JC2 PRELIMINARY EXAMINATION

Higher 2

MATHEMATICS **9740/01**

Paper 1 **15th September 2016**

3 Hours

Additional Materials: Cover Sheet
Answer Papers
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages.

1 The nth term of a sequence is given by $u_n = \dfrac{4^n n^2}{(n+1)(n+2)}$, for $n \geq 1$.

The sum of the first n terms is denoted by S_n. Use the method of mathematical induction to show that $S_n = \left(\dfrac{n-1}{n+2}\right)\left(\dfrac{4^{n+1}}{3}\right) + \dfrac{2}{3}$ for all positive integers n. **[5]**

2 Using partial fractions, find $\displaystyle\int_{-2}^{2} \frac{17x^2+23x+12}{(3x+4)(x^2+4)}\,dx$, leaving your answer in exact form. **[6]**

3 A curve C has parametric equations

$$x = \frac{1}{2}(\sin t\cos t + t), \quad y = \frac{1}{2}t - \frac{1}{4}\sin 2t, \quad \text{for } -\frac{\pi}{2} < t \leq 0.$$

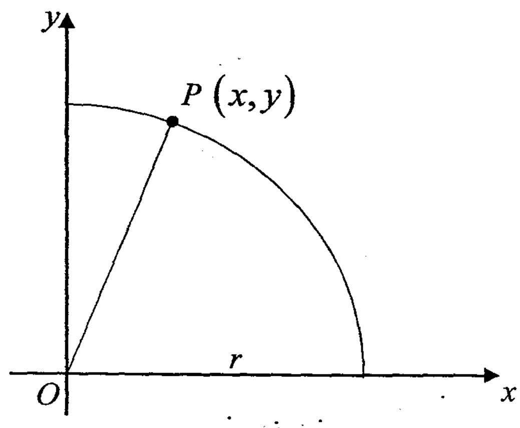
The tangent to the curve at the point P has gradient 1. Find the equation of the normal at P.

The region bounded by this normal, the curve C and the x-axis is rotated through 2π radians about the x-axis. Find, to 5 decimal places, the volume of the solid obtained. What can be said about the tangents to the curve as t approaches 0? **[7]**

4 Referred to the origin, the points A and B have position vectors **a** and **b** respectively. A point C is such that $OACB$ forms a parallelogram. Given that M is the mid-point of AC, find the position vector of point N if M lies on ON produced such that $OM : ON$ is in the ratio 3:2. Hence show that A, B and N are collinear. **[4]**

Point P is on AB is such that MP is perpendicular to AB. Given that angle AOB is 60^o, $|\mathbf{a}| = 2$ and $|\mathbf{b}| = 3$, find the position vector of P in terms of **a** and **b**. **[4]**

5



A particle P moves along the curve with equation $x^2+y^2=r^2$, where $x \geq 0, y \geq 0$, and r is a constant. By letting $m=\tan\left(\sin^{-1}\frac{y}{r}\right)$, find an expression for $\frac{\mathrm{d}m}{\mathrm{d}y}$ in terms of y and r.

Given that the rate of change of y with respect to time t is 0.1% of r, show that $\frac{\mathrm{d}m}{\mathrm{d}t}=\left(\frac{r}{10\sqrt{r^2-y^2}}\right)^3$.

State the geometrical meaning of $\frac{\mathrm{d}m}{\mathrm{d}t}$. [7]

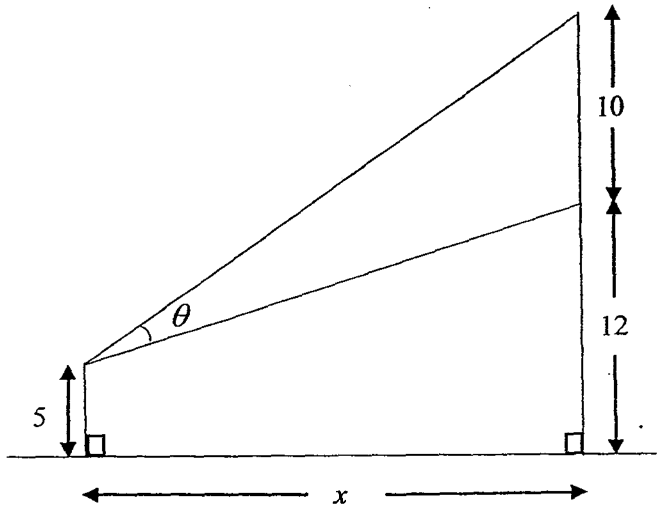
6 (i) Show that $\frac{r^2+r-1}{(r+2)!}=\frac{A}{r!}+\frac{B}{(r+1)!}+\frac{C}{(r+2)!}$, where A, B and C are constants to be determined. [2]

(ii) Hence find $\sum_{r=1}^{n}\frac{r^2+r-1}{(r+2)!}$ in terms of n. (There is no need to express your answer as a single algebraic function.) [3]

(iii) Explain why $\sum_{r=1}^{n}\frac{r^2-1}{(r+2)!}<\frac{1}{2}$. [2]

(iv) Use your answer to part (ii) to find $\sum_{r=4}^{n}\frac{r^2-3r+1}{r!}$ in terms of n. [3]

7



A 10 feet tall statue is mounted on a 12 feet tall pedestal. A boy is standing x feet away from the pedestal. His eyes are 5 feet above ground level, and the angle subtended by the statue from the boy's eyes is θ radians (see diagram).

Prove that

$$\tan\theta = \frac{10x}{119+x^2}.$$

Hence, or otherwise, find the exact value of x for which θ is maximum and justify that this value of x gives the maximum value of θ.

Deduce, to the nearest degree, the maximum angle subtended by the statue from the boy's eyes. **[9]**

8 **(i)** Find the fourth roots of $-1+\sqrt{3}\,\mathrm{i}$, giving the roots in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\le\pi$. **[3]**

(ii) Hence, or otherwise, write down the roots of the equation $(1+z)^4+1-\mathrm{i}\sqrt{3}=0$ and show the roots Z_i, $i=1,2,3,4$ on an Argand diagram. **[3]**

(iii) Illustrate, using the same Argand diagram, the locus of a point Q representing the complex number v, where $\left|v+1-4\sqrt{3}-4\mathrm{i}\right|=2$.

Hence find the exact greatest and least possible values of Z_iQ. **[4]**

9 Two biologists are investigating the growth of a certain bacteria of size x hundred thousand at time t days. It is known that the number of bacteria initially is 20% of a, where a is a positive constant.

(i) One biologist believes that x and t are related by the differential equation $\dfrac{\mathrm{d}x}{\mathrm{d}t} = x(a - x)$. Given that the number of bacteria increases to 50% of a when $t = \ln 2$ days, show that $x = \dfrac{2}{4\mathrm{e}^{-2t} + 1}$. **[7]**

(ii) Another biologist believes that x and t are related by the differential equation $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} = 10 - 9t^2$. Find the general solution of this differential equation and sketch three members of the family of solution curves. **[5]**

10 **(a)** **(i)** Express $\sin x + \sqrt{3}\cos x$ in the form $R\sin(x + \alpha)$ where R and α are exact positive constants to be found. **[1]**

The function f is defined by $\mathrm{f} : x \mapsto \sin x + \sqrt{3}\cos x, \ \dfrac{\pi}{6} \le x \le k$.

(ii) Find the largest exact value of k such that f has an inverse. Hence define f^{-1} in similar form and write down the set of values of x for which $\mathrm{f}\mathrm{f}^{-1}(x) = \mathrm{f}^{-1}\mathrm{f}(x)$. **[5]**

(b) The function g is defined by $\mathrm{g} : x \mapsto 2 - \dfrac{5x}{1 + x^2}, x \in \mathbb{R}$.

(i) Use an algebraic method to find the range of g. **[3]**

(ii) State a sequence of transformations which transform the graph of $y = \mathrm{g}(x)$ to the graph of $y = \dfrac{10x}{4 + x^2}$. **[3]**

11 The line l_1 passes through the point A with coordinates (1, 2, 1) and is parallel to the vector $\mathbf{i} + a\mathbf{j} + 2\mathbf{k}$, where $a \in \mathbb{R}$. The line l_2 has equation $x - 3 = \dfrac{y}{2} = \dfrac{z-5}{3}$. It is given that l_1 and l_2 intersect at point B.

(i) Find the value of a. **[4]**

(ii) The plane p_1 contains the point A and is perpendicular to l_2. Find the exact shortest distance from point B to p_1. Hence find the acute angle between l_1 and p_1. **[5]**

(iii) Find a cartesian equation of plane p_2 that is perpendicular to p_1 and contains l_1. **[3]**

(iv) Find the acute angle between p_2 and the xy-plane. **[2]**

Solutions to P1 Prelim 2016

1

Let P_n denote the proposition $S_n=\left(\dfrac{n-1}{n+2}\right)\left(\dfrac{4^{n+1}}{3}\right)+\dfrac{2}{3}$ for $n\in\mathbb{Z}^+$.

When $n=1$, LHS $=S_1=u_1=\dfrac{4^1(1)^2}{(2)(3)}=\dfrac{2}{3}$.

$$\text{RHS}=\left(\frac{1-1}{1+2}\right)\left(\frac{4^2}{3}\right)+\frac{2}{3}=\frac{2}{3}=\text{LHS}.$$

$\therefore P_1$ is true.

Assume P_k is true for some $k\in\mathbb{Z}^+$, i.e. $S_k=\left(\dfrac{k-1}{k+2}\right)\left(\dfrac{4^{k+1}}{3}\right)+\dfrac{2}{3}$.

To prove P_{k+1} is also true, i.e. $S_{k+1}=\left(\dfrac{k}{k+3}\right)\left(\dfrac{4^{k+2}}{3}\right)+\dfrac{2}{3}$.

$$\begin{aligned}
\text{LHS}&=S_{k+1}\\
&=S_k+u_{k+1}\\
&=\left(\frac{k-1}{k+2}\right)\left(\frac{4^{k+1}}{3}\right)+\frac{2}{3}+\frac{4^{k+1}(k+1)^2}{(k+2)(k+3)}\\
&=\left(\frac{4^{k+1}}{3(k+2)}\right)\left[k-1+\frac{3(k+1)^2}{k+3}\right]+\frac{2}{3}\\
&=\left(\frac{4^{k+1}}{3(k+2)}\right)\left[\frac{(k-1)(k+3)+3(k+1)^2}{k+3}\right]+\frac{2}{3}\\
&=\left(\frac{4^{k+1}}{3(k+2)}\right)\left[\frac{k^2+2k-3+3k^2+6k+3}{k+3}\right]+\frac{2}{3}\\
&=\left(\frac{4^{k+1}}{3(k+2)}\right)\left[\frac{4k^2+8k}{k+3}\right]+\frac{2}{3}\\
&=\left(\frac{4^{k+2}}{3(k+2)}\right)\left[\frac{k(k+2)}{k+3}\right]+\frac{2}{3}=\left(\frac{k}{k+3}\right)\left(\frac{4^{k+2}}{3}\right)+\frac{2}{3}=\text{RHS}.
\end{aligned}$$

Since P_1 is true and P_k is true $\Rightarrow P_{k+1}$ is true, by Mathematical Induction, P_n is true for all $n\in\mathbb{Z}^+$.

2	$\dfrac{17x^2+23x+12}{(3x+4)(x^2+4)} \equiv \dfrac{A}{3x+4}+\dfrac{Bx+C}{x^2+4}$ Solving, $A=2$, $B=5$ and $C=1$ $\displaystyle\int_{-2}^{2}\frac{17x^2+23x+12}{(3x+4)(x^2+4)}\,dx=\int_{-2}^{2}\frac{2}{3x+4}+\frac{5x+1}{x^2+4}\,dx$ $\displaystyle=\int_{-2}^{2}\frac{2}{3x+4}+\frac{5}{2}\left(\frac{2x}{x^2+4}\right)+\frac{1}{x^2+4}\,dx$ $\displaystyle=\frac{2}{3}\Big[\ln\lvert 3x+4\rvert\Big]_{-2}^{2}+\frac{5}{2}\Big[\ln(x^2+4)\Big]_{-2}^{2}+\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2}$ $=\dfrac{2}{3}\ln 5+\dfrac{\pi}{4}$
3	$x=\dfrac{1}{2}(\sin t\cos t+t)=\dfrac{1}{2}t+\dfrac{1}{4}\sin 2t$ and $y=\dfrac{1}{2}t-\dfrac{1}{4}\sin 2t$ $\dfrac{dx}{dt}=\dfrac{1}{2}\cos 2t+\dfrac{1}{2}$ and $\dfrac{dy}{dt}=\dfrac{1}{2}-\dfrac{1}{2}\cos 2t$ $\dfrac{dy}{dx}=\dfrac{1-\cos 2t}{1+\cos 2t}=\dfrac{1-(1-2\sin^2 t)}{(2\cos^2 t-1)+1}=\tan^2 t$ When $\dfrac{dy}{dx}=1\Rightarrow\tan^2 t=1\Rightarrow t=\pm\dfrac{\pi}{4}$ Since $t<0$, $t=-\dfrac{\pi}{4}$, and $x=-\dfrac{1}{4}-\dfrac{\pi}{8}$ and $y=\dfrac{1}{4}-\dfrac{\pi}{8}$ Equation of normal is $y-\left(\dfrac{1}{4}-\dfrac{\pi}{8}\right)=-\left[x-\left(\dfrac{1}{4}-\dfrac{\pi}{8}\right)\right]$ $y=-x-\dfrac{\pi}{4}$ When $y=0$, $x=-\dfrac{\pi}{4}$. Volume required is $\displaystyle=\frac{1}{3}\pi\left(\frac{1}{4}-\frac{\pi}{8}\right)^2\left(\frac{\pi}{8}-\frac{1}{4}\right)+\pi\int_{-\frac{\pi}{4}}^{0}\left(\frac{1}{2}t-\frac{1}{4}\sin 2t\right)^2\left(\frac{1}{2}\cos 2t+\frac{1}{2}\right)dt$ $=0.00759$ (5 d.p.). $\dfrac{dy}{dx}=\tan^2 t\approx t^2\to 0$ as t approaches 0. Therefore the tangents are parallel to the x-axis.
4	$\overrightarrow{OM}=\dfrac{1}{2}\left(\overrightarrow{OA}+\overrightarrow{OC}\right)=\dfrac{1}{2}(2\mathbf{a}+\mathbf{b})$ $\overrightarrow{ON}=\dfrac{2}{3}\overrightarrow{OM}=\dfrac{2}{3}\times\dfrac{1}{2}(2\mathbf{a}+\mathbf{b})=\dfrac{1}{3}(2\mathbf{a}+\mathbf{b})$

$\overrightarrow{AN} = \overrightarrow{ON} - \overrightarrow{OA} = \frac{1}{3}(2\mathbf{a}+\mathbf{b}) - \mathbf{a} = \frac{1}{3}(\mathbf{b}-\mathbf{a}) = \frac{1}{3}\overrightarrow{AB}$

Since $\overrightarrow{AN}$ is parallel to $\overrightarrow{AB}$ and A is the common point, hence A, B and N are collinear. B1

Since P is on AB, $\overrightarrow{OP} = \mathbf{a} + \lambda(\mathbf{b}-\mathbf{a}), \textit{where } \lambda \in \mathbb{R}$

$\overrightarrow{MP} \cdot \overrightarrow{AB} = 0$

$$\left[\mathbf{a} + \lambda(\mathbf{b}-\mathbf{a}) - \frac{1}{2}(2\mathbf{a}+\mathbf{b})\right] \cdot (\mathbf{b}-\mathbf{a}) = 0$$

$$\left[\left(\lambda - \frac{1}{2}\right)\mathbf{b} - \lambda\mathbf{a}\right] \cdot (\mathbf{b}-\mathbf{a}) = 0$$

$$\left(\lambda - \frac{1}{2}\right)|\mathbf{b}|^2 - \left(\lambda - \frac{1}{2}\right)\mathbf{a}\cdot\mathbf{b} - \lambda\mathbf{a}\cdot\mathbf{b} + \lambda|\mathbf{a}|^2 = 0$$

But $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos AOB = 2\times 3\cos 60^o = 3$

Hence, $9\left(\lambda - \frac{1}{2}\right) - 3\left(\lambda - \frac{1}{2}\right) - 3\lambda + 4\lambda = 0$

$$\lambda = \frac{3}{7}$$

$$\overrightarrow{OP} = \mathbf{a} + \frac{3}{7}(\mathbf{b}-\mathbf{a}) = \frac{1}{7}(4\mathbf{a}+3\mathbf{b})$$

<u>Alternative method:</u>

$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos AOB = 2\times 3\cos 60^o = 3$

Using cosine formula $|\mathbf{b}\text{-}\mathbf{a}| = \sqrt{|\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos 60} = \sqrt{7}$

$$\overrightarrow{AP} = \left(\overrightarrow{AM}\cdot\frac{\overrightarrow{AB}}{|\overrightarrow{AB}|}\right)\frac{\overrightarrow{AB}}{|\overrightarrow{AB}|}$$

$$= \left(\frac{1}{2}\mathbf{b}\cdot\frac{(\mathbf{b}-\mathbf{a})}{|\mathbf{b}-\mathbf{a}|}\right)\frac{\mathbf{b}-\mathbf{a}}{|\mathbf{b}-\mathbf{a}|}$$

$$= \frac{1}{2}\left(\frac{\mathbf{b}\cdot\mathbf{b}\text{-}\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}-\mathbf{a}|^2}\right)(\mathbf{b}-\mathbf{a})$$

$$= \frac{1}{2}\cdot\frac{6}{7}(\mathbf{b}-\mathbf{a})$$

$$= \frac{3}{7}(\mathbf{b}-\mathbf{a})$$

$$\overrightarrow{OP} = \mathbf{a} + \frac{3}{7}(\mathbf{b}-\mathbf{a}) = \frac{1}{7}(4\mathbf{a}+3\mathbf{b})$$

5	1. Let $\theta=\sin^{-1}\frac{y}{r}\Rightarrow\sin\theta=\frac{y}{r}\Rightarrow\cos\theta=\sqrt{1-\frac{y^2}{r^2}}$, Diff wrt y: $\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}y}=\frac{1}{r}\Rightarrow\frac{\mathrm{d}\theta}{\mathrm{d}y}=\frac{1}{r\cos\theta}$ $\therefore m=\tan\theta\Rightarrow\frac{\mathrm{d}m}{\mathrm{d}y}=\sec^2\theta\frac{\mathrm{d}\theta}{\mathrm{d}y}=\frac{1}{r\cos^3\theta}$ ie, $\frac{\mathrm{d}m}{\mathrm{d}y}=\frac{1}{r\left(\frac{r^2-y^2}{r^2}\right)^{\frac{3}{2}}}$ $=\frac{r^2}{\left(r^2-y^2\right)^{\frac{3}{2}}}$ Using $\frac{\mathrm{d}m}{\mathrm{d}t}=\frac{\mathrm{d}m}{\mathrm{d}y}\times\frac{\mathrm{d}y}{\mathrm{d}t}$, we have $\frac{\mathrm{d}m}{\mathrm{d}t}=\frac{r^2}{\left(r^2-y^2\right)^{\frac{3}{2}}}\times\frac{r}{1000}=\frac{r^3}{10^3\left(\sqrt{r^2-y^2}\right)^3}$ $=\left(\frac{r}{10\sqrt{r^2-y^2}}\right)^3$ $\frac{\mathrm{d}m}{\mathrm{d}t}$ is the rate of change of the gradient of the line OP **Alternate method 1:** $m=\tan\left(\sin^{-1}\frac{y}{r}\right)$ $\frac{\mathrm{d}m}{\mathrm{d}y}=\sec^2\left(\sin^{-1}\frac{y}{r}\right)\left[\frac{1}{\sqrt{1-\left(\frac{y}{r}\right)^2}}\left(\frac{1}{r}\right)\right]$ $=\frac{1}{\cos^2\left(\sin^{-1}\frac{y}{r}\right)}\left[\frac{1}{\sqrt{r^2-y^2}}\right]$ $=\frac{1}{\cos^2\theta}\left[\frac{1}{\sqrt{r^2-y^2}}\right]$ where $\theta=\sin^{-1}\frac{y}{r}\Rightarrow\sin\theta=\frac{y}{r}$ $=\frac{1}{(x/r)^2}\left[\frac{1}{\sqrt{r^2-y^2}}\right]=\frac{r^2}{x^2}\left[\frac{1}{\sqrt{r^2-y^2}}\right]$

$$= \frac{r^2}{r^2 - y^2}\left[\frac{1}{\sqrt{r^2 - y^2}}\right]$$

$$= \frac{r^2}{\left(r^2 - y^2\right)^{3/2}}$$

Using $\frac{\mathrm{d}m}{\mathrm{d}t} = \frac{\mathrm{d}m}{\mathrm{d}y} \times \frac{\mathrm{d}y}{\mathrm{d}t}$

$$= \frac{r^2}{\left(r^2 - y^2\right)^{3/2}} \times \frac{r}{1000}$$

$$= \frac{r^3}{10^3\left(r^2 - y^2\right)^{3/2}}$$

$$= \left(\frac{r}{10\sqrt{r^2 - y^2}}\right)^3$$

$\frac{\mathrm{d}m}{\mathrm{d}t}$ is the rate of change of the gradient of the line OP.

Alternate method 2:

$$m = \tan\left(\sin^{-1}\frac{y}{r}\right)$$

$$\tan^{-1} m = \sin^{-1}\frac{y}{r}$$

$$\left(\frac{1}{1+m^2}\right)\frac{\mathrm{d}m}{\mathrm{d}y} = \frac{1}{\sqrt{1-\left(\frac{y}{r}\right)^2}}\left(\frac{1}{r}\right)$$

$$\frac{\mathrm{d}m}{\mathrm{d}y} = \frac{1+m^2}{\sqrt{r^2 - y^2}}$$

$$= \frac{1+\tan^2\theta}{\sqrt{r^2 - y^2}} \qquad \text{where } \theta = \sin^{-1}\frac{y}{r} \;\Rightarrow\; \sin\theta = \frac{y}{r}$$

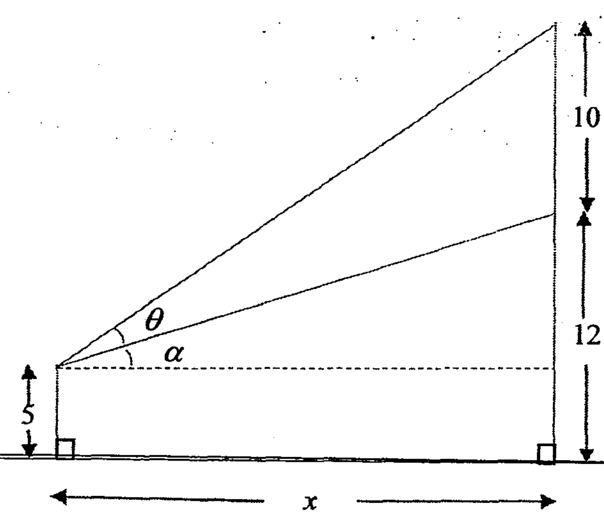
$$= \frac{1+\left(y/x\right)^2}{\sqrt{r^2 - y^2}}$$

$$= \frac{1+\frac{y^2}{x^2}}{\sqrt{r^2 - y^2}}$$

$$= \frac{1+\frac{y^2}{r^2 - y^2}}{\sqrt{r^2 - y^2}}$$

	$=\dfrac{\left(r^2-y^2\right)+y^2}{\left(r^2-y^2\right)\sqrt{r^2-y^2}}$ $=\dfrac{r^2}{\left(r^2-y^2\right)^{3/2}}$ Second part is similar to the above.
	Alternate method 3: $m=\tan\left(\sin^{-1}\dfrac{y}{r}\right)=\tan\theta,\quad \text{where } \theta=\sin^{-1}\dfrac{y}{r}$ $m=\dfrac{y}{x}=\dfrac{y}{\sqrt{r^2-y^2}}$ $\dfrac{dm}{dy}=\dfrac{\sqrt{r^2-y^2}-y\left(\frac{1}{2}\right)\left(r^2-y^2\right)^{-1/2}(-2y)}{\left(r^2-y^2\right)}$ $=\dfrac{\sqrt{r^2-y^2}+y^2\left(r^2-y^2\right)^{-1/2}}{\left(r^2-y^2\right)}$ $=\dfrac{\left(r^2-y^2\right)+y^2}{\left(r^2-y^2\right)^{1/2}\left(r^2-y^2\right)}$ $=\dfrac{r^2}{\left(r^2-y^2\right)^{3/2}}$ Second part is similar to the above.
6(i)	$\dfrac{r^2+r-1}{(r+2)!}=\dfrac{A}{r!}+\dfrac{B}{(r+1)!}+\dfrac{C}{(r+2)!}$ $\dfrac{r^2+r-1}{(r+2)!}=\dfrac{A(r+1)(r+2)+B(r+2)+C}{(r+2)!}$ By comparing coefficients, $A=1, B=-2, C=1$ $\therefore \dfrac{r^2+r-1}{(r+2)!}=\dfrac{1}{r!}-\dfrac{2}{(r+1)!}+\dfrac{1}{(r+2)!}$

(ii)	$$\sum_{r=1}^{n}\frac{r^2+r-1}{(r+2)!}=\sum_{r=1}^{n}\left(\frac{1}{r!}-\frac{2}{(r+1)!}+\frac{1}{(r+2)!}\right)$$ $$=\Big[\frac{1}{1!}-\frac{2}{2!}+\frac{1}{3!}$$ $$+\frac{1}{2!}-\frac{2}{3!}+\frac{1}{4!}$$ $$+\frac{1}{3!}-\frac{2}{4!}+\frac{1}{5!}$$ $$+\frac{1}{4!}-\frac{2}{5!}+\frac{1}{6!}$$ $$\vdots$$ $$+\frac{1}{(n-2)!}-\frac{2}{(n-1)!}+\frac{1}{n!}$$ $$+\frac{1}{(n-1)!}-\frac{2}{n!}+\frac{1}{(n+1)!}$$ $$+\frac{1}{n!}-\frac{2}{(n+1)!}+\frac{1}{(n+2)!}\Big]$$ $$=\frac{1}{2}-\frac{1}{(n+1)!}+\frac{1}{(n+2)!}$$
(iii)	Since $r^2-1<r^2+r-1$ for $r>0$, so we have $$\sum_{r=1}^{n}\frac{r^2-1}{(r+2)!}<\sum_{r=1}^{n}\frac{r^2+r-1}{(r+2)!}=\frac{1}{2}-\frac{1}{(n+1)!}+\frac{1}{(n+2)!}$$ $$=\frac{1}{2}-\left[\frac{n+1}{(n+2)!}\right]<\frac{1}{2}$$ $$\left(\because \frac{n+1}{(n+2)!}>0 \text{ as } n\in\mathbb{Z}^+\right)$$
(iv)	$$\sum_{r=4}^{n}\frac{r^2-3r+1}{r!}$$ Replace r with $(k+2)$ $$=\sum_{k=2}^{n-2}\frac{(k+2)^2-3(k+2)+1}{(k+2)!}$$

	$=\sum_{k=2}^{n-2}\frac{k^2+k-1}{(k+2)!}$ $=\sum_{k=1}^{n-2}\frac{k^2+k-1}{(k+2)!}-\left(\frac{1}{6}\right)$ $=\frac{1}{2}-\frac{1}{(n-1)!}+\frac{1}{n!}-\frac{1}{6}$ $=\frac{1}{3}-\frac{1}{(n-1)!}+\frac{1}{n!}$ Alternatively, consider $\sum_{r=1}^{n}\frac{r^2+r-1}{(r+2)!}$ and sub. $r=k-2$. So we have $\sum_{r=1}^{n}\frac{r^2+r-1}{(r+2)!}=\sum_{k=3}^{n+2}\frac{k^2-3k+1}{k!}$. $\Rightarrow\sum_{k=3}^{n+2}\frac{k^2-3k+1}{k!}=\frac{1}{2}-\frac{1}{(n+1)!}+\frac{1}{(n+2)!}$ $\Rightarrow\sum_{k=4}^{n+2}\frac{k^2-3k+1}{k!}=\frac{1}{2}-\frac{1}{(n+1)!}+\frac{1}{(n+2)!}-\left(\frac{3^2-3(3)+1}{3!}\right)$ $=\frac{1}{3}-\frac{1}{(n+1)!}+\frac{1}{(n+2)!}$ $\therefore\sum_{k=4}^{n}\frac{k^2-3k+1}{k!}=\frac{1}{3}-\frac{1}{(n-1)!}+\frac{1}{n!}$
7	

From diagram, $\tan\alpha = \frac{7}{x}$

and $\tan(\theta+\alpha) = \frac{17}{x}$

$$\tan\theta = \tan((\theta+\alpha)-\alpha)$$
$$= \frac{\tan(\theta+\alpha)-\tan\alpha}{1-\tan(\theta+\alpha)\tan\alpha}$$
$$= \frac{\frac{17}{x}-\frac{7}{x}}{1-\frac{17}{x}(\frac{7}{x})}$$
$$= \frac{\frac{10}{x}}{1-\frac{119}{x^2}} = \frac{10x}{x^2+119} \quad \text{(shown)}$$

Differentiating wrt x,

$$\sec^2\theta\frac{d\theta}{dx} = \frac{(119+x^2)(10)-10x(2x)}{(119+x^2)^2}$$
$$= \frac{1190-10x^2}{(119+x^2)^2}$$

For maximum angle,

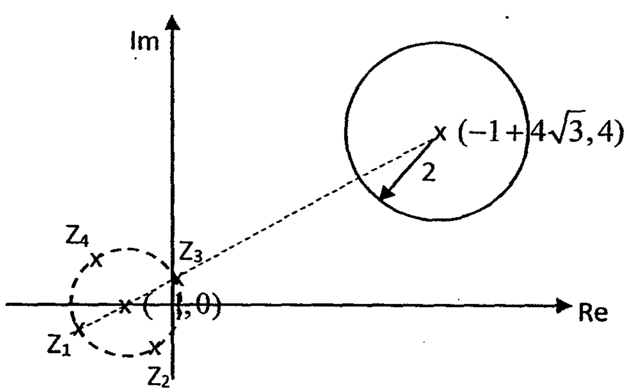
$$\frac{d\theta}{dx} = 0 \Rightarrow 10x^2 = 1190$$

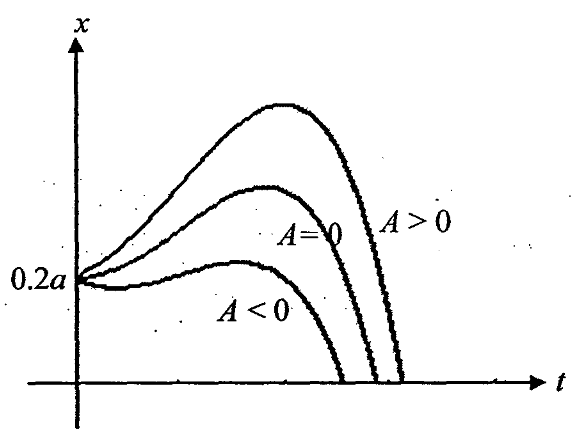
Since $x>0$, $x=\sqrt{119}$

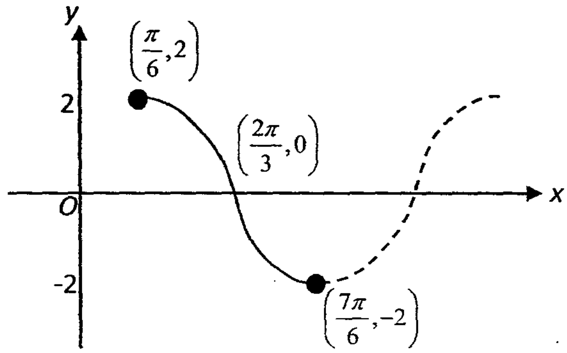
Since $\sec^2\theta > 0$,

	$(\sqrt{119})^-$	$(\sqrt{119})$	$(\sqrt{119})^+$
Sign of $\frac{d\theta}{dx}$	+	0	-
slope	/	—	\

Using sign test of derivatives at vicinity of $x=\sqrt{119}$, it can be shown that the angle is maximum.

	Therefore, $\tan\theta = \dfrac{10\sqrt{119}}{119+119} = \dfrac{5\sqrt{119}}{119} \Rightarrow \theta \approx 25^{o}$
8	*(i)* $w^4 = -1+\sqrt{3}\,\mathrm{i}$ $w^4 = 2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$ $w = 2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\frac{1}{4}\left(\frac{2\pi}{3}+2k\pi\right)}$ $k=-2,-1,0,1,$ $w = 2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(-\frac{5\pi}{6}\right)},\ 2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{3}\right)},\ 2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{6}\right)},\ 2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}\right)}$ *(ii)* $(1+z)^4+1-\mathrm{i}\sqrt{3}=0 \Rightarrow (1+z)^4 = -1+\mathrm{i}\sqrt{3}$ $z = w-1$ $z = 2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(-\frac{5\pi}{6}\right)}-1,\ 2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{3}\right)}-1,\ 2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{6}\right)}-1,\ 2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}\right)}-1$ $z_1 = 2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(-\frac{5\pi}{6}\right)}-1,\ z_2 = 2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{3}\right)}-1, z_3 = 2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{6}\right)}-1,\ z_4 = 2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}\right)}-1$  *(iii)* *Least possible* $Z_iQ = Z_3Q = 8-2-2^{1/4} = 6-2^{1/4}$ *Greatest possible* $Z_iQ = Z_1Q = 8+2+2^{1/4} = 10-2^{1/4}$
9	(i) $\dfrac{\mathrm{d}x}{\mathrm{d}t} = x(a-x)$ $\int \dfrac{\mathrm{d}x}{x(a-x)} = \int \mathrm{d}t$ $\int \dfrac{\frac{1}{a}}{x}+\dfrac{\frac{1}{a}}{a-x}\,\mathrm{d}x = \int \mathrm{d}t$ $\dfrac{1}{a}\ln\lvert x\rvert - \dfrac{1}{a}\ln\lvert a-x\rvert = t+C$ $\dfrac{1}{a}\ln\left\lvert\dfrac{x}{a-x}\right\rvert = t+C$ $\ln\left\lvert\dfrac{x}{a-x}\right\rvert = at+aC$ $\dfrac{x}{a-x} = A\mathrm{e}^{at}$, where $A = \pm e^{aC}$

	When $t=0, x=0.2a$ $\frac{0.2a}{0.8a}=A$ $A=\frac{1}{4}$ When $t=\ln 2, x=0.5a$ $\frac{0.5a}{0.5a}=\frac{1}{4}e^{a\ln 2}$ $4=e^{a\ln 2}$ $2^a=4$ $a=2$ Subst. values of A and a, $\frac{x}{2-x}=\frac{1}{4}e^{2t}$ $4x=2e^{2t}-xe^{2t}$ $x\left(4+e^{2t}\right)=2e^{2t}$ $x=\frac{2e^{2t}}{\left(4+e^{2t}\right)}=\frac{2}{4e^{-2t}+1}$ (shown) (ii) $\frac{d^2x}{dt^2}=10-9t^2$ $\frac{dx}{dt}=10t-3t^3+A$ $x=5t^2-\frac{3}{4}t^4+At+B$ When $t=0, x=0.2a$ $B=0.2a$ $x=5t^2-\frac{3}{4}t^4+At+0.2a$ 
10 (a)(i)	$\sin x+\sqrt{3}\cos x=2\sin\left(x+\frac{\pi}{3}\right)$

(ii)	 For f to have an inverse, f must be one-to-one. Hence largest $k=\frac{7\pi}{6}$. Consider $y=2\sin\left(x+\frac{\pi}{3}\right) \Rightarrow x=\sin^{-1}\left(\frac{y}{2}\right)-\frac{\pi}{3}$ So $\mathrm{f}^{-1}: x \mapsto \sin^{-1}\left(\frac{x}{2}\right)-\frac{\pi}{3}, \quad -2 \le x \le 2$. For $\mathrm{ff}^{-1}(x)=\mathrm{f}^{-1}\mathrm{f}(x)$, we must have $D_f \cap D_{f^{-1}}$. So the solution set is $x \in \left[\frac{\pi}{6},2\right]$.
(b)(i)	Consider $y=2-\frac{5x}{1+x^2}$ $\Rightarrow 2-y=\frac{5x}{1+x^2}$ $\Rightarrow (2-y)\left(1+x^2\right)=5x$ $\Rightarrow (2-y)x^2-5x+(2-y)=0$ $D=(-5)^2-4(2-y)(2-y)\ge 0$ $25-4(2-y)^2 \ge 0$ $(5-2(2-y))(5+2(2-y))\ge 0$

	$(1+2y)(9-2y)\geq 0$ $\therefore -\frac{1}{2}\leq y\leq \frac{9}{2}$ So range of g $=\left[-\frac{1}{2},\frac{9}{2}\right]$
(ii)	$g(x)=2-\frac{5x}{1+x^2}$ $g\left(-\frac{x}{2}\right)=2-\frac{5\left(-\frac{x}{2}\right)}{1+\left(-\frac{x}{2}\right)^2}=2+\frac{10x}{4+x^2}$ $g\left(-\frac{x}{2}\right)-2=\frac{10x}{4+x^2}$ Scale the graph of g by factor 2 parallel to the x-axis followed by a reflection in the y-axis followed by a translation of -2 units in the direction of y-axis. Or $g(x)=2-\frac{5x}{1+x^2} \quad \rightarrow -g\left(\frac{x}{2}\right)=-\left[2-\frac{5\left(\frac{x}{2}\right)}{1+\left(\frac{x}{2}\right)^2}\right]=-2+\frac{10x}{4+x^2}$ Scale the graph of g $\rightarrow -g\left(\frac{x}{2}\right)+2=\frac{10x}{4+x^2}$ by factor 2 parallel to the x-axis followed by a reflection in the x-axis followed by a translation of 2 units in the direction of y-axis.
11	(i) $l_1:\mathbf{r}=\begin{pmatrix}1\\2\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\a\\2\end{pmatrix},\lambda\in\mathbb{R}$ $l_2:\mathbf{r}=\begin{pmatrix}3\\0\\5\end{pmatrix}+\mu\begin{pmatrix}1\\2\\3\end{pmatrix},\mu\in\mathbb{R}$ Equating the equations of the 2 lines, $1+\lambda=3+\mu$ $2+a\lambda=2\mu$ $1+2\lambda=5+3\mu$

Solving, $\mu = 0, \lambda = 2$

$\therefore a = -1$

(ii) Coordinates of B is (3, 0, 5)

Shortest distance from B to p_1 $= \dfrac{\left|\overrightarrow{AB} \cdot \begin{pmatrix}1\\2\\3\end{pmatrix}\right|}{\sqrt{14}}$

$$= \frac{\left|\left[\begin{pmatrix}3\\0\\5\end{pmatrix} - \begin{pmatrix}1\\2\\1\end{pmatrix}\right] \cdot \begin{pmatrix}1\\2\\3\end{pmatrix}\right|}{\sqrt{14}} = \frac{\left|\begin{pmatrix}2\\-2\\4\end{pmatrix} \cdot \begin{pmatrix}1\\2\\3\end{pmatrix}\right|}{\sqrt{14}} = \frac{10}{\sqrt{14}}$$

Alternative solution 1:

Equation of plane p_1 : $\mathbf{r} \cdot \dfrac{\begin{pmatrix}1\\2\\3\end{pmatrix}}{\sqrt{14}} = \dfrac{\begin{pmatrix}1\\2\\2\end{pmatrix} \cdot \begin{pmatrix}1\\2\\3\end{pmatrix}}{\sqrt{14}} = \dfrac{8}{\sqrt{14}}$

Equation of plane parallel to p_1 and containing point B is

$$\mathbf{r} \cdot \frac{\begin{pmatrix}1\\2\\3\end{pmatrix}}{\sqrt{14}} = \frac{\begin{pmatrix}3\\0\\5\end{pmatrix} \cdot \begin{pmatrix}1\\2\\3\end{pmatrix}}{\sqrt{14}} = \frac{18}{\sqrt{14}}$$

Shortest distance from B to p_1 is $\dfrac{18}{\sqrt{14}} - \dfrac{8}{\sqrt{14}} = \dfrac{10}{\sqrt{14}}$

Alternative solution 2:

Let F be the foot of perpendicular from point B to plane p_1

Line l_2 meets plane p_1 at point F

$$\overrightarrow{OF} = \begin{pmatrix}3\\0\\5\end{pmatrix} + \lambda \begin{pmatrix}1\\2\\3\end{pmatrix} \text{ for some } \lambda \in \mathbb{R}$$

$$\left[\begin{pmatrix}3\\0\\5\end{pmatrix} + \mu \begin{pmatrix}1\\2\\3\end{pmatrix}\right] \cdot \begin{pmatrix}1\\2\\3\end{pmatrix} = 8$$

$3 + 15 + \mu(1 + 4 + 9) = 8$

$\mu = -\dfrac{5}{7}$

Shortest distance from B to p_1 is $\left|\overrightarrow{BF}\right| = \left|-\dfrac{5}{7}\begin{pmatrix}1\\2\\3\end{pmatrix}\right| = \dfrac{5}{7}\sqrt{1+4+9} = \dfrac{5}{7}\sqrt{14}$

$$|\overrightarrow{AB}| = \sqrt{2^2 + (-2)^2 + 4^2} = \sqrt{24}$$

$$\sin\theta = \frac{10}{\sqrt{14}}\frac{1}{\sqrt{24}}$$

$\theta \approx 33.1^0$

(iii)A normal vector to p_2 is $\begin{pmatrix}1\\2\\3\end{pmatrix} \times \begin{pmatrix}1\\-1\\2\end{pmatrix} = \begin{pmatrix}7\\1\\-3\end{pmatrix}$

Equation of p_2 is $r\cdot\begin{pmatrix}7\\1\\-3\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix}\cdot\begin{pmatrix}7\\1\\-3\end{pmatrix} = 6$

Cartesian equation of p_2 is $7x + y - 3z = 6$

(iv) Equation of x-y plane is $z = 0$

Let α be the acute angle between p_2 and x-y plane.

$$\cos\alpha = \frac{\left|\begin{pmatrix}7\\1\\-3\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right|}{\sqrt{59}} = \frac{3}{\sqrt{59}}$$

$\therefore \alpha = 67.0^{\circ}$

JC2 PRELIMINARY EXAMINATION

Higher 2

MATHEMATICS **9740/02**

Paper 2 **22nd September 2016**

3 Hours

Additional Materials: Cover Sheet
Answer Papers
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

Section A: Pure Mathematics [40 marks]

1 The first four terms of a sequence of numbers are 10, 6, 5 and 7. S_n is the sum of the first n terms of this sequence. Given that S_n is a cubic polynomial in n, find S_n in terms of n. **[4]**

Show that $U_n = \frac{3}{2}n^2 - \frac{17}{2}n + 17$, where U_n denotes the n^{th} term of the sequence. **[2]**

Find the set of values of n for which $S_n < 3U_n$. **[2]**

2 On separate diagrams, draw sketches of the graphs of

(i) $y = \dfrac{x^2(3-x)}{1+x}$,

(ii) $y^2 = \dfrac{x^2(3-x)}{1+x}$,

including the coordinates of the points where the graphs cross the axes and the equations of any asymptotes. You should show the features of the graphs at the points where it crosses the x-axis clearly.

Show that the area of the region enclosed by the graph in **(ii)** may be expressed in the form $2\displaystyle\int_0^3 \frac{3x-x^2}{\sqrt{(4-(x-1)^2)}}\,\mathrm{d}x$.

By using the substitution $x-1 = 2\sin\theta$, evaluate this area exactly. **[10]**

3 **(a)** Solve $z^3 - 2(2-\mathrm{i})z^2 + (8-3\mathrm{i})z - 5 + \mathrm{i} = 0$, given that one of the three roots is real. **[5]**

(b) The complex number u is given by $u = \cos\theta + \mathrm{i}\sin\theta$, where $0 < \theta < \frac{\pi}{2}$.

(i) Show that $1 - u^2 = -2\mathrm{i}u\sin\theta$ and hence find the modulus and argument of $1-u^2$ in terms of θ. **[4]**

(ii) Given that $\left(1-u^2\right)^{10}$ is real and negative, find the possible values of θ in terms of π. **[3]**

4 [In this question, you may use the result that for a circle with radius r, a sector with angle θ has arc length $r\theta$ and area $\frac{1}{2}r^2\theta$.]

(a) A circle of radius r is divided into 16 sectors of decreasing arc length. Let L_n and A_n be the arc length and the area of the nth sector respectively. Suppose L_n is an arithmetic sequence with first term r and common difference d.

(i) Show that $d = \left(\frac{\pi - 8}{60}\right)r$. **[2]**

(ii) Show that A_n is an arithmetic sequence. **[3]**

(b) Let G_n be the area of a sector of a circle with radius a. Suppose that G_n is a geometric sequence with first term a and common ratio r, where $0 < r < 1$.

(i) If N sectors are needed to form the circle, show that r satisfies the equation

$$r^N - \pi a r + (\pi a - 1) = 0.$$ **[3]**

(ii) If an infinite number of sectors are needed to form the circle, find r in terms of a. **[2]**

Section B: Statistics [60 marks]

5 A company sells a certain brand of baby milk powder and would like to gather feedback on their product. Explain why quota sampling is appropriate in this situation and describe briefly how a sample of 50 could be chosen using quota sampling. **[3]**

The company wishes to randomly reward 5 customers with free milk vouchers through a lucky draw. Suppose that 2000 customers qualify for the draw, show that there will be equal probability of a particular customer being the first to be selected or the third to be selected for the free milk vouchers. **[2]**

6 The mass, in grams, of an ice-cube has the distribution $N(\mu, \sigma^2)$. The mean mass of a random sample of n ice-cubes is denoted by $\overline{X}$. It is given that $P(\overline{X} < 35.0) = 0.97725$ and $P(\overline{X} \geq 20.0) = 0.84134$.

(i) Obtain an expression for σ in terms of n. **[3]**

(ii) Find $P(\overline{X} > 32)$. **[2]**

Assume now that the mass of an ice-cube has the distribution $N(25, 50)$.

An ice dispenser discharges 15 ice cubes each time into a cup. State the distribution of the mass of a discharge of 15 ice cubes. **[1]**

(iii) Find the mass exceeded by 10% of these discharges, correct to 1 decimal place. **[2]**

(iv) Find the probability that the mass of the first discharge of ice-cubes is more than the second discharge. **[2]**

7 A team of 5 men and 5 women is to be picked from 8 men and 9 women such that two of the 9 women, Ann and Lucy, must both be selected or not at all. Find the number of ways in which this can be done. **[2]**

Assume now the team is selected and Ann, Carrie and Lucy are included.

(i) The selected team is to form a queue. Find the number of possible arrangements if Ann and Lucy are to occupy both the second and the sixth positions and no two people of the same gender are to stand next to each other. **[3]**

(ii) On another occasion, the selected team is required to be seated at a round table with 10 chairs of different colours. If only Carrie can be seated between Ann and Lucy, find the number of possible arrangements. **[3]**

8 Two teams, the Ramblers and the Strollers, meet annually for a quiz which always has a winner. If the Ramblers wins the quiz, the probability of them winning the following year is 0.7. If the Strollers wins the quiz, the probability of them winning the following year is 0.5.

The Ramblers won the quiz in 2015.

(i) Find the probability that the Strollers will win in 2018. **[2]**

(ii) If the Strollers were to win in 2018, what is the probability that it will be their first win for at least three years since 2015? **[2]**

(iii) Assuming that the Strollers wins in 2018, find the smallest value of n such that the probability of the Ramblers winning the quiz for n consecutive years after 2018 is less than 5%. **[3]**

9 It is believed that the probability p of a randomly chosen pregnant woman giving birth to a Down Syndrome child is related to the woman's age x, in years. The table gives observed values of p for 6 different values of x.

x	20	25	30	35	40	45
p	0.00023	0.00067	0.00125	0.00333	0.01000	0.03330

(i) Sketch the scatter diagram for the given data. **[1]**

(ii) Find, correct to 4 decimal places, the product moment correlation coefficient between

(a) p and x ,

(b) $\ln p$ and x ,

(c) p and x^2 . **[2]**

(iii) Using the most appropriate case from part **(ii)**, find the equation which best models the probability of a pregnant woman giving birth to a Down Syndrome child at different ages. **[2]**

(iv) Hence, estimate the expected number of children with Down Syndrome that will be born to 5000 randomly chosen pregnant women of age 32. **[2]**

10 At an early stage in analysing the marks, x, scored by a large number of candidates in an examination paper, the Examination Board takes the scores from a random sample of 250 candidates. The results are summarised as follows:

$$\sum x = 11872 \quad \text{and} \quad \sum x^2 = 646193$$

(i) Calculate unbiased estimates of the population mean and variance to 3 decimal places. **[2]**

(ii) In a 1-tail test of the null hypothesis $\mu = 49.5$, the alternative hypothesis is accepted. State the alternative hypothesis and find an inequality satisfied by the significance level of the test. **[4]**

(iii) It is subsequently found that the population mean and standard deviation for the examination paper are 45.292 and 18.761 respectively. Find the probability that in a random sample of size 250, the sample mean is at least as high as the one found in the sample above. **[2]**

11 On a typical weekday morning, customers arrive at the post office independently and at a rate of 3 per 10 minute period.

(i) State, in context, a condition needed for the number of customers who arrived at the post office during a randomly chosen period of 30 minutes to be well modelled by a Poisson distribution. **[1]**

(ii) Find the probability that no more than 4 customers arrive between 11.00 a.m. and 11.30 a.m. **[2]**

(iii) The period from 11.00 a.m. to 11.30 a.m. on a Tuesday morning is divided into 6 periods of 5 minutes each. Find the probability that no customers arrive in at most one of these periods. **[2]**

The post office opens for 3.5 hours each in the morning and afternoon and it is noted that on a typical weekday afternoon, customers arrive at the post office independently and at a rate of 1 per 10 minute period. Arrivals of customers take place independently at random times.

(iv) Show that the probability that the number of customers who arrived in the afternoon is within one standard deviation from the mean is 0.675, correct to 3 decimal places. **[3]**

(v) Find the probability that more than 38 customers arrived in a morning given that a total of 40 customers arrived in a day. **[4]**

(vi) Using a suitable approximation, estimate the probability that more than 100 customers arrive at the post office in a day. **[3]**

Solutions to P2 Prelim 2016:

1.

$S_n = an^3 + bn^2 + cn + d$

$10 = a + b + c + d$

$16 = 8a + 4b + 2c + d$

$21 = 27a + 9b + 3c + d$

$28 = 64a + 16b + 4c + d$

$a = \frac{1}{2},\ b = -\frac{7}{2},\ c = 13,\ d = 0$

$S_n = \frac{1}{2}n^3 - \frac{7}{2}n^2 + 13n$

$$\begin{aligned} U_n &= S_n - S_{n-1} \\ &= \frac{1}{2}n^3 - \frac{7}{2}n^2 + 13n - \left(\frac{1}{2}(n-1)^3 - \frac{7}{2}(n-1)^2 + 13(n-1)\right) \\ &= \frac{3}{2}n^2 - \frac{17}{2}n + 17 \end{aligned}$$

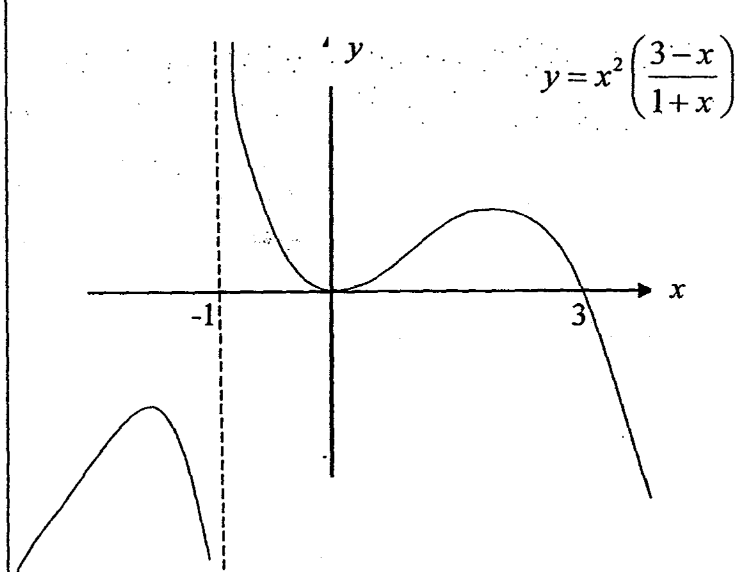
$S_n < 3U_n$

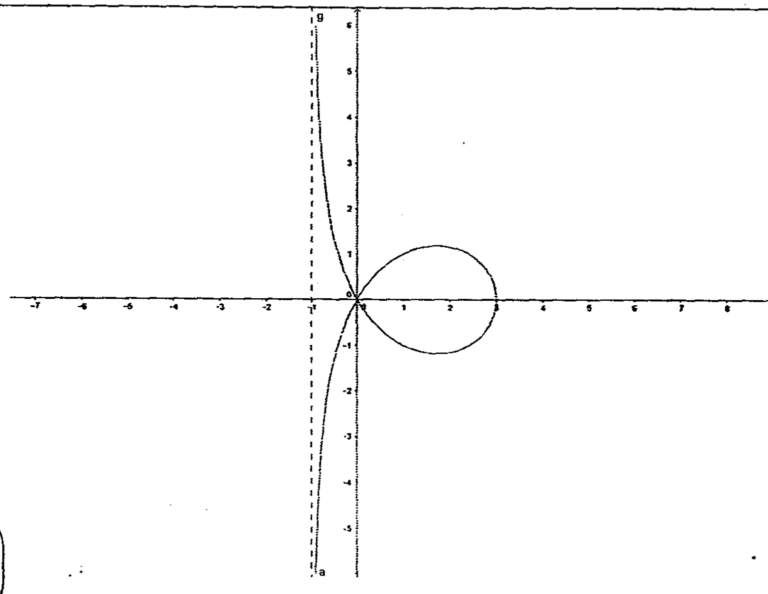
$\frac{1}{2}n^3 - \frac{7}{2}n^2 + 13n < 3\left(\frac{3}{2}n^2 - \frac{17}{2}n + 17\right)$

n	S_n	$3U_n$
1	10	30
2	16	18
7	91	93

From GC, $\{n : n \in \mathbb{Z}^+, n = 1, 2, 7\}$

2





$y^2 = x^2\left(\frac{3-x}{1+x}\right)$

$$\text{Area enclosed by loop} = 2\int_0^3 x\sqrt{\frac{3-x}{1+x}}\,\mathrm{d}x$$

$$= 2\int_0^3 x\sqrt{\frac{(3-x)(3-x)}{(1+x)(3-x)}}\,\mathrm{d}x$$

$$= 2\int_0^3 \frac{x(3-x)}{\sqrt{3+2x-x^2}}\,\mathrm{d}x$$

$$= 2\int_0^3 \frac{3x-x^2}{\sqrt{(4-(x-1)^2)}}\,\mathrm{d}x$$

Using $x-1=2\sin\theta$, we have $\frac{\mathrm{d}x}{\mathrm{d}\theta}=2\cos\theta$

Also, when $x=0$, $\sin\theta=-\frac{1}{2}\Rightarrow\theta=-\frac{\pi}{6}$

And when $x=3$, $\sin\theta=1\Rightarrow\theta=\frac{\pi}{2}$

$$\text{Therefore area} = 2\int_0^3 \frac{3x-x^2}{\sqrt{(4-(x-1)^2)}}\,\mathrm{d}x$$

$$= 2\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{3(2\sin\theta+1)-(2\sin\theta+1)^2}{\sqrt{(4-(2\sin\theta)^2)}}(2\cos\theta)\,\mathrm{d}\theta$$

$$= 2\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \left(2\sin\theta+2-4\sin^2\theta\right)\mathrm{d}\theta$$

$$=4\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\sin\theta+\left(1-2\sin^2\theta\right)\right)\mathrm{d}\theta$$

$$= 4\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} (\sin\theta+\cos 2\theta)\,\mathrm{d}\theta$$

$$= 4\left[-\cos\theta+\frac{\sin 2\theta}{2}\right]_{-\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= 4\left[0-\left(-\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{4}\right)\right]=3\sqrt{3}$$

3(a)	$z^3-2(2-\mathrm{i})z^2+(8-3\mathrm{i})z-5+\mathrm{i}=0$ Let $z=x$ be the real root. $x^3-2(2-\mathrm{i})x^2+(8-3\mathrm{i})x-5+\mathrm{i}=0$ $x^3-4x^2+2\mathrm{i}x^2+8x-3\mathrm{i}x-5+\mathrm{i}=0$ $(x^3-4x^2+8x-5)+(2x^2-3x+1)\mathrm{i}=0$ Since $z=x$ is a root, $x^3-4x^2+8x-5=0$ and $2x^2-3x+1=0$ From GC: $x=1$ Therefore, the real root is $z=1$ $z^3-2(2-\mathrm{i})z^2+(8-3\mathrm{i})z-5+\mathrm{i}=0$ $(z-1)(z^2+Az+(5-\mathrm{i}))=0$ $(z-1)(z^2+(-3+2\mathrm{i})z+(5-\mathrm{i}))=0$ $z=1$ or $z^2+(-3+2\mathrm{i})z+(5-\mathrm{i})=0$ $z=\dfrac{-(-3+2\mathrm{i})\pm\sqrt{(-3+2\mathrm{i})^2-4(5-\mathrm{i})}}{2}$ $=\dfrac{-(-3+2\mathrm{i})\pm(1-4\mathrm{i})}{2}$ $=2-3\mathrm{i}$ or $1+\mathrm{i}$ Roots: 1, $2-3\mathrm{i}$, $1+\mathrm{i}$
3(b)	$1-u^2=1-(\cos\theta+\mathrm{i}\sin\theta)^2$ $=1-\cos^2\theta+\sin^2\theta-2\mathrm{i}\sin\theta\cos\theta$ $=2\sin^2\theta-2\mathrm{i}\sin\theta\cos\theta$ $=2\sin\theta(\sin\theta-2\mathrm{i}\cos\theta)$ $=-2\mathrm{i}\sin\theta(\cos\theta+\mathrm{i}\sin\theta)$ $=-2\mathrm{i}u\sin\theta$ **Alternative** $u=\cos\theta+\mathrm{i}\sin\theta=\mathrm{e}^{\theta\mathrm{i}}$ $1-u^2=1-\mathrm{e}^{2\theta\mathrm{i}}$ $=\mathrm{e}^{\theta\mathrm{i}}(\mathrm{e}^{-\theta\mathrm{i}}-\mathrm{e}^{\theta\mathrm{i}})$ $=u(\cos\theta-\mathrm{i}\sin\theta-\mathrm{i}\sin\theta-\cos\theta)$ $=u(-2\mathrm{i}\sin\theta)$ $=-2\mathrm{i}u\sin\theta$ $\lvert 1-u^2\rvert=\lvert -2\mathrm{i}u\sin\theta\rvert=\lvert -2\sin\theta\rvert\lvert\mathrm{i}\rvert\lvert u\rvert$ $=2\sin\theta$

$$\arg\left(1-u^2\right)=\arg\left(-2iu\sin\theta\right)$$
$$=\arg\left(-2i\sin\theta\right)+\arg\left(u\right)$$
$$=-\frac{\pi}{2}+\theta$$

$\left(1-u^2\right)^{10}$ is real and negative: $\arg\left(1-u^2\right)^{10}=10\arg\left(1-u^2\right)=\left(2k+1\right)\pi, k\in\mathbb{Z}$

$$10\left(-\frac{\pi}{2}+\theta\right)=\left(2k+1\right)\pi$$
$$-5\pi+10\theta=\left(2k+1\right)\pi$$
$$\theta=\frac{\left(2k+6\right)\pi}{10}$$

$0<\theta<\frac{\pi}{2}$: $\theta=\frac{1}{5}\pi,\frac{2}{5}\pi$

Alternative

$$\left(1-u^2\right)^{10}=\left(2\sin\theta \mathrm{e}^{-\frac{\pi}{2}+\theta}\right)^{10}$$
$$=\left(2^{10}\sin^{10}\theta\right)\left(\cos\left(-5\pi+10\theta\right)+i\sin\left(-5\pi+10\theta\right)\right)$$

Since $\left(1-u^2\right)^{10}$ is real and negative, and $2^{10}\sin^{10}\theta>0$,

$\sin\left(-5\pi+10\theta\right)=0$ and $\cos\left(-5\pi+10\theta\right)<0$

$$-5\pi+10\theta=k\pi \quad, k\in\mathbb{Z}$$
$$\theta=\frac{\left(k+5\right)\pi}{10}$$

$0<\theta<\frac{\pi}{2}$: $\theta=\frac{1}{10}\pi,\frac{1}{5}\pi,\frac{3}{10}\pi,\frac{2}{5}\pi$

Only when $\theta=\frac{1}{5}\pi,\frac{2}{5}\pi$ will $\cos\left(-5\pi+10\theta\right)<0$.

Therefore, $\theta=\frac{1}{5}\pi,\frac{2}{5}\pi$.

4(a)(i) Since $S_{16}=2\pi r$, thus

$$\frac{16}{2}(2r+15d)=2\pi r$$
$$\Rightarrow 2r+15d=\frac{\pi r}{4}$$
$$\Rightarrow 15d=\left(\frac{\pi-8}{4}\right)r$$
$$\Rightarrow d=\left(\frac{\pi-8}{60}\right)r$$

(a)(ii) Since $L_n=r\theta_n$ and $A_n=\frac{1}{2}r^2\theta_n$, thus $A_n=\frac{1}{2}rL_n$.

	$A_{n+1}-A_n=\frac{1}{2}rL_{n+1}-\frac{1}{2}rL_n$ $=\frac{1}{2}r\left(L_{n+1}-L_n\right)$ $=\frac{1}{2}rd=\text{constant}$ for all $n=2,\ldots,15$ Thus A_n is an arithmetic sequence.
(b)(i)	Since $S_N=\pi a^2$, we have $\frac{a(r^N-1)}{r-1}=\pi a^2$ $\Rightarrow r^N-1=\pi ar-\pi a$ $\Rightarrow r^N-\pi ar+(\pi a-1)=0$
(b)(ii)	Since $0<r<1$, we have $r^N\to 0$ as $N\to\infty$. Thus $-\pi ar+(\pi a-1)=0$ $\Rightarrow r=\frac{\pi a-1}{\pi a}$ Alternative: Since $S_\infty=\pi a^2$, we have $\frac{a}{1-r}=\pi a^2$ $\Rightarrow 1-r=\frac{1}{\pi a}$ $\Rightarrow r=\frac{\pi a-1}{\pi a}$
5	No sampling frame. Station an interviewer at the exits of a local supermarket store during peak hours and he is free to choose 25 male and 25 female customers who buy the products. P(a particular consumer is the first to be selected) = $\frac{1}{2000}$ P(a particular consumer is the third to be selected) $=\frac{1999}{2000}\frac{1998}{1999}\frac{1}{1998}=\frac{1}{2000}$ (shown)
6	(i) $P(\bar{X}<35.0)=0.97725$ $P(Z<\frac{35-\mu}{\sigma/\sqrt{n}})=0.97725$ $\frac{35-\mu}{\sigma/\sqrt{n}}=2$ -----[1] $P(\bar{X}<20.0)=0.15866$

$P(Z < \dfrac{20-\mu}{\sigma/\sqrt{n}}) = 0.15866$

$\dfrac{20-\mu}{\sigma/\sqrt{n}} = -1 ----[2]$

Eqn [1] −[2]: $\dfrac{3\sigma}{\sqrt{n}} = 15$

$\sigma = 5\sqrt{n}$

(ii)

$\mu = 25$, $\bar{X} \sim N(25, 5^2)$ since $\dfrac{\sigma}{\sqrt{n}} = 5$

$P(\bar{X} > 32) = 0.0808$

let M be the mass of a randomly chosen discharge of 15 ice cubes. $M \sim N(375, 750)$

(iii)

$P(M > a) = 0.1$
$P(M \le a) = 0.9$
$a = 410.1$

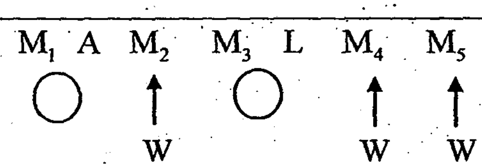
(iv)

$M_1 - M_2 \sim N(0, 1500)$
$P(M_1 > M_2) = P(M_1 - M_2 > 0) = 0.5$

7 5M, [3W, A, L]:
No. of selections = $({}^8C_5)({}^7C_3)$ = 1960
5M, 5W (exclude A and L):
No. of selections = $({}^8C_5)({}^7C_5)$ = 1176

Total number of selections
= $({}^8C_5)({}^7C_3) + ({}^8C_5)({}^7C_5)$
= 1960 + 1176
= 3136

(i)



Arrange 5M: No. of ways = 5! = 120

Arrange A and L: No. of ways = 2! = 2

Arrange 3W: No. of ways = 1(3!) = 6

Total no. of arrangements = $(5!)\left[(2!)\times(3!)\right]$

= 1440

(ii) No of arrangements $(8-1)!(2)(10)$ = 100800

8(i).	P(Strollers will win in 2018) $=(0.7)(0.7)(0.3)+(0.7)(0.3)(0.5)$ $+(0.3)(0.5)(0.3)+(0.3)(0.5)(0.5)$ $=0.372$
(ii)	Let event A denotes "Strollers first win for at least three years" and event B denotes "Strollers win in 2018" $\mathrm{P}(A\mid B)=\dfrac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}$ $=\dfrac{(0.7)(0.7)(0.3)}{0.372}$ $=0.39516$ $=0.395 \quad (3\text{ s.f.})$
(iii)	$(0.5)(0.7)^{n-1}<0.05$ $(0.7)^{n-1}<0.1$ $n-1>\dfrac{\ln(0.1)}{\ln(0.7)}=6.4557$ $n>7.4557$ Hence, the smallest value is $n=8$. Alternative method: $(0.5)(0.7)^{n-1}<0.05$ Using GC: n / probability: 7 / 0.05582; 8 / 0.04418 <0.05; 9 / 0.02882 Hence, the smallest value is $n=8$. Or $(0.7)^{n-1}<0.1$ n / probability: 7 / 0.11765; 8 / 0.08235 <0.01; 9 / 0.05465 Hence, the smallest value is $n=8$.

Using GC (first table):

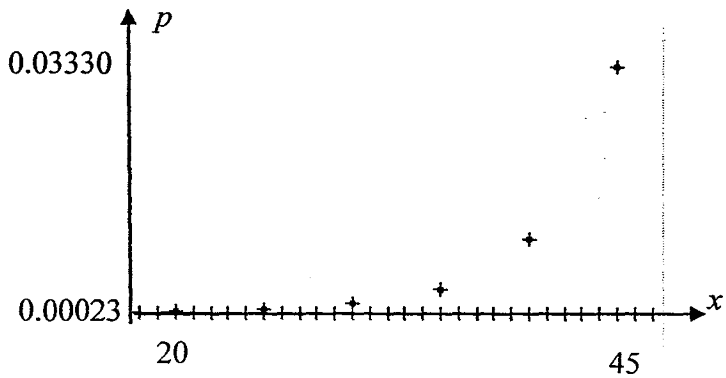
n	probability
7	0.05582
8	0.04418 <0.05
9	0.02882

Or (second table):

n	probability
7	0.11765
8	0.08235 <0.01
9	0.05465

9

(i)



(ii)(a) $r = 0.8130$ (b) $r = 0.9960$ (c) $r = 0.8667$

(iii)Since $|r|$ is closest to 1 for model (b), equation would be

$\ln p = 0.19409x - 12.322$

(iv)When $x = 32$,

$p = 0.0022181$

Expected number $= 5000(0.0022181) = 11.1$

10

(i) Unbiased estimate of pop mean =

$$\bar{x} = \frac{\sum x}{n} = \frac{11872}{250} = 47.488$$

Unbiased estimate of population variance, $s^2 = \frac{1}{249}(646193 - \frac{11872^2}{250}) = 330.986$

(i) To test

$H_o : \mu = 49.5$

$H_1 : \mu < 49.5$

At $\alpha\%$ significance level

Since n= 250 is large, by Central limit Theorem, $\bar{X} \sim N(49.5, \frac{330.986}{250})$ approx.

Since H0 is rejected,

p-value $= P(\bar{X} < 47.488) = 0.040179 < \frac{\alpha}{100}$

$\alpha > 4.02$

(ii) Since n=250 is large, by Central Limit Theorem,

$\bar{X} \sim N(45.292, \frac{18.761^2}{250})$ approximately

$P(\bar{X} \geq 47.488) \approx 0.0321$

11(i) The mean number of customers who arrived at the village post office during a random chosen 30 minutes period must be a constant.

(ii) Let X be the random variable denoting the number of customers who arrive at the village post office between 11.00 a.m. and 11.30 a.m.

	i.e. $X \sim P_o(9)$ $P(X \le 4) = 0.054964 = 0.0550 \quad (3 \text{ s.f.})$
(iii)	Let Y be the random variable denoting the number of customers who arrive at the village post office in 5 minutes i.e. $Y \sim P_o(1.5)$ $P(Y = 0) = 0.22313$ Let W be the random variable denoting the number of periods (of 5 minutes each) out of 6 where $Y = 0$ i.e. $W \sim B(6, 0.22313)$ $P(W \le 1) = 0.59867 = 0.599 \quad (3 \text{ s.f.})$
(iv)	Let U be the random variable denoting the number of customers who arrive at the village post office in 3.5 hours in the afternoon i.e. $U \sim Po(21)$ $P(\mu - \sigma < U < \mu + \sigma) = P(16.4 < U < 25.6)$ $= P(U \le 25) - P(U \le 16)$ $= 0.675$
(v)	Let T be the random variable denoting the number of customers who arrive at the village post office in 3.5 hours in the morning $T + U$ □ Po(84) $P(T > 38 \mid T + U = 40) = \dfrac{P(T > 38 \text{ and } T + U = 40)}{P(T + U = 40)}$ $= \dfrac{P(T = 39)P(U = 1) + P(T = 40)P(U = 0)}{P(T + U = 40)}$ $= 1.44 \times 10^{-4}$
(vi)	$T + U \sim Po(84)$ Since $\lambda = 84 > 10$, hence use the normal distribution for approximation i.e. $T + U \sim N\left(126, \left(\sqrt{126}\right)^2\right)$ approximately $P(T + U > 100) \xrightarrow{c.c.} P(T + U > 100.5) = 0.0359 \quad (3 \text{ s.f.})$