3}x^3+...)$ $\Rightarrow \frac{-\sec^2 x}{1-\tan x}=-1-x-2x^2+...$ $\Rightarrow \frac{\sec^2 x}{1-\tan x}=1+x+2x^2+...$	**√M1**- Replace with $(-x)$ <u>both sides</u> *Either order* ↕ **√M1** – differentiate wrt x<u>both sides</u> **A1**
		Total Marks: 8

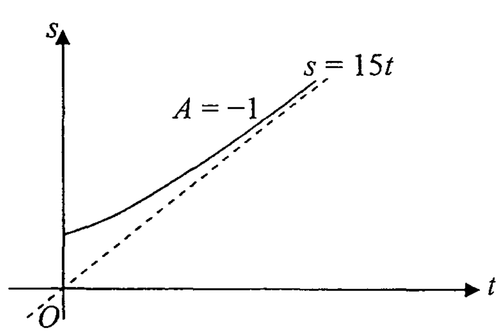
Qn	Suggested Solution		Mark Scheme
6(a)	$\int \frac{1}{2e^t + e^{-t}}\,dt$ $= \int \frac{1}{2x + \frac{1}{x}}\left(\frac{1}{x}\right)dx$ $= \int \frac{1}{2x^2+1}\,dx$ $= \frac{1}{2}\int \frac{1}{x^2 + \left(\frac{1}{\sqrt{2}}\right)^2}\,dx$ $= \frac{\sqrt{2}}{2}\tan^{-1}(\sqrt{2}x) + C$ $= \frac{\sqrt{2}}{2}\tan^{-1}(\sqrt{2}e^t) + C$	$x = e^t$ $\frac{dx}{dt} = e^t$ "$dt = \frac{1}{x}dx$"	**B1** correct substituti **M1** see $\tan^{-1}(\sqrt{2}x)$ **A1** correct expressior terms of t SR: Deduct one mark no arbitrary constant
6(b)	Let $\frac{4+x}{(1-x)(4+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{4+x^2}$. Then $A(4+x^2) + (Bx+C)(1-x) = 4+x$ $x = 1 : A = 1$ $x^2 : A - B = 0 \therefore B = 1$ constant: $4A + C = 4 \therefore C = 0$ $\frac{4+x}{(1-x)(4+x^2)} = \frac{1}{1-x} + \frac{x}{4+x^2}$. $\int_2^n \frac{4+x}{(1-x)(4+x^2)}\,dx$ $= \int_2^n \left(\frac{1}{1-x} + \frac{x}{4+x^2}\right)dx$ $= \left[-\ln\lvert 1-x\rvert + \frac{1}{2}\ln\left(4+x^2\right)\right]_2^n$ $= -\ln\lvert 1-n\rvert + \frac{1}{2}\ln\left(4+n^2\right) - \frac{1}{2}\ln 8$ $= -\frac{1}{2}\ln\lvert 1-n\rvert^2 + \frac{1}{2}\ln\left(4+n^2\right) - \frac{1}{2}\ln 8$ $= \frac{1}{2}\ln\left(\frac{4+n^2}{8(n-1)^2}\right)$		**M1** comparing coefficients or substitu some values of x **A1** $\sqrt{}$**M1** either one correc integration involving l **A1**

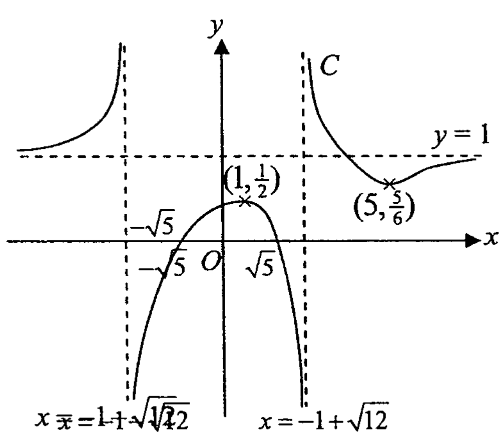
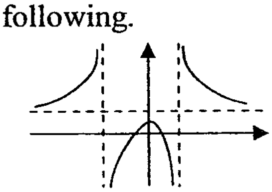
	The required area $= \lim_{n\to\infty}\left[-\frac{1}{2}\ln\left(\frac{4+n^2}{8(n-1)^2}\right)\right]$ $= \lim_{n\to\infty}\left[-\frac{1}{2}\ln\left(\frac{4+n^2}{8(n^2-2n+1)}\right)\right]$ $\lim_{n\to\infty}\left[-\frac{1}{2}\ln\left(\frac{\frac{4}{n^2}+1}{8(1-\frac{2}{n}+\frac{1}{n^2})}\right)\right]$ $=-\frac{1}{2}\ln\left(\frac{1}{8}\right)$ $[\because \text{since } \frac{1}{n} \text{ and} \frac{1}{n^2} \to 0 \text{ as } n\to\infty]$ $=\frac{3}{2}\ln 2$	**M1** apply $n\to\infty$ (condone missing "–") **A1** Do not award if "–" not addressed in working. Simplification not needed.
	Alternative The required area $= \lim_{n\to\infty}\left[-\frac{1}{2}\ln\left(\frac{4+n^2}{8(n-1)^2}\right)\right]$ $= \lim_{n\to\infty}\left[-\frac{1}{2}\ln\left(\frac{\frac{4}{n^2}+1}{8(1-\frac{1}{n})^2}\right)\right]$ $=-\frac{1}{2}\ln\left(\frac{1}{8}\right)$ $[\because \text{since } \frac{1}{n} \text{ and} \frac{4}{n^2} \to 0 \text{ as } n\to\infty]$ $=\frac{3}{2}\ln 2$	**M1** apply $n\to\infty$ (condone missing "–") **A1** Do not award if "–" not addressed in working. Simplification not needed.
		Total Marks: 9

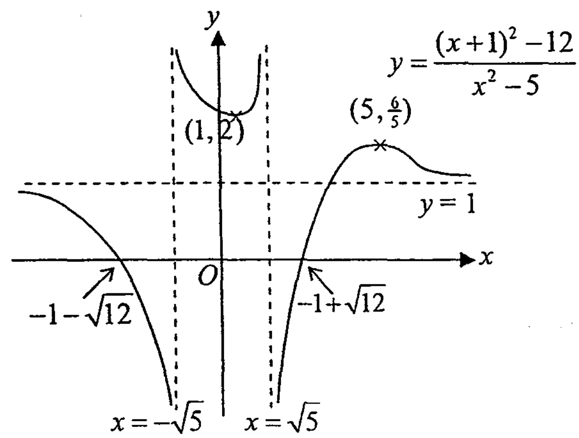
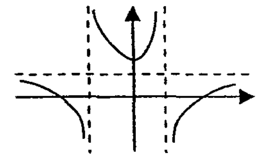
Qn	Suggested Solution	Mark Scheme
7(i)	$x=\dfrac{\theta}{\sqrt{1-\theta^2}}$ $\Rightarrow \dfrac{\mathrm{d}x}{\mathrm{d}\theta}=\dfrac{(1-\theta^2)^{\frac{1}{2}}-\theta(\frac{1}{2})(-2\theta)(1-\theta^2)^{-\frac{1}{2}}}{(1-\theta^2)}$ $=\dfrac{(1-\theta^2)^{-\frac{1}{2}}\left[(1-\theta^2)+\theta^2\right]}{(1-\theta^2)}$ $=(1-\theta^2)^{-\frac{3}{2}}$ $y=\sin^{-1}\theta$ $\Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}\theta}=\dfrac{1}{\sqrt{1-\theta^2}}=(1-\theta^2)^{-\frac{1}{2}}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{(1-\theta^2)^{-\frac{1}{2}}}{(1-\theta^2)^{-\frac{3}{2}}}$ $=1-\theta^2$ (shown) As $\theta\to\pm1$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}\to 0$. The tangents becomes parallel to the x-axis as $\theta\to\pm1$.	**B1** – $\dfrac{\mathrm{d}x}{\mathrm{d}\theta}$ correct. **M1** – $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{d}y}{\mathrm{d}\theta}\Big/\dfrac{\mathrm{d}x}{\mathrm{d}\theta}$ with attempts to find $\dfrac{\mathrm{d}y}{\mathrm{d}\theta}$ and $\dfrac{\mathrm{d}x}{\mathrm{d}\theta}$ **AG1** **B1**
(ii)	 Note: as $\theta\to\pm1$, $x\to\pm\infty$ and $y\to\pm\frac{1}{2}\pi$.	**G1** – Shape of graph (pass through origin) **G1** – Both asymptotes correct.
(iii)	Since $\theta^2\geq 0$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=1-\theta^2$ is maximum at $\theta=0$. At $\theta=0$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=1$, $x=y=0$. Equation of tangent at (0, 0): $y=x$. Number of real roots of the equation $\sin^{-1}x-\dfrac{kx}{\sqrt{(1-x^2)}}=0$ is equal to the number of intersection points between the line $y=kx$ and the curve C. Hence, required set of positive constants $k=\{k\in\square : k\geq 1\}$	**B1** – Correct maximum gradient of 1 at $\theta=0$. **B1** – Correct equation of tangent. **B1** – $y=kx$ **B1** – Correct range of values of k in set notation.
		Total Marks: 10

Qn	Suggested Solution	Mark Scheme
8(i)	Let P(n)be the proposition $u_n = 2-\ln(n+1)$ for $n \ge 0$. When $n=0$, LHS of P(0) $= u_0 = 2$ (given) RHS of P(0) $= 2-\ln(0+1) = 2 =$ LHS of P(0) $\therefore$ P(0) is true. Assuming that P(k) is true for some $k \ge 0$ i.e. $u_k = 2-\ln(k+1)$, To show that P(k + 1) is true i.e. $u_{k+1} = 2-\ln(k+2)$. LHS of P($k+1$) $= u_{k+1}$ $= u_k + \ln\left(\frac{k+1}{k+2}\right)$ $= 2-\ln(k+1)+\ln\left(\frac{k+1}{k+2}\right)$ $= 2-\ln(k+1)+\ln(k+1)-\ln(k+2)$ $= 2-\ln(k+2)$ $=$ RHS $\therefore$ P(k) is true $\Rightarrow$ P($k+1$) is true. Since P(0) is true, and P(k) is true $\Rightarrow$ P($k+1$) is true, hence by mathematical induction, P(n) is true for all $n \ge 0$.	**B1** – prove that P_0 is true. **B1** – use recurrence relation and assumption **B1** – show RHS **B1** – correct conclusion statement.
(ii)	Consider $u_r - u_{r-1} = \ln\frac{r}{r+1} = \ln(r)-\ln(r+1)$ $\sum_{r=1}^{n}(u_r - u_{r-1}) = \sum_{r=1}^{n}(\ln(r)-\ln(r+1))$ $[u_1 - u_0 = [\ln(1)-\ln(2)$ $+ u_2 - u_1 \quad +\ln(2)-\ln(3)$ $+ u_3 - u_2 \quad +\ln(3)-\ln(4)$ $\vdots \quad \vdots$ $+ u_{n-1} - u_{n-2} \quad +\ln(n-1)-\ln(n)$ $+ u_n - u_{n-1}] \quad +\ln(n)-\ln(n+1)]$ $u_n - u_0 = \ln(1)-\ln(n+1)$ $u_n = 2-\ln(n+1)$	**B1** **M1** – MOD LHS listed correctly with at least first 2 and last 1 cancellation **M1** – MOD RHS listed correctly with at least first 2 and last 1 cancellationor equivalent method **SR:** Deduct at most 1 mark if insufficient cancellation **AG1**

	Alternative for RHS $\sum_{r=1}^{n}\left(\ln\frac{r}{r+1}\right)$ $=\ln\frac{1}{2}+\ln\frac{2}{3}+\ln\frac{3}{4}+\ldots+\ln\frac{n}{n+1}$ $=\ln\left(\frac{1}{2}\frac{2}{3}\frac{3}{4}\ldots\frac{n}{n+1}\right)$ $=\ln\frac{1}{n+1}$ $=-\ln(n+1)$	
(iv)	Since $n<N$, $2-\ln(n+1)>2-\ln(N+1)$ $\sum_{n=0}^{N}[2-\ln(n+1)]>\sum_{n=0}^{N}[2-\ln(N+1)]$ $\sum_{n=0}^{N}u_n>[2-\ln(N+1)]\sum_{n=0}^{N}1$ $\sum_{n=0}^{N}u_n>(N+1)[2-\ln(N+1)]$	**M1** – Either see "$2-\ln(n+1)>2-\ln($ " Or "$[2-\ln(N+1)]\sum_{n=0}^{N}1$" **AG1** – apply inequalit concepts to get to the s result.
		Total mark

Qn	Suggested Solution	Mark Scheme
9i)	$\frac{du}{dt}=k(u-a)$, where k is a constant. Given $\frac{du}{dt}=1$ when $u=14.5$ and $\frac{du}{dt}=2$ when $u=14$, $1=k(14.5-a)$ -----(1) $2=k(14-a)$ ------(2) From GC, $k=-2$ and $ak=-30$ $\therefore a=15$ $\therefore \frac{du}{dt}=-2(u-15)$ (shown)	**B1** – correct formulation. Condone if did not state k is a constant. **M1** – solve simultaneously using GC or equivalent methods **AG1** – correct result from GC (see $ak=-30$)
(ii)	$\int \frac{1}{u-15}\,du=-2\int dt$ $\ln\|u-15\|=-2t+C$ $\|u-15\|=e^{-2t+C}$ $u=15+Ae^{-2t}$ where $A=\pm e^{C}$	**M1** – Separate variables **M1** – Correct integration, condone without modulus sign and arbitrary constant **A1** – Correct removal of modulus sign and express u in terms of t.
(iii)	As $t\to\infty$, $e^{-2t}\to 0$, $u\to 15$ Joseph will eventually reach a steady speed of 15 km/h.	**B1**– award correct conclusion in context of question.
(iv)	$s=\int u\,dt$ $=\int(15+Ae^{-2t})\,dt$ $=15t-\frac{A}{2}e^{-2t}+D$	**√M1** – Integrate expression for u in part **(ii)** with respect to t. **A1**
(v)	For graph to tends towards an asymptote that passes through the origin, $D=0$. I.e. $s=15t-\frac{A}{2}e^{-2t}$ For $A=-1$, $s=15t+\frac{1}{2}e^{-2t}$ 	**B1** – $D=0$ or asymptote is $y=15t$ **G1** – correct graph (negative A values only since $s>0$) with linear asymptote. Condone wrong equation of linear asymptote since mark awarded earlier. Do not award for wrong labelling of axis.
		Total marks : 11

Qn	Suggested Solution	Mark Scheme
10(i)	$(x+1)^2-12=0 \Rightarrow x=-1\pm\sqrt{12}$ Asymptotes are: $y=1,\ x=-1-\sqrt{12},\ x=-1+\sqrt{12}$	**B1** – HA **B1** – Both VA
(ii)	$y=\dfrac{x^2-5}{(x+1)^2-12}=\dfrac{x^2-5}{x^2+2x-11}$ $y(x^2+2x-11)=x^2-5$ $(y-1)x^2+2yx+5-11y=0$ For no values of x, there are no real solutions for the above quadratic equation. Discriminant $=4y^2+4(y-1)(11y-5)<0$ $12y^2-16y+5<0$ $(6y-5)(2y-1)<0$ $\therefore \frac{1}{2}<y<\frac{5}{6}$ (shown)	**M1** – form quadratic equation **M1** – consider discriminant. Condone wrong sign for discriminant. **AG1** - discriminant < 0 and solve inequalities **SR:** Do not accept if students find max, min point by differentiation.
(iii)	At turning points of C, When $y=\frac{1}{2}$, $x=1$; When $y=\frac{5}{6}$, $x=5$. Coordinates of turning points $(1,\frac{1}{2})$ and $(5,\frac{5}{6})$. 	**B1** – 3 sections of graph with one HA and 2 VA. Based on GC, students should at least get the following.  **B1** –correct turning points **B1**–correct shape, x-intercepts and asymptotes

(iv)		**B1** – 3 sections of graph with one HA and 2 VA. Based on GC, students should at least get the following.  √**B1** – evidence of VA ↔ *x*-intercepts OR $(a, b) \leftrightarrow (a, 1/b)$ **B1** – shape correct and all details correct.
(v)	$y=\dfrac{(1-x)^2-5}{((1-x)+1)^2-12}=\dfrac{(x-1)^2-5}{(x-2)^2-12}.$ Therefore *x* is replaced with 1 – *x*. $y=\mathrm{f}(x)\to y=\mathrm{f}(x+1)\to y=\mathrm{f}((-x)+1)=\mathrm{f}(1-x)$ Sequence of transformation: *C* is translated by 1 unit in the negative *x*-direction and then reflected in the *y*-axis OR $y=\mathrm{f}(x)\to y=\mathrm{f}(-x)\to y=\mathrm{f}(-(x-1))=\mathrm{f}(1-x)$ *C* is reflected in the *y*-axis and then translated 1 unit in the positive *x*-direction.	**B1** – identifycorrect replacement **B1** – both transformation correct according to the respective order
		Total : 13 marks

Qn	Suggested Solution	Mark Scheme
		SR: Deduct Max 1 mark if answer not in coordinate form.
11(i)	Plane *ABD*: $4x+y+2z=16$ When *A* is on the *x*-axis, $y=z=0$. $4x=16 \Rightarrow x=4$ $A(4, 0, 0)$	**B1**
(ii)	Plane *CBD*: $7x-11y-5z=-23$ Line *BD* is the line of intersection between planes *ABD* and *CBD*. From GC, $\mathbf{r}=\begin{pmatrix}3\\4\\0\end{pmatrix}+\lambda'\begin{pmatrix}-\frac{1}{3}\\-\frac{2}{3}\\1\end{pmatrix}$ $l_{BD}:\mathbf{r}=\begin{pmatrix}3\\4\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\-3\end{pmatrix}, \lambda\in\mathbb{R}$	**M1** – findline of intersection of planes *ABD* and *CBD*. **A1** – any equivalent form
(iii)	Equation of *xy*-plane : $\mathbf{r}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}=0\Rightarrow z=0$ Using $\overline{OB}=\begin{pmatrix}3\\4\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\-3\end{pmatrix}\Rightarrow 3\lambda=0\Rightarrow\lambda=0$ $\therefore\overline{OB}=\begin{pmatrix}3\\4\\0\end{pmatrix}$ $B(3, 4, 0)$ **Alternative** *B* is the point of intersection between planes *ABD*, *CBD* and *xy*-plane $4x+y+2z=16$ $7x-11y-5z=-23$ $z=0$ Using GC, $B(3, 4, 0)$	**B1** – $z=0$ **B1** **B1** – $z=0$ **B1**
(iv)	$l_{AD}:\mathbf{r}=\begin{pmatrix}4\\0\\0\end{pmatrix}+\mu\begin{pmatrix}-2\\2\\3\end{pmatrix}$ *D* is the point of intersection between l_{AD} and plane *CBD* $\left[\begin{pmatrix}4\\0\\0\end{pmatrix}+\mu\begin{pmatrix}-2\\2\\3\end{pmatrix}\right]\cdot\begin{pmatrix}7\\-11\\-5\end{pmatrix}=-23$ $28+\mu(-51)=-23$ $\mu=1$	**B1** – Correct vector equation of l_{AD} **M1** – Solve simultaneously

	$\overrightarrow{OD}=\begin{pmatrix}4\\0\\0\end{pmatrix}+\begin{pmatrix}-2\\2\\3\end{pmatrix}=\begin{pmatrix}2\\2\\3\end{pmatrix}$ $D(2, 2, 3)$ **Alternative** D is the point of intersection between l_{AD} and l_{BD}. $\begin{pmatrix}4\\0\\0\end{pmatrix}+\mu\begin{pmatrix}-2\\2\\3\end{pmatrix}=\begin{pmatrix}3\\4\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\-3\end{pmatrix}$ $\lambda+2\mu=1$ $2\lambda-2\mu=-4$ $-3\lambda-3\mu=0\Rightarrow\lambda=-\mu$ $\therefore \mu=1$	**A1** **B1** – Correct vector equation of l_{AD} **M1** – solvesimultaneously
(v)	$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}-1\\4\\0\end{pmatrix}$ $\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=\begin{pmatrix}-5\\1\\1\end{pmatrix}$ Area of triangle ABC = $\frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|=\frac{1}{2}\left\|\begin{pmatrix}-1\\4\\0\end{pmatrix}\times\begin{pmatrix}-5\\1\\1\end{pmatrix}\right\|=\frac{1}{2}\left\|\begin{pmatrix}4\\1\\19\end{pmatrix}\right\|$ Distance from D to plane ABC $=\frac{\left\|\overrightarrow{AD}\cdot\begin{pmatrix}4\\1\\19\end{pmatrix}\right\|}{\left\|\begin{pmatrix}4\\1\\19\end{pmatrix}\right\|}=\frac{\left\|\begin{pmatrix}-2\\2\\3\end{pmatrix}\cdot\begin{pmatrix}4\\1\\19\end{pmatrix}\right\|}{\left\|\begin{pmatrix}4\\1\\19\end{pmatrix}\right\|}=\frac{51}{\left\|\begin{pmatrix}4\\1\\19\end{pmatrix}\right\|}$ **Alternative** $\therefore$ Plane $ABC: \mathbf{r}\cdot\begin{pmatrix}4\\1\\19\end{pmatrix}=16$ Distance from D to plane ABC $=\frac{\left\|\begin{pmatrix}2\\2\\3\end{pmatrix}\cdot\begin{pmatrix}4\\1\\19\end{pmatrix}-16\right\|}{\left\|\begin{pmatrix}4\\1\\19\end{pmatrix}\right\|}=\frac{\|67-16\|}{\left\|\begin{pmatrix}4\\1\\19\end{pmatrix}\right\|}=\frac{51}{\left\|\begin{pmatrix}4\\1\\19\end{pmatrix}\right\|}$	√**M1 – find two vectors**rep any two sides of triangle ABC **M1** – area of triangle $\frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|$ or equivalent **B1** – $\begin{pmatrix}4\\1\\19\end{pmatrix}$ **M1** – shortest distance $=\left\|\overrightarrow{AD}\cdot\hat{\mathbf{n}}\right\|$ **M1** – shortest distance $=\frac{\left\|\overrightarrow{OD}\cdot\mathbf{n}-\overrightarrow{OA}.\mathbf{n}\right\|}{\|\mathbf{n}\|}$

	Volume of tetrahedron $OABC = \frac{1}{3} \times \frac{1}{2} \left\| \begin{pmatrix} 4 \\ 1 \\ 19 \end{pmatrix} \right\| \frac{51}{\left\| \begin{pmatrix} 4 \\ 1 \\ 19 \end{pmatrix} \right\|} = \frac{51}{6}$ units^3	**A1**
		Total : 13 marks

Name:		Index Number:		Class:	

Preliminary Examination
Year 6

MATHEMATICS (Higher 2) 9740/02

Paper 2 **26September 2016**

3 hours

Additional Materials: Answer Paper
Graph paper
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your Name, Index Number and Class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

For teachers' use:

Qn	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Total
Score												
Max Score	8	9	11	12	3	6	7	10	10	12	12	100

Section A: Pure Mathematics [40 marks]

1 The function f is defined by

$$\mathrm{f}: x \mapsto \pi \sin\left(\tfrac{1}{2}x\right), \quad x \in \mathbb{R}, \ 0 \le x \le a,$$

where a is a positive constant.

(i) State the largest exact value of a for which the function f^{-1} exists. [1]

For the rest of the question, use the value of a found in part **(i)**.

(ii) Write down the equation of the line in which the graph of $y = \mathrm{f}(x)$ must be reflected in order to obtain the graph of $y = \mathrm{f}^{-1}(x)$ and hence verify that 0 and π are solutions to the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. [2]

(iii) Find the area of the region bounded by the graphs of f and f^{-1}, giving your answer in terms of π. [3]

The function g is defined by

$$\mathrm{g}: x \mapsto |x - 1|, \quad x \in \mathbb{R}.$$

(iv) Find the exact range of the composite function gf. [2]

2 The angle between two unit vectors **a** and **b** is $\cos^{-1}\frac{1}{4}$. Relative to the origin O, the position vector of a point P on a line l is given by $\overrightarrow{OP} = \mathbf{a} + \lambda(\mathbf{a} + 2\mathbf{b}),\ \lambda \in \mathbb{R}$ and the point C has position vector $\mathbf{a} - \mathbf{b}$.

(i) By considering scalar product,show that $CP^2 = 6\lambda^2 + \frac{9}{2}\lambda + 1$. [4]

(ii) Deduce the exact shortest distance of C to l and write down the position vector of the point F, the foot of the perpendicular from C to l,in terms of **a** and **b**. [3]

(iii) Find the equation of the plane that contains l and is perpendicular to CF in the form $\mathbf{r}.\mathbf{n} = d$ where $\mathbf{n}$ is expressed in terms of **a** and **b** and d is a constant. [2]

3 The number of bacteria (in millions) in Pond A at the start of the nth week, before any chemical treatment, is given by u_n. Pond A is treated at the start of each week with Chemical A, which kills 70% of all bacteria instantly. At the end of each week, 6 million new bacteria is reproduced.

(i) Write down a recurrence relation of the form $u_{n+1} = au_n + b$, where a and b are constants to be determined. [1]

(ii) Show that $u_n = 0.3^{n-1}u_1 + \frac{60}{7}(1 - 0.3^{n-1})$. [2]

The number of bacteria (in millions) in Pond B at the start of the nth week, before any chemical treatment, is given by v_n. Pond B is treated at the start of each week with Chemical B. It is known that v_n follows the recurrence relation

$$v_{n+1} = 0.01v_n^{\ 2} + 6.$$

It is given that if the sequence $v_1, v_2, v_3, \ldots$ converges to a limit, it converges to either α or β, where $\alpha < \beta$.

(iii) Find α and β. Explain whether v_n necessarily converges to α or β. [3]

(iv) If $u_1 = v_1 = 30$, determine which chemical would be more effective in killing the bacteria in the long run. [2]

Pond C is treated with Chemical C. To account for the bacteria's increasing resistance to the chemical, the dosage of Chemical C is increased by 5 ml each week. The first dose is 20 ml.

(v) How many weeks does it take to finish the first 3 litres of chemical in the treatment of Pond C? [3]

4 **Do not use a graphic calculator in answering this question.**

(a) On a single Argand diagram, sketch the locus of z satisfying both inequalities $|z + 1 - 2\text{i}| \le 2$ and $\frac{1}{4}\pi \le \arg(z+1) \le \frac{1}{2}\pi$. Hence find the range of $\arg(z)$. [5]

(b) Solve the equation

$$w^6 = 64,$$

giving the roots in the form $re^{\text{i}\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. [2]

(i) Hence write down the roots of the equation $(z - 1 - \text{i}\sqrt{3})^6 = 64$ in the form $a + re^{\text{i}\theta}$, where a is a complex number in cartesian form, $r > 0$ and $-\pi < \theta \le \pi$. Show the roots on an Argand diagram. [3]

(ii) Of the roots found in part **(b)(i)**, find in cartesian form the root with the largest modulus. [2]

Section B: Statistics [60 marks]

5 The Land Transport Authority (LTA) wishes to gather feedback on the quality of train s
a new train station.

(i) The LTA decides to station a team of surveyors at the gantries to survey the commuters passing through the train station. State, with a reason, whether th(described is quota sampling.

(ii) The LTA decides to obtain a random sample instead to survey 5% of the commu particular day. Describe how a systematic sample can be carried out in this context.

6 John plays for his school's soccer team. There is a probability of 0.15 that he scores ii and a probability of 0.3 that his parents are present at a game. When he scores in a game a probability of 0.2 that his parents are present.

(i) Show that the probability that he scores in a game when his parents are present is 0.1

(ii) State, with justification, whether his parents' presence at a game will affect his ch scoring in the game.

Games are equally likely to be home or away games. In a home game, there is a proba 0.24 that John does not score and his parents are present.

(iii) Find the least and greatest values of the probability that a game is a home game parents are not present at the game.

7 A committee decides to meet on four days in a span of four weeks. Find the probability committee meets on two Tuesdays and two Saturdays if

(i) committee meetings are equally likely to be held on any day in the four weeks,

(ii) committee meetings are held once a week. The probability of holding a meeting on from Monday to Friday is $\frac{1}{9}$ and the probability of holding a meeting on either Satu Sunday is $\frac{2}{9}$.

The committee of ten sits in a circle at a meeting.

(iii) Find the probability that the two committee vice-heads are seated together and they seated next to the committee head.

8 A research is being conducted to study the growth of car population over time. The data below shows the population of the car, y millions after x years of study from the start of the research:

Years (x)	5	7	9	14	18	23	27
Car Population (y millions)	7.2	10.5	11.6	13.0	14.5	15.5	15.7

(i) Draw a scatter diagram for the data, labelling the axes. [1]

(ii) State, with a reason, which of the following models is appropriate:

A: $y = a + bx^2$, **B**: $y = a + b\ln x$,

where$b$ is positive. [2]

Based on the appropriate model chosen in part **(ii)**,

(iii) calculate the value of the product moment correlation coefficient.State, with a reason, whether this valuewould be different if y is recorded in thousands instead. [2]

(iv) calculate the least square estimates of a and b and write down the corresponding regression line. Obtain the value of the car population after 20 years of study. [3]

(v) give an interpretation of the value of $a$in the context of the question. Comment on the reliability of the value of a. [2]

9 In this question you should state clearly all distributions that you use, together with the values of the appropriate parameters.

(a) The queuing time, in minutes, for flight passengers at the Economy and Business class check-in counters have independent normal distributions with means and standard deviations as shown in the table.

Check-in Counter	Mean queuing time	Standard deviation
Economy class	11.6	4.2
Business class	3.2	0.9

(i) Find the probability that the queueing time of a randomly chosen Economy class passenger is within 5 minutes of the total queueing time of 2 randomly chosen Business class passengers. [4]

(ii) The queueing time of 8 randomly chosen Business class passengers are taken. Find the probability that the shortest queuing time among all 8 passengers is no less than 2 minutes. [2]

(b) The probability that a passenger books a flight and does not turn up is 0.05. The airline decides to allow for over-booking by selling more tickets than the number of seats available.

For a particular flight with 350 available seats, n tickets were sold, where $n > 350$. By using a suitable approximation, show that if the flight is to have no more than 1% chance of having insufficient seats, the number of tickets sold must satisfy the approximate inequality

$$350.5 - 0.95n \geq 2.3263\sqrt{(0.0475n)}.$$ [4]

10 A manufacturer claims that ropes with a certain diameter produced by his factory have mean breaking strength of at least 169.7 kN. Recently, a new material is used to produce the ropes. A random sample of 8 coils of the rope made with the new material is taken and the breaking strength of each coil of rope, x kN, is measured as follows.

171.3	168.5	166.5	164.4	170.0	165.1	170.1	167.2

(i) Find the unbiased estimates of the population mean and variance. [2]

(ii) Stating a necessary assumption, test at the 5% significance level whether the manufacturer's claim is valid after the change in material. [5]

Instead of using the new material, the manufacturer decides to change the weaving process of the ropes. The manufacturer claims that the mean breaking strength is now μ_0 kN. The population variance is found to be 29.16 $(\text{kN})^2$. A random sample of 50 coils of the rope made using the new process is taken and the mean breaking strength, $\bar{y}$ kN, is found to be 171 kN.

(iii) Find the set of values of μ_0 for which the mean breaking strength does not differ from the claim when tested at the 1% significance level. [4]

(iv) Explain, in the context of the question, the meaning of 'at the 1% significance level'. [1]

11 (a) A restaurant has 15 tables consisting of 6 rectangular tables and 9 round tables. During the restaurant's opening hours, the rectangular tables are occupied, on average 80 percent of the time, and the round tables are occupied, on average 65 percent of the time. You may assume that the tables in the restaurant are occupied independently of each other.

(i) If a customer walks into the restaurant at a randomly selected time, what is the probability that 4 rectangular tables and 7 round tables are occupied? [2]

(ii) Give a reason in context why the assumption made above may not be valid. [1]

(b) A café sells both coffee and tea. The number of cups of coffee and tea sold in a randomly chosen 20-minute period have independent Poisson distributions with means 5 and 3.5 respectively.

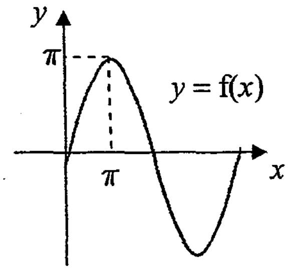
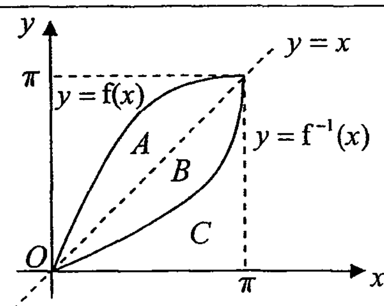
(i) In a particular 20-minute period, at least 7 cups of beverages are sold. Find the probability that at least6 cups of tea are sold during the 20-minute period. [4]

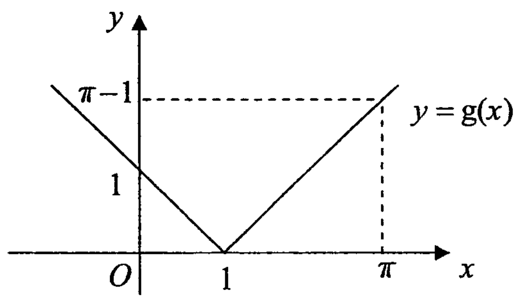
(ii) Let p_k denote the probability that k cups of coffee are sold in a 20-minute period.

Show that $\dfrac{p_{k+1}}{p_k} = \dfrac{5}{k+1}$ and deduce that $p_{k+1} > p_k$, when $k<4$. [3]

Hence find the most probable number(s) of cups of coffee sold in a 20-minute period.

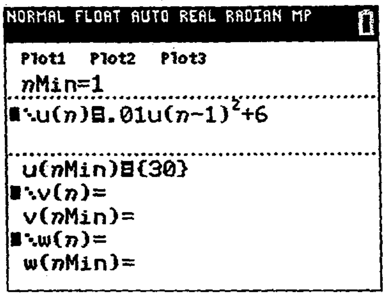
2016 Year 6 H2 Math Prelim Exam Paper 2 Mark Scheme

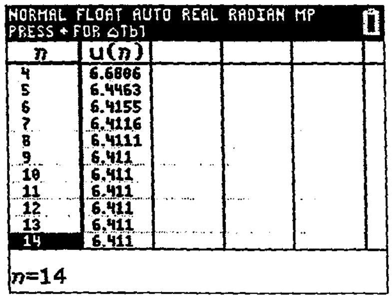
Qn	Suggested Solution	Mark Scheme
1(i)	Largest $a = \pi$  	**B1**
(ii)	The line is $y = x$. $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ Since the points of intersection lies on $y = x$, $\mathrm{f}(x) = x$ $\pi\sin\left(\frac{1}{2}x\right) = x$ When $x = 0$: LHS $= \pi\sin(0) = 0 =$ RHS When $x = \pi$: LHS $= \pi\sin\left(\frac{1}{2}\pi\right) = \pi =$ RHS $\therefore$ 0 and π are solutions to the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$	**B1** **AG1** – Form $\pi\sin\left(\frac{1}{2}x\right) = x$ & check LHS = RHS for both solutions
(iii)	 Required area $= A + B = 2A$ (by symmetry) Area of $B + C = \frac{1}{2}\pi^2$ (area of triangle) Area of $A + B + C = \int_0^{\pi} \mathrm{f}(x)\ \mathrm{d}x$ $= \int_0^{\pi} \pi\sin\left(\frac{1}{2}x\right) \mathrm{d}x$ $= \left[-2\pi\cos\left(\frac{1}{2}x\right)\right]_0^{\pi}$ $= 2\pi$ $\therefore$ Area bounded by the graphs of f and f^{-1} $= 2\left(2\pi - \frac{1}{2}\pi^2\right) = 4\pi - \pi^2$	**B1** – Correctly identify area required, i.e $A + B$, or use symmetry property **M1** – Attempt to find $\int_0^{\pi} \pi\sin\left(\frac{1}{2}x\right) \mathrm{d}x$, condone either missing "–" or "2" but not both **A1** – In terms of π

	Alternative Required area $= 2\int_0^{\pi} f(x) - x \, dx$ $= 2\int_0^{\pi} \pi\sin\left(\frac{1}{2}x\right) - x \, dx$ $= 2\left[-2\pi\cos\left(\frac{1}{2}x\right) - \frac{x^2}{2}\right]_0^{\pi}$ $= 2\left(2\pi - \frac{\pi^2}{2}\right) = 4\pi - \pi^2$	**B1**– $2\int_0^{\pi} f(x) - x \, dx$ **M1** – Attempt to find $\int_0^{\pi} \pi\sin\left(\frac{1}{2}x\right) dx$, condone either missing "–" or "2" but not both **A1** – In terms of π
(iv)	 $[0,\pi] \xrightarrow{f} [0,\pi] \xrightarrow{g} [0,\pi-1]$ $\therefore R_{gf} = [0,\pi-1]$	**M1** – 2 stage mapping or sketch graph of gf. **A1** – Exact answer Also accept $[0,\|\pi-1\|]$
		Total marks : 8

Qn	Suggested Solution	Mark Scheme
2(i)	$\overline{CP} = \mathbf{a} + \lambda(\mathbf{a}+2\mathbf{b}) - (\mathbf{a}-\mathbf{b}) = \mathbf{b} + \lambda(\mathbf{a}+2\mathbf{b})$ $CP^2 = [\mathbf{b}+\lambda(\mathbf{a}+2\mathbf{b})].[\mathbf{b}+\lambda(\mathbf{a}+2\mathbf{b})]$ $= \mathbf{b}.\mathbf{b} + \lambda^2(\mathbf{a}+2\mathbf{b}).(\mathbf{a}+2\mathbf{b}) + 2\lambda\mathbf{b}.(\mathbf{a}+2\mathbf{b})$ $= \mathbf{b}.\mathbf{b} + \lambda^2(\mathbf{a}.\mathbf{a}+4\mathbf{a}.\mathbf{b}+4\mathbf{b}.\mathbf{b}) + 2\lambda(\mathbf{b}.\mathbf{a}+2\mathbf{b}.\mathbf{b})$ $= 1+\lambda^2(1+1+4)+2\lambda\left(\frac{1}{4}+2\right)$ as $\mathbf{a}.\mathbf{a} = \mathbf{b}.\mathbf{b} = 1$ and $\mathbf{a}.\mathbf{b} = \frac{1}{4}$ $= 6\lambda^2 + \frac{9}{2}\lambda + 1$ (shown)	**M1** – $\overline{CP}\cdot\overline{CP}$ **M1** – distributive law **M1** – use of $\mathbf{a}.\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$ **AG1**
(ii)	$CP^2 = 6\left[\lambda^2 + \frac{3}{4}\lambda\right] + 1$ $= 6\left(\lambda+\frac{3}{8}\right)^2 + 1 - \frac{54}{64} = 6\left(\lambda+\frac{3}{8}\right)^2 + \frac{5}{32}$ $CP = \sqrt{6\left(\lambda+\frac{3}{8}\right)^2 + \frac{5}{32}}$ The perpendicular distance from C to l occurs when P is nearest to l, that is when CP is least or $\lambda = -\frac{3}{8}$. Least CP is $\frac{\sqrt{10}}{8}$. P is F in this case. $\overline{OF} = \mathbf{a} - \frac{3}{8}(\mathbf{a}+2\mathbf{b}) = \frac{1}{8}(5\mathbf{a}-6\mathbf{b})$ **Alternative to find minimum *CP*:** CP is minimum when CP^2 is minimum: $\frac{\mathrm{d}}{\mathrm{d}x}(CP^2) = 12\lambda + \frac{9}{2}$ When $\frac{\mathrm{d}}{\mathrm{d}x}(CP^2) = 0, 12\lambda + \frac{9}{2} = 0$ $\therefore \lambda = -\frac{3}{8}$. Since CP^2 is quadratic and coefficient of $\lambda^2 > 0$, CP^2 is minimum at $\lambda = -\frac{3}{8}$ $\therefore$ perpendicular distance from C to l occur when $\lambda = -\frac{3}{8}$.	**M1** – complete the square **A1** **B1** **M1** – Differentiation and note that coefficient of $\lambda^2 > 0$.
(iii)	$\overline{CF} = \frac{1}{8}(5\mathbf{a}-6\mathbf{b}) - (\mathbf{a}-\mathbf{b}) = \frac{1}{8}(-3\mathbf{a}+2\mathbf{b})$ Equation of plane $\mathbf{r}.(-3\mathbf{a}+2\mathbf{b}) = \mathbf{a}.(-3\mathbf{a}+2\mathbf{b}) = -3 + \frac{2}{4} = -\frac{5}{2}$	√**M1** – See $\mathbf{r}\cdot(2\mathbf{b}-3\mathbf{a}) = D$ **A1**
		Total marks : 9

Qn	Suggested Solution	Mark Scheme
3(i)	$u_{n+1}=0.3u_n+6$	B1
(ii)	$u_n=0.3u_{n-1}+6$ $=0.3(0.3u_{n-2}+6)+6$ $=0.3^2u_{n-2}+0.3(6)+6$ $=0.3^2(0.3u_{n-3}+6)+0.3(6)+6$ $=0.3^3u_{n-3}+0.3^2(6)+0.3(6)+6$ $\vdots$ $=0.3^{n-1}u_1+0.3^{n-2}(6)+0.3^{n-3}(6)+\ldots+0.3(6)+6$ $=0.3^{n-1}u_1+\frac{6(1-0.3^{n-1})}{1-0.3}$ $=0.3^{n-1}u_1+\frac{60}{7}(1-0.3^{n-1})$ **Alternative** $u_2=0.3u_1+6$ $u_3=0.3u_2+6$ $=0.3(0.3u_1+6)+6$ $=0.3^2u_1+0.3(6)+6$ $\vdots$ $u_n=0.3^{n-1}u_1+0.3^{n-2}(6)+0.3^{n-3}(6)+\ldots+0.3(6)+6$ $=0.3^{n-1}u_1+\frac{6(1-0.3^{n-1})}{1-0.3}$ $=0.3^{n-1}u_1+\frac{60}{7}(1-0.3^{n-1})$	**M1** – backward substitution, with enough terms to deduce GP pattern **AG1** **M1** –forward substitution, with enough terms to deduce GP pattern **AG1**
(iii)	As $n\to\infty$, $v_{n+1}\to L$ and $v_n\to L$. $\therefore L=0.01L^2+6$ $0.01L^2-L+6=0$ From G.C., $L=6.4110$ or 93.588 $\therefore \alpha=6.41,\ \beta=93.6$ (3 s.f.) v_n may not necessarily converge to a limit as we do not know what is the value of its starting term v_1 (or initial number of bacteria).	**M1** – substitute v_n and v_{n-1} with limit. **A1** **B1**

<table>
<tr><td></td><td></td><td></td></tr>
<tr>
<td>(iv)</td>
<td>
As $n \to \infty$,
$u_n \to 0u_1 + \frac{60}{7}(1-0) = 8.57$ (3 s.f.)


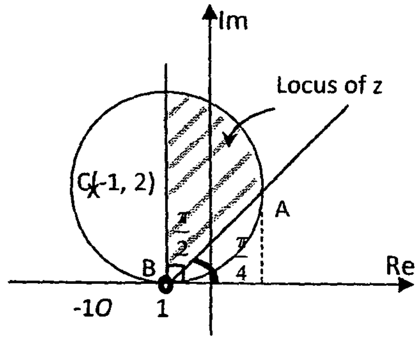
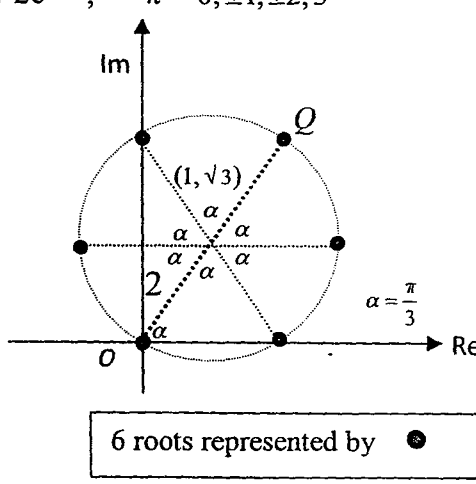


From GC, $v_n \to 6.41$ (3 s.f.)
Thus, Chemical B is more effective in the long run.
</td>
<td>
M1 – find limit of either u_n or v_n.

A1
</td>
</tr>
<tr>
<td>(v)</td>
<td>
The amount of Chemical C used per week is an Arithmetic Progression with first term $a = 20$, common difference $d = 5$.
Want: $S_n \geq 3000$
$\frac{n}{2}[2(20)+(n-1)(5)] \geq 3000$
$n^2 + 7n - 1200 \geq 0$
From G.C.,
<table>
<tr><th>n</th><th>$n^2 + 7n - 1200$</th></tr>
<tr><td>31</td><td>-22</td></tr>
<tr><td>32</td><td>48</td></tr>
<tr><td>33</td><td>120</td></tr>
</table>
$\therefore$ it takes 32weeks
Alternative: From GC, $n \geq 31.3$ (to 3 s.f.) $\Rightarrow$ It takes 32 wks
</td>
<td>
B1 –formulate inequality/ equation.

M1 – use AP sum formula and attempt to solve inequality/ equation (AP formula must be correct)

A1 – conclusion.
</td>
</tr>
<tr><td></td><td></td><td>Total Marks: 11</td></tr>
</table>

Qn	Suggested Solution	Mark Scheme
4(a)	 $\lvert z+1-2i\rvert=\lvert z-(-1+2i)\rvert$ ie, $\lvert z-(-1+2i)\rvert\le 2$ Min arg (z) occurs at $A=\tan^{-1}\frac{2}{1}=\tan^{-1}2$ Max arg (z) occurs at B Hence, $\tan^{-1}2\le \arg(z) < \pi$	**G1** – Circle, centre $(-1,2)$ touching $(-1,0)$ **G1** – Two half-lines with c starting point $(-1,0)$ **G1** – Enclosed region indic SR: Deduct max 1 mk if diagram not clearly labellec or $(-1,0)$ not excluded **M1**– Either min or max con (exact) **A1** – Correct range, π exclu (condone 1.11 in place of $\tan^{-1}2$)
(b)	$w^6=2^6$ $\Rightarrow w^6=2^6e^{2k\pi i}$ $\therefore w=2e^{i\frac{k\pi}{3}},\quad k=0,\pm1,\pm2,3$	**M1** – Correct application of Moivre's Theorem. Condone wrong conversion of comple number to polar form **A1** – correct expression of z $2e^{i\frac{k\pi}{3}}$ (with correct k)
(i)	$z=1+i\sqrt{3}+2e^{i\frac{k\pi}{3}},\quad k=0,\pm1,\pm2,3$ 	**√B1**–$z=1+i\sqrt{3}+re^{i\theta}$, condo wrong values of k from previo part. **B1** – Six roots on locus of circ centred at $(1,\sqrt{3})$. Condone wrong radius. **B1** – Correct radius of circle. Three symmetric pairs of root equally spaced out with angle between consecutive roots correctly labelled. One root at origin, one on each axis.
(ii)	$\lvert z\rvert$ is maximum at Q, where Q is the point representing the root when $k=1$ and OQ is the diameter of the circle	**B1**- Correct identification of with k=1

$$z = 1 + i\sqrt{3} + 2e^{i\frac{\pi}{3}}$$
$$= 1 + i\sqrt{3} + 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$
$$= 1 + i\sqrt{3} + 1 + i\sqrt{3}$$
$$= 2 + i\,2\sqrt{3}$$

B1- correct z in cartesian form.

<u>**Alternative**</u>

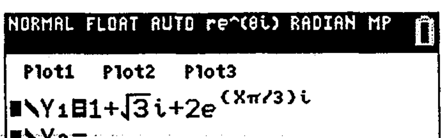
$$z = 4e^{i\frac{\pi}{3}}$$
$$= 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$
$$= 2 + i2\sqrt{3}$$

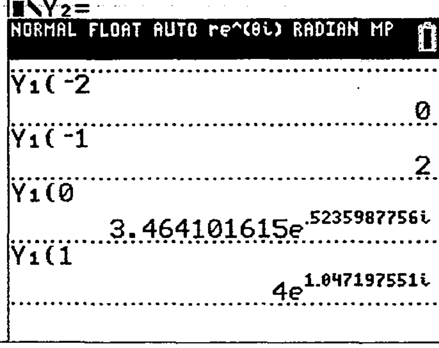
--------- **OR** -----------

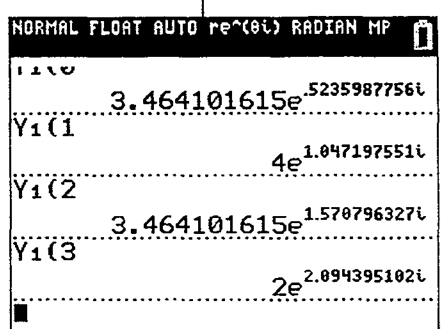
B1- Correct identification of root with k=1

B1- correct z in cartesian form.

<u>**For Reference: Use GC to CHECK the location of roots (Not to be used for sol presentation as GC not allowed**</u>

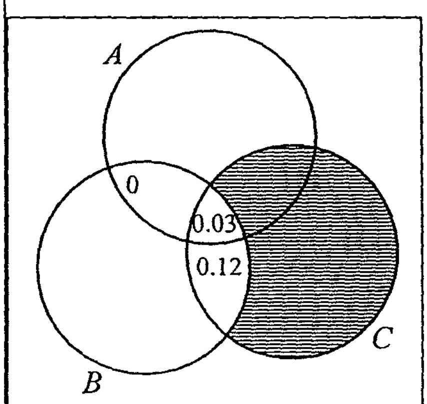
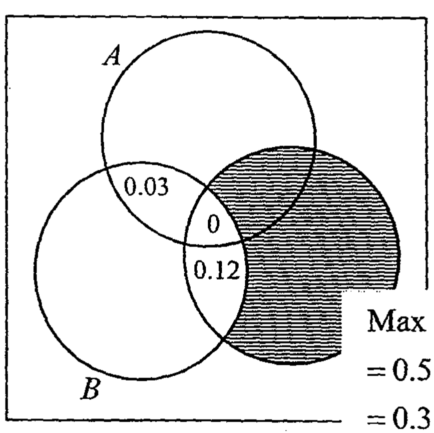
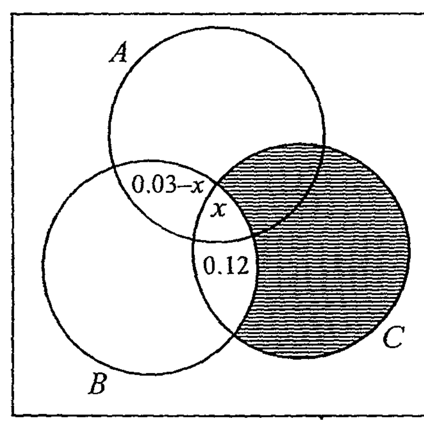






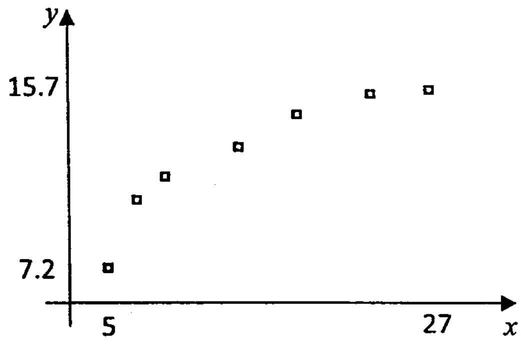
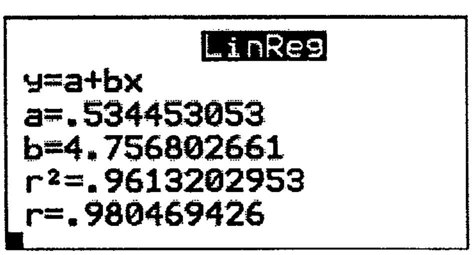
Total marks : 12

Qn	Suggested Solution	Mark Scheme
5(i)	No, the people are surveyed without consideration of the stratum e.g. age group they belong to.	**B1** –Appropriate reason, in context of question SR: Do not accept "sample is non-random"
(ii)	To obtain a systematic sample of 5%, we can first randomly select the first commuter from the first 20 commuters entering the train station by stationing the surveyors at the gantries and thereafter select every 20th commuter thereafter entering the train station from the start to the end of that particular day.	**B1** – Identify sampling interval 20. √**B1** – Randomly select first patron from the first k patrons where k = sampling interval identified and systematically select the next patron according to the sampling interval. SR: Description wrong for all 2 points above but selecting from consistent interval, award 1 out of 2.
		Total marks : 3

Qn	Suggested Solution	Mark Scheme
6(i)	Let A denote the event that John scores in a game. Let B denote the event that John's parents are present at a game. Given that $\text{P}(A)=0.15, \text{P}(B)=0.3, \text{P}(B\mid A)=0.2$, $\text{P}(A\mid B)$ $=\dfrac{\text{P}(A\cap B)}{\text{P}(B)}$ $=\dfrac{\text{P}(B\mid A)\times \text{P}(A)}{\text{P}(B)}$ $=\dfrac{0.2\times 0.15}{0.3}=\dfrac{0.03}{0.3}$ $=0.1$ (shown)	**M1** – See either conditional probability formulae for P(A|B) and $\text{P}(A\cap B)$ **AG1**
(ii)	Since $\text{P}(A\mid B)=0.1\neq \text{P}(A)=0.15$ his parent's presence in a game will affect his chances of scoring in the game. **Alternative** Since $\text{P}(B\mid A)=0.2\neq \text{P}(B)=0.3$ his parent's presence in a game will affect his chances of scoring in the game.	**B1** – Correct conclusion with appropriate comparison of probabilities.
(iii)	Let C denote the event that the game is ahome game. $\text{P}(A'\cap B\cap C)=\text{P}((A'\cap B)\mid C)\times \text{P}(C)=0.24\times 0.5=0.12$  Least $\text{P}(B'\cap C)$ $=0.5-0.03-0.12$ $=0.35$  Max $\text{P}(B'\cap C)$ $=0.5-0.12$ $=0.38$	**M1** – Apply conditional probability formula to find $\text{P}(A'\cap B\cap C)$.  Note: $0\le x\le 0.03$ **A1** – Correct least value **A1** – Correct greatest value

			Total n

Qn	Suggested Solution	Mark Scheme
7(i)	Required probability $=\dfrac{\left({}^{4}C_{2}\right)^{2}}{{}^{28}C_{4}}$ $=\dfrac{4}{2275}$	**M1** – Either denominator or numerator correct **A1** – Also accept 0.00176 (3 s.f.)
(ii)	Required probability $=\left(\dfrac{1}{9}\right)^{2}\times\left(\dfrac{2}{9}\right)^{2}\times\dfrac{4!}{2!2!}$ $=\dfrac{8}{2187}$	**M1** – See $\left(\dfrac{1}{9}\right)^{2}\times\left(\dfrac{2}{9}\right)^{2}$ **A1** – Also accept 0.00366 (3 s.f.)
(iii)	No. of ways to seat the remaining members $=(7-1)!$ $=720$ No. of ways to slot in the committee head and the 2 vice-heads as a pair $={}^{7}P_{2}=42$ No. of ways to arrange the 2 vice-heads = 2 Required probability $=\dfrac{720\times42\times2}{(10-1)!}=\dfrac{1}{6}$	**B1** – 720 **M1**– Attempt to use slotting method. Condone ${}^{8}P_{2}$, ${}^{7}C_{2}$. **A1**
	Alternative No. of ways in which the vice-heads are seated together $=(9-1)!\times2!$ $=80640$ No of ways in which the vice-heads are seated together and the head is seated next to one of them $=(8-1)!\times2!\times2$ $=20160$ Required probability $=\dfrac{80640-20160}{(10-1)!}=\dfrac{1}{6}$	**B1** – Either unrestricted problem or complement case computed correctly. **M1**– Attempt to use complement method. Unrestricted problem should be identified as the no. of ways in which the vice-heads are seated together. **A1**
	Alternative No. of ways in which the vice-heads are seated together with the rest of the committee excluding the committee head $=(8-1)!\times2!$ No. of ways to slot in the committee head $={}^{6}C_{1}$ Required probability $=\dfrac{(8-1)!(2)(6)}{(10-1)!}=\dfrac{1}{6}$	**B1** – (8-1)! **M1**– Attempt to use slotting method. Condone ${}^{7}C_{1}$. **A1**
		Total marks : 7

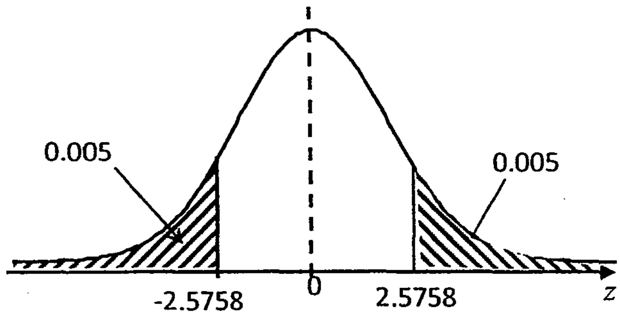
Qn	Suggested Solution	Mark Scheme
8(i)		**B1**– Must see the following: (i) Labelled axes (ii) Min/max values shown (iii) x-values and y-values appropriately spaced (iv) curvilinear shape with y increases at a decreasing rate, evident from the last 4 points especially (v) Note last point positions higher
(ii)	Model **B**: $y = a + b \ln x$ is appropriate but not Model A. From the scatter diagram, as x increases, y increases at a **decreasing rate,** which is consistent with Model B but not Model A which predicts an increasing rate of increase for y.	**B1** – Correct model chosen **B1** – Reason explained with comparison of models (model B must be discussed). ***SR: Do NOT*** accept reason based on r as value of r for a linear model is quite high too. Further, question did not suggest reference to r.
(iii)	Screenshot for reference:  $r = 0.98047$ (5 s.f.) ≈ 0.980 (3 s.f.) The value of r would not be different as it is unaffected when **data is scaled**	**B1** – Accept answer in 3 s.f. or 5 s.f. **B1**
(iv)	$a = 0.534$, $b = 4.76$ $y = 0.53445 + 4.7568 \ln x$ (5 s.f.) (or $y = 0.534 + 4.76 \ln x$) When $x = 20$,	**B1**– Correct a and b (accept both 3s.f. or 5 s.f.) **B1** – Correct equation, note "$\ln x$" (accept both 3s.f. or 5 s.f.) **B1** – $y = 14.8$

	$y = 0.53445 + 4.7568\ln(20)$ $= 14.785$ $= 14.8$ (3 s.f.) Car population is 14.8 millions	**SR: Do not award if students wrote "car population is 14.8" See ER**
(v)	Value of a represents the predicted car population **after 1 year** of study. [If wrong model chosen: Value of a represents the predicted car population at the start of the study.] The value of a is unreliable (invalid) as it is a value obtained from an extrapolation at Year 1 (i.e, $x = 1$) which is outside the data range of Year 5 to Year 27; the linear relationship may not hold.	**B1** – Comment in context [Not awarded if quoted as "at the start" or "before the start" or "in the 1st year"] **[B1 follow thru]** **B1** – Answer with elaboration [Not awarded if "a" or "it" is outside the data range]
		Total Marks: 10

Qn	Suggested Solution	Mark
9(a) (i)	Let A and B denote the queuing times of a randomly chosen passenger at Economy and Business class counters respectively. $A\sim N(11.6, 4.2^2)$ $B\sim N(3.2, 0.9^2)$ Find $P(\lvert A-(B_1+B_2)\rvert<5)$ $A-(B_1+B_2)\sim N(11.6-2\times3.2,\ 4.2^2+2\times0.9^2)$ i.e. $A-(B_1+B_2)\sim N(5.2,\ 19.26)$ $\therefore P(\lvert A-(B_1+B_2)\rvert<5)=P(-5<A-(B_1+B_2)<5)$ ≈ 0.47177 $=0.472$ (3 s.f.)	**B1** – e $A-(B$ **B1** – c $A-(B_1$ $\sim N(5.2$ **M1** – $P(\lvert A-$ **A1**
(ii)	Required probability $=P(B_1\geq 2)\times P(B_2\geq 2)\times ...\times P(B_8\geq 2)$ $=[P(B\geq 2)]^8$ ≈ 0.46526 $=0.465$ (3 s.f.)	**M1** – use of strict i **A1**
(b)	Let X denote the number of passengers who <u>turned up</u> for their flight, out of n passengers who bought the n tickets. $X\sim B(n, 0.95)$ Since $n>350$ is sufficiently large such that $np=0.95n>5$ and $nq=0.05n>5$, $X\sim N(0.95n, 0.0475n)$ approximately $P(\text{Flight is overbooked})\leq 0.01$ $\Rightarrow P(X>350)\leq 0.01$ $\Rightarrow P(X>350.5)\leq 0.01$ (continuity correction) $\Rightarrow P(X<350.5)\geq 0.99$ $\Rightarrow P\left(Z<\dfrac{350.5-0.95n}{\sqrt{0.0475n}}\right)\geq 0.99$ $\Rightarrow \dfrac{350.5-0.95n}{\sqrt{0.0475n}}\geq 2.3263$ $\Rightarrow 350.5-0.95n\geq 2.3263\sqrt{0.0475n}$ (shown)	**B1** – state distributio conditions **B1** – state a normal distr **M1** – attem inequality v correction **AG1** – stan z-value for n

	<u>Alternative to show approx inequality:</u> Let X denote the number of passengers who <u>did not turn up</u> for their flight, out of n passengers who bought the n tickets. $X \sim \mathrm{B}(n, 0.05)$ Since $n > 350$ is sufficiently large such that $np = 0.05n > 5$ and $nq = 0.95n > 5$, $X \sim \mathrm{N}(0.05n, 0.0475n)$ approximately $\mathrm{P}(\text{Flight is overbooked}) \le 0.01$ $\Rightarrow \mathrm{P}(X < n - 350) \le 0.01$ $\Rightarrow \mathrm{P}(X \le n - 351) \le 0.01$ $\Rightarrow \mathrm{P}(X < n - 350.5) \le 0.01$ (continuity correction) $\Rightarrow \mathrm{P}\left(Z < \frac{n - 350.5 - 0.05n}{\sqrt{0.0475n}}\right) \le 0.01$ $\Rightarrow \frac{0.95n - 350.5}{\sqrt{0.0475n}} \le -2.3263$ $\Rightarrow 0.95n - 350.5 \le -2.3263\sqrt{0.0475n}$ $\Rightarrow 350.5 - 0.95n \ge 2.3263\sqrt{0.0475n}$ (shown)	**B1** – state original distribution and check conditions **B1** – state approximate normal distribution. **M1** – attempt to formulate inequality with continuity correction **AG1** – standardise to obtain z-value for an inequality in n
		Total Marks: 10

Qn	Suggested Solution	Mark Scheme
10(i)	Unbiased estimate of population mean, $\bar{x} = 167.89 = 168$ (3 s.f.) Unbiased estimate of population variance, $s^2 = 2.4988^2 = 6.24$ (3 s.f.)	**B1** **B1**
(ii)	Assume that the breaking strength of each coil of rope is normally distributed. $H_0 : \mu = 169.7$ $H_1 : \mu < 169.7$ Perform 1-tail test at 5% significance level. Under H_0, $\frac{\bar{X}-169.7}{S/\sqrt{8}} \sim t(8-1)$. Using t-test, p-value = 0.039673. Since p-value ≤ 0.05, we reject H_0 and conclude that there is sufficient evidence at the 5% significance level that the mean breaking strength is less than 169.7 kg, i.e. the manufacturer's claim is not valid.	**B1** – normal assumption **B1** – correct H_0 and H_1 **B1** – sampling distribution (t-distribution; must see $\frac{\bar{X}-169.7}{S/\sqrt{8}}$ SR: Condone if S is substituted in the sampling distribution with the value calculated in (i)) **B1** – correct p-value **B1** – conclusion (must refer to mean breaking strength) SR: Give mark for correct conclusion if student obtain inaccurate p-value of 0.0478 or 0.0479 due to using rounded-off intermediate values.

(iii)	$H_0 : \mu = \mu_0$ $H_1 : \mu \neq \mu_0$ Under H_0, $\bar{Y} \sim N(\mu_0, \frac{29.16}{50})$ approximately by Central Limit Theorem.  Given $\bar{y} = 171$, and H_0 is not rejected, $-2.5758 < \frac{171-\mu_0}{\sqrt{\frac{29.16}{50}}} < 2.5758$ $\Rightarrow 169.03 < \mu_0 < 172.97$ $\therefore$ set of values of μ_0 is: $\{\mu_0 \in \square : 169 < \mu_0 < 173\}$	**B1** – correct $\bar{Y}$ distribution (must see CLT) **B1** – see 2.5758 **B1** – correct criterion that p-value > 0.01 or equivalent (two-tail) **B1**– Also accept answers (169,173) or [169,173] or $\{\mu_0 \in \square : 169 \leq \mu_0 \leq 173\}$
(iv)	Testing at the 1% significance level means that there is a probability of 0.01 of concluding that the mean breaking strength differs from the claim when it is actually unchanged.	**B1** – With keywords (underlined)
		Total Marks: 12

Qn	Suggested Solution	Mark Scheme
11(a) (i)	Let X and Y be the number of rectangular tables and round tables that are occupied. $X \sim B(6,0.8)$ $Y \sim B(9,0.65)$ Required probability $= P(X=4)\,P(Y=7)$ $=0.24576 \times 0.21619$ $=0.0531$ (3 s.f.)	**B1** – correctbinomial distributions **B1**
(ii)	Accept any answer in context that explains <u>why the probability that a table is occupied may be affected by another table that is occupied</u>, such as: • Customers may arrive as a big group that requires them to be split into two separate tables next to each other. OR • The restaurant may choose to seat the customers at tables in a particular section first. <u>Do not accept</u> reasons that fail to directly address the issue of independence stated in the question: • Customers may prefer to be seated at the rectangular tables instead of the round tables, or vice versa • Arriving customers may be asked to share/join a table that is partially occupied.	**B1** Not accepted as this addresses the difference in success probabilities i.e. 0.8 vs. 0.65 instead. (or non-constant success probabilities) Not accepted as this addresses the appropriateness of the binomial model instead.
(b) (i)	Let C and T be the number of cups of coffee and tea sold in 20 minutes, respectively. $C \sim P_o(5)$ $T \sim P_o(3.5)$ $C+T \sim P_o(8.5)$ $P(T \geq 6 \mid C+T \geq 7) = \dfrac{P(\{T \geq 6\} \cap \{C+T \geq 7\})}{P(C+T \geq 7)}$ $= \dfrac{P(T=6)\,P(C \geq 1) + P(T \geq 7)P(C \geq 0)}{P(C+T \geq 7)}$ $= \dfrac{P(T=6)\,[1-P(C=0)] + [1-P(T \leq 6)]}{1-P(C+T \leq 6)}$ $= \dfrac{0.077098(0.99326) + 0.065288}{0.74382}$ $= \dfrac{0.14187}{0.74382} = 0.191$ (3 s.f.)	**B1** - $C+T \sim P_o(8.5)$ **M1** – consider probability for intersection of events for conditional probability with correct denominator **B1** – correct simplification for at least one of the two cases in numerator **A1**

(ii)	Using $p_k = e^{-\lambda}\frac{\lambda^k}{k!}$ for $\lambda = 5$, $\frac{p_{k+1}}{p_k} = \frac{\left(e^{-5}\frac{5^{k+1}}{(k+1)!}\right)}{\left(e^{-5}\frac{5^k}{k!}\right)} = \frac{5^{k+1}k!}{5^k(k+1)!}$	M1 – apply $e^{-\lambda}\frac{\lambda^k}{k!}$
	$= \frac{5\cdot 5^k k!}{5^k(k+1)k!} = \frac{5}{k+1}$ (shown)	AG1 – simplify ratio
	When $k<4$, $k+1<5$, $\Rightarrow \frac{5}{k+1} > 1 \Rightarrow \frac{p_{k+1}}{p_k} > 1 \Rightarrow p_{k+1} > p_k$.	AG1 – use $k<4$ to show $\frac{5}{k+1} > 1$
	When $k<4$, i.e. $k = 0,1,2,3$ $p_{k+1} > p_k \Rightarrow p_4 > p_3 > p_2 > p_1 > p_0$. When $k>4$, i.e. $k = 5,6,7,...$ $p_{k+1} < p_k \Rightarrow p_5 > p_6 > p_7 \cdots$. When $k=4$, $p_{k+1} = p_k \Rightarrow p_4 = p_5$.	M1 – show either $p_{k+1} < p_k$ for $k>4$ or $p_{k+1} = p_k$ for $k=4$, or explain qualitatively the first increasing, then decreasing trend of Poisson probabilities.
	From above, $p_0 < p_1 < \cdots < p_4 = p_5 > p_6 > p_7 > p_8 > \cdots$ (Thus p_k is greatest when $k = 4$ and 5) The most probable number of cups of coffee sold (i.e. the mode) are 4 and 5.	A1
		Total Marks: 12