Preliminary Examination

MATHEMATICS
Higher 2 **9740/01**

Paper 1

Monday **29 August 2016** **3 hours**

Additional materials : Answer paper
List of Formulae (MF15)
Cover Sheet

READ THESE INSTRUCTIONS FIRST

Write your name, civics group and index number on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.

Answer **all** the questions. Total marks : **100**

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
At the end of the examination, fasten all your work securely together.

This document consists of 7 printed pages including this page.

1 The sum $\sum_{r=1}^{n} \frac{1}{(3r-2)(3r+1)}$ is denoted by S_n.

(i) By using the method of differences, find an expression for S_n in terms of n. [3]

(ii) Hence find the value of S_n as n tends to infinity. [1]

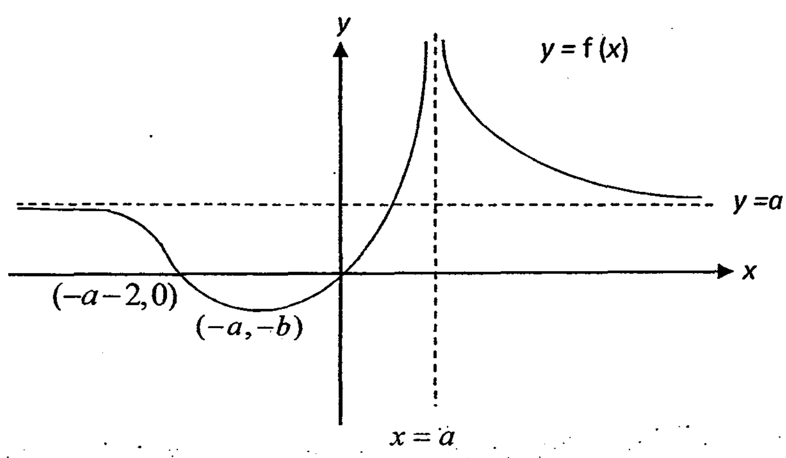
(iii) Find the smallest value of n for which S_n is within 2×10^{-4} of the sum to infinity. [3]

(iv) Using your answer in part (i), find $\sum_{r=0}^{n} \frac{1}{(3r+1)(3r+4)}$ and deduce that

$\sum_{r=0}^{n} \frac{1}{(3r+4)^2} < \frac{1}{3}$. [4]

2 (a) By completing the square, or otherwise, state precisely a sequence of geometrical transformations which would transform the graph of $y=\ln(4x^2-16x+15)$ onto the graph of $y=\frac{1}{2}\ln(4x^2-1)$. [3]

(b) The diagram shows the graph of $y=\mathrm{f}(x)$. The graph passes through the origin and the point $(-a-2,0)$. It has a minimum point at $(-a,-b)$, $a>1$, $b>1$. The graph also has a vertical asymptote $x=a$ and a horizontal asymptote $y=a$.



Sketch, on separate diagrams, the graph of:

(i) $y=\frac{1}{\mathrm{f}(x)}$; [2]

(ii) $y=\mathrm{f}'(x)$, [2]

showing clearly all the asymptotes, turning points and axes intercepts.

3 **(a)** John deposits $\$x$ into a bank at the beginning of each year. The bank pays interest at a fixed rate of 5% of the amount at the end of each year. John then withdraws the interest as soon as it is added. Find, in terms of x and N, the total amount of interest he will collect at the beginning of $(N+1)$th year. **[3]**

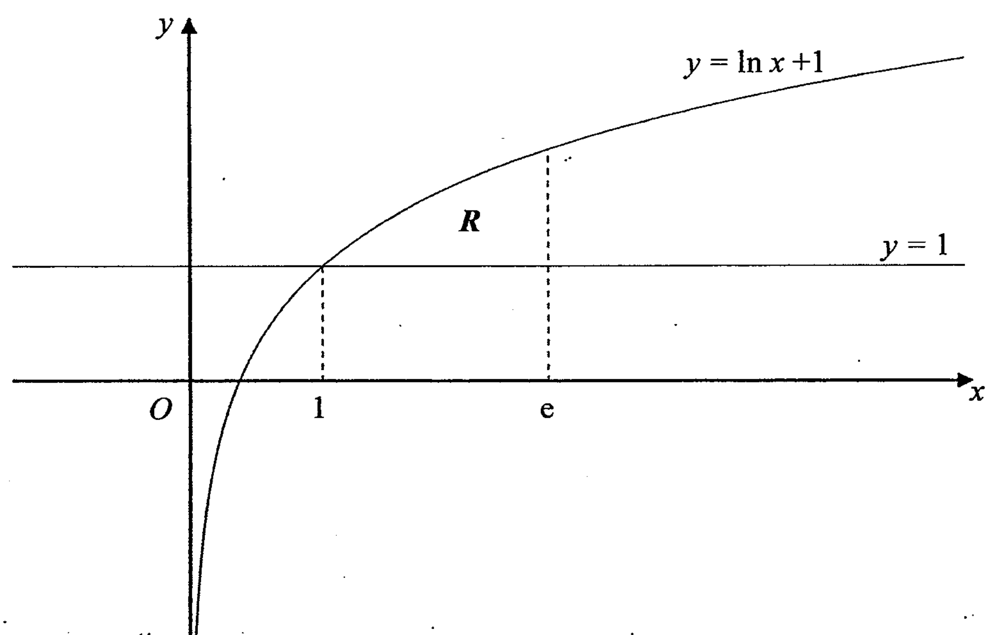
(b) An agricultural farm has 2000kg of vegetables. At the end of each week, the farm sells 10% of the vegetables and grows another 80kg on the farm.

(i) Find the amount of vegetables the farm has at the end of nth week, expressing your answer in the form $A(B^n)+C$, where A, B and C are constants to be determined. **[3]**

(ii) At which week will the amount of vegetables in the farm be first less than 835kg? **[2]**

4 **(i)** Find $\int u^2 e^u \, du$. **[3]**

(ii) The curve C has equation $y = \ln x + 1$ as shown below.



The region R is bounded by the curve C and the lines $x = 1$, $x = \mathrm{e}$ and $y = 1$.

Write down the equation of the curve by translating C one unit in the negative y-direction.

Hence, using the substitution $u = \ln x$, evaluate the exact volume generated when R is rotated completely about the line $y = 1$ by 2π radians. **[4]**

5 The functions f and g are defined by

$$\mathrm{f}: x \mapsto \mathrm{e}^{2x} - 2\mathrm{e}^{x} + 3, \quad x \in \mathbf{R}$$

$$\mathrm{g}: x \mapsto \ln(2 - x), \quad x \in \mathbf{R}, x < 2$$

(i) By sketching a graph, explain why the inverse function f^{-1} does not exist. **[2]**

(ii) Given that the domain of f is restricted to $(-\infty, a]$, state the maximum value of a for which f^{-1} exist. **[1]**

(iii) Using the value of a found in **(ii)** and by completing the square, find the inverse function f^{-1}. **[3]**

(iv) Find the exact range of gf^{-1}. **[2]**

6 Given that $y = \sqrt{4 + \sin 2x}$, show that $y\dfrac{\mathrm{d}y}{\mathrm{d}x} = \cos 2x$. **[1]**

(i) By further differentiation of the above result, find the Maclaurin series for y in ascending powers of x up to and including the term in x^2. **[3]**

(ii) Verify the correctness of the series found in **(i)** by using an appropriate standard series expansion. **[2]**

(iii) Deduce from part **(i)** the approximate value of $\displaystyle\int_0^{0.1} \sqrt{4 - \sin 2x}\,\mathrm{d}x$, giving your answer to 5 significant figures. **[2]**

7 Relative to the origin O, the position vectors of two points A and B are $\mathbf{a}$ and $\mathbf{b}$ respectively, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors. The vector $\mathbf{a}$ is a unit vector which is perpendicular to $\alpha\mathbf{a} + \beta\mathbf{b}$, where $\alpha > 1$ and $\beta > 1$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\dfrac{5\pi}{6}$.

(i) Show that $|\mathbf{b}| = \dfrac{2\sqrt{3}}{3}\left(\dfrac{\alpha}{\beta}\right)$. **[3]**

(ii) Give the geometrical interpretation of $|\mathbf{a} \times \mathbf{b}|$ and find its value in terms of α and β. **[3]**

(iii) The point M divides AB in the ratio $\lambda : 1 - \lambda$ where $0 < \lambda < 1$. The point N is such that $OMNB$ is a parallelogram. Find $\overrightarrow{ON}$ in terms of $\mathbf{a}$ and $\mathbf{b}$ and the area of the triangle OAN in terms of λ, α and β. **[5]**

8 The variables w, x and y are connected by the following differential equations:

$$\frac{dw}{dx} = -\frac{3}{2}w - 2 \quad (A)$$

$$\frac{dy}{dx} = w \quad (B)$$

(i) Solve equation (A) to find w in terms of x. **[3]**

(ii) Hence find y in terms of x. **[2]**

(iii) The result in part **(ii)** represents a family of curves. Some members of the family are straight lines. Write down the equation of one of these lines. On a single diagram, sketch your line together with a non-linear member of the family of curves that has your line as an asymptote, indicating clearly any axes intercepts. **[3]**

9 **(a)** **(i)** Solve $z^3 = 1 - i\sqrt{3}$, giving your answers in the form $re^{i\theta}$, where $r > 0$, and $-\pi < \theta \le \pi$. **[3]**

(ii) Show that

$$\left(z^n - 2e^{i\theta}\right)\left(z^n - 2e^{-i\theta}\right) = z^{2n} - 4z^n\cos\theta + 4,$$

Hence find the roots of the equation

$z^6 - 2z^3 + 4 = 0$ in the form of $re^{i\theta}$, where $r > 0$, and $-\pi < \theta \le \pi$. **[3]**

(b) Given that $z = \cos\theta + i\sin\theta$, show that $1 - z^2 = (-2i\sin\theta)z$. Given also that $0 < \theta < \pi$, find the modulus and argument of $1 - z^2$ in terms of θ. **[5]**

10

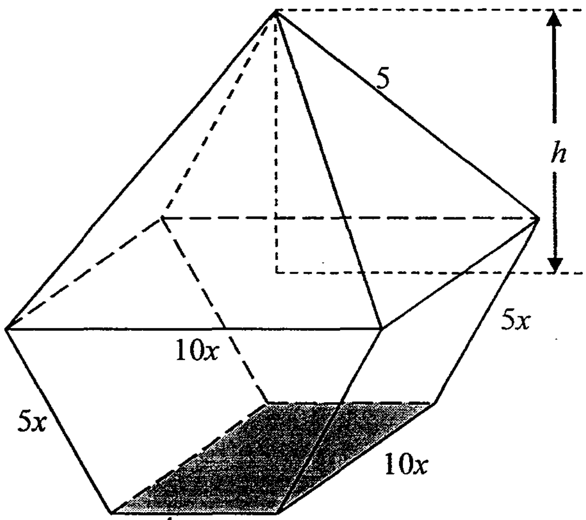


Figure 1

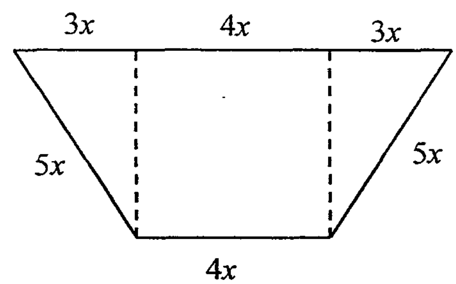


Figure 2

A designer decided to build a model as shown in **Figure 1** above, consisting of a base and a top. The base is made up of a prism with a cross-section of a trapezium where the length of the parallel sides are $4x$ cm and $10x$ cm (**Figure 2**). The top is a right pyramid with a square base of sides $10x$ cm, height h cm and a fixed slant height of 5 cm.

(i) Find an expression for the volume of the model, V, in terms of x. Given that $x = x_1$ is the value of x which gives the maximum value of V, show that x_1 satisfies the equation $13563x^4 - 7719x^2 + 625 = 0$. **[6]**

(ii) Find the two solutions to the equation in part **(i)** for which $x > 0$, giving your answers correct to 5 decimal places. **[2]**

(iii) Using both the solutions found in part **(ii)**, show that one of the values does not give a stationary value of V. Hence, write down the value of x_1. **[3]**

[Area of trapezium $= \frac{1}{2}$(sum of parallel sides)$\times$height ;

Volume of pyramid $= \frac{1}{3}$base area $\times$ height]

11 The line l_1 and the planes p_1 and p_2 have equations as follows:

$$l_1 \ : \ x-5=-y-1, z=4;$$

$$p_1 \ : \ xa+z=5a+4,$$

$$p_2 : \quad \mathbf{r}=\lambda\begin{pmatrix}1\\1\\0\end{pmatrix}+\mu\begin{pmatrix}0\\1\\1\end{pmatrix}$$

where a is a positive constant and λ and μ are real numbers.

(i) Given that the acute angle between l_1 and p_1 is $\frac{\pi}{6}$, show that $a=1$. **[2]**

(ii) The planes p_1 and p_2 meet in the line l_2. Find a vector equation of l_2. **[2]**

(iii) Hence, find the values of α and β such that the system of equations

$$x+z=9$$
$$x+z=y$$
$$5x+4y+\alpha z=\beta$$

has

(a) more than one solution;

(b) exactly one solution.

If $\alpha=5$, $\beta=10$, give a geometrical interpretation of the relationship between the 3 equations. Explain your answer. **[6]**

End of Paper

2016 Prelim Paper 1 Solution

1(i)	$\sum_{r=1}^{n} \frac{1}{(3r-2)(3r+1)}$ $= \frac{1}{3}\sum_{r=1}^{n}\left(\frac{1}{3r-2} - \frac{1}{3r+1}\right)$ $= \frac{1}{3}[\ 1 - \frac{1}{4}$ $+ \frac{1}{4} - \frac{1}{7}$ $+ \ \vdots$ $+ \frac{1}{3n-5} - \frac{1}{3n-2}$ $+ \frac{1}{3n-2} - \frac{1}{3n+1}]$ $= \frac{1}{3}\left(1 - \frac{1}{3n+1}\right)$
1(ii)	As $n \to \infty, S_n \to \frac{1}{3}$ since $\frac{1}{3n+1} \to 0$
(iii)	$\lvert S_n - S \rvert < 2\times10^{-4}$ $\left\lvert \frac{1}{3}\left(1 - \frac{1}{3n+1}\right) - \frac{1}{3} \right\rvert < 2\times10^{-4}$ $\left\lvert -\frac{1}{3}\left(\frac{1}{3n+1}\right) \right\rvert < 2\times10^{-4}$ $\frac{1}{3}\left(\frac{1}{3n+1}\right) < 2\times10^{-4}$ since $n \in \mathbb{Z}^+$ $\frac{1}{3n+1} < \frac{3}{5000}$ $3n+1 > \frac{5000}{3}$ $n > 555.2$ Hence, smallest $n = 556$.

(iv)

$$\sum_{r=0}^{n}\frac{1}{(3r+1)(3r+4)}$$

Let $r=k-1$

$3r+1=3(k-1)+1=3k-2$

$3r+4=3(k-1)+4=3k+1$

When $r=0, k=1,$

When $r=n, k=n+1$

$$\therefore \sum_{r=0}^{n}\frac{1}{(3r+1)(3r+4)}$$

$$=\sum_{k=1}^{n+1}\frac{1}{(3k-2)(3k+1)}$$

$$=\frac{1}{3}\left[1-\frac{1}{3(n+1)+1}\right] \text{ from (i)}$$

$$=\frac{1}{3}\left(1-\frac{1}{3n+4}\right)$$

$(3r+4)^2>(3r+1)(3r+4)$ for $r\geq 0, r\in \square$

$$\frac{1}{(3r+4)^2}<\frac{1}{(3r+1)(3r+4)}$$

$$\sum_{r=0}^{n}\frac{1}{(3r+4)^2}<\sum_{r=0}^{n}\frac{1}{(3r+1)(3r+4)}<\frac{1}{3}$$

since $1-\dfrac{1}{3n+4}<1$ for $n\in \square^{+}\cup\{0\}$

2 (a)

$y=\ln(4x^2-16x+15)$

$=\ln(4(x^2-4x)+15)$

$=\ln(4(x-2)^2-16+15)$

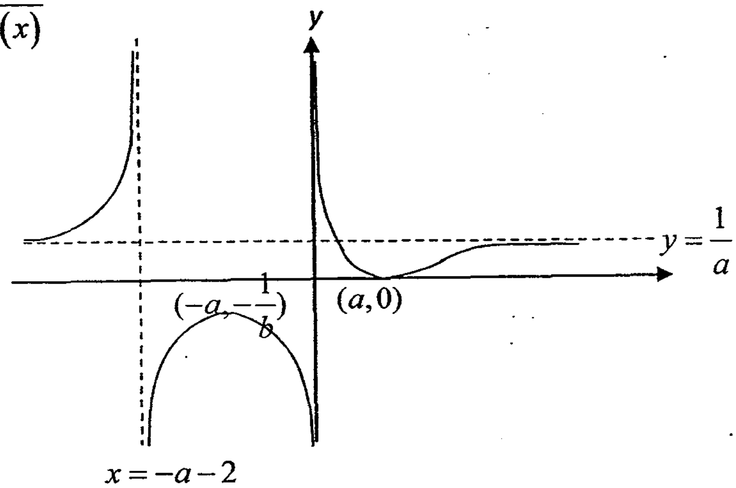
$=\ln(4(x-2)^2-1)$

Therefore to transform $y=\ln(4(x-2)^2-1)$ to $y=\dfrac{1}{2}\ln(4x^2-1)$ means:

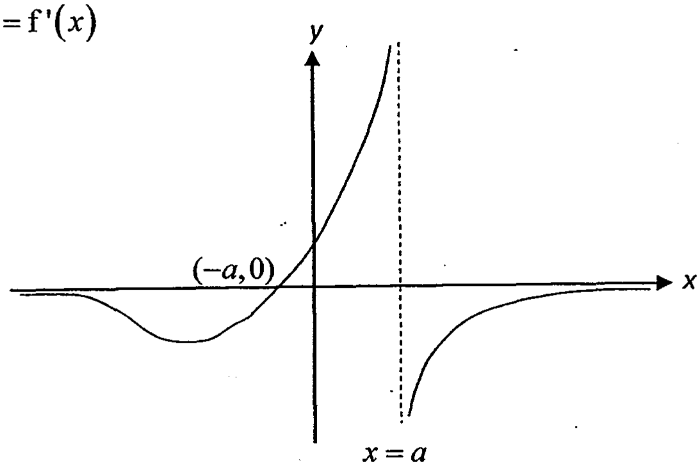
A translation to the left by two units in the direction of the positive $x-$ axis followed by;

A scale parallel to $y-$ axis by scale factor of 1/2.

(b)(i) $y=\dfrac{1}{f(x)}$



(ii) $y=f'(x)$



3(a)

Year No	Amount withdrawn at the beginning of N years
1	0
2	$0.05x$
3	$0.05(2x)$
...	
$N+1$	$T_N=0.05Nx$

$S_{N+1} = 0.05x(1+2+3...N)$

$= \dfrac{0.05xN(N+1)}{2}$

$= \dfrac{xN(N+1)}{40}$

(i)

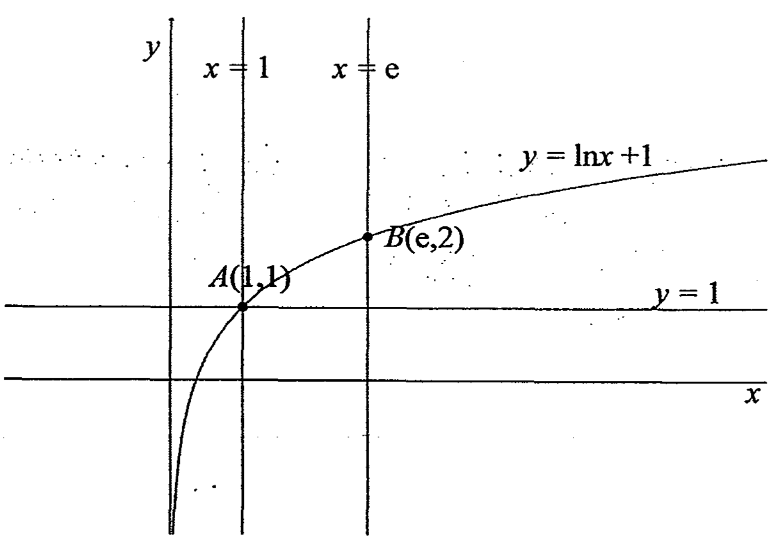
Week No	Amount of vegetables on the farm
1	2000 (0.9) +80
2	$=((2000\times0.9)+80)\times0.9+80$ $=(2000\times0.9^2)+(80\times0.9)+80$
3	$=((2000\times0.9^2)+(80\times0.9)+80)\times0.9)+80$ $=(2000\times0.9^3)+(80\times0.9^2)+(80\times0.9)+80$
….	
n	$=(2000\times0.9^n)+(80\times0.9^{n-1})...+(80\times0.9)+80$ $=(2000\times0.9^n)+80\left[0.9^{n-1}+0.9^{n-2}+...1\right]$ $=(2000\times0.9^n)+80\left[\dfrac{(1-0.9^n)}{1-0.9}\right]$ $=(2000\times0.9^n)+800(1-0.9^n)$ $=(1200\times0.9^n)+800$

$A = 1200$, $B = 0.9$, $C = 800$

(ii)

$(1200\times0.9^n)+800<835$

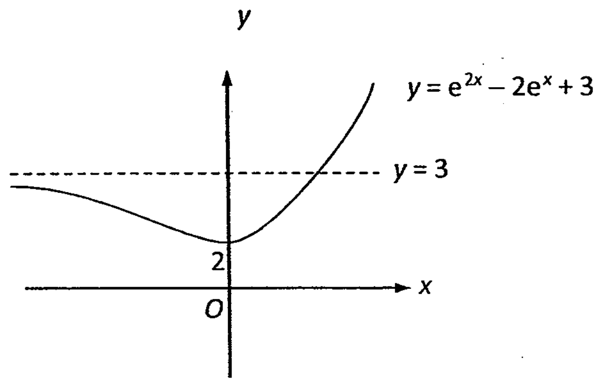
Solving,

	$(1200\times 0.9^n) < 35$ $0.9^n < \dfrac{35}{1200}$ $n > 33.549$ Least number is 34 weeks
4(i)	$U = u^2 \quad \dfrac{dV}{du} = e^u$ $\dfrac{dU}{du} = 2u \quad V = e^u$ $\int u^2 e^u \, du$ $= \left[u^2 e^u\right] - 2\int ue^u \, du$ $U = u \quad \dfrac{dV}{du} = e^u$ $\dfrac{dU}{du} = 1 \quad V = e^u$ $u^2 e^u - 2\int ue^u \, du$ $= u^2 e^u - 2\left\{\left[ue^u\right] - \int e^u \, du\right\}$ $= u^2 e^u - 2ue^u + 2e^u + C$, where C was an arbitrary constant
4	At $x = 1, y = 1$ At $x = e, y = 2$ 

	After translations, the graph is: $y_1 = \ln x$, with A' (1 , 0) and B' (e , 1) $u = \ln x$ $x = e^u$ $\frac{\mathrm{dx}}{\mathrm{du}} = e^u$ $x = 1, u = 0$ $x = e; u = 1$ $V = \pi\int_1^e \left[(\ln x)^2\right] \mathrm{dx}$ $= \pi\int_0^1 u^2 \frac{\mathrm{dx}}{\mathrm{du}} \mathrm{du}$ $= \pi\int_0^1 u^2 e^u \mathrm{d}u$ $= \pi\left[u^2 e^u - 2ue^u + 2e^u\right]_0^1$ $= \pi\left[e - 2e + 2e - 2\right]$ $= \pi(e-2)$ units³

5

(i)



As the horizontal line $y = k$, $k \in (2,3)$ cuts the graph $y = \mathrm{f}(x)$ more than once, f is not one - one. Hence f^{-1} does not exist.

(ii) Max $a = 0$.

(iii) Let $y = \mathrm{e}^{2x} - 2\mathrm{e}^{x} + 3$

$y = (\mathrm{e}^{x})^2 - 2\mathrm{e}^{x} + 3$

$y = (\mathrm{e}^{x} - 1)^2 + 2$

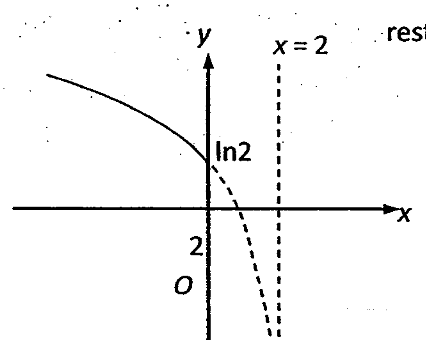
$x = \ln(1 + \sqrt{y-2})$ (rejected, since $x \le 0$).

or $\ln(1 - \sqrt{y-2})$

$\therefore \mathrm{f}^{-1} : x \mapsto \ln(1 - \sqrt{x-2}),\ x \in \square,\ 2 \le x < 3.$

(iv) $[2,3) \xrightarrow{\mathrm{f}^{-1}} (-\infty, 0] \xrightarrow{\mathrm{g}} [\ln 2, \infty)$

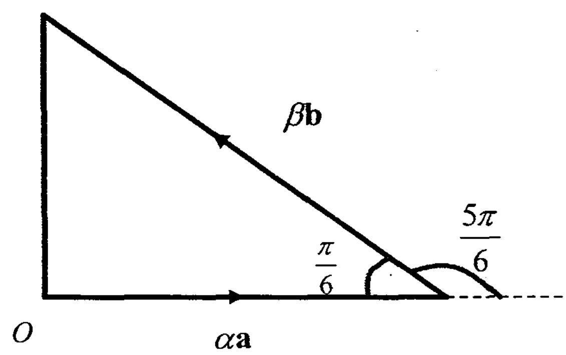
$R_{\mathrm{gf}^{-1}} = [\ln 2, \infty)$



Graph of g when D_g is restricted to $(-\infty, 0]$

6	$y=\sqrt{4+\sin 2x}$ $y^2=4+\sin 2x$ Differentiating implicitly with respect to x, $2y\frac{dy}{dx}=2\cos 2x$ $y\frac{dy}{dx}=\cos 2x$ .... (1)
(i)	Differentiating (1) implicitly with respect to x, $y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=-2\sin 2x$ .... (2) When $x=0$, $y=\sqrt{4+0}=2$ From (1) $\frac{dy}{dx}=\frac{1}{2}$ From (2) $2\frac{d^2y}{dx^2}+\left(\frac{1}{2}\right)^2=0 \Rightarrow \frac{d^2y}{dx^2}=-\frac{1}{8}$ The Maclaurin's Series of y is $y=2+\frac{1}{2}x-\frac{1}{8}\times\frac{x^2}{2!}+\cdots$ $y\approx 2+\frac{1}{2}x-\frac{x^2}{16}$, up to and including the term in x^2
(ii)	By using the standard series of $\sin x$, $\sin 2x\approx 2x$ $\sqrt{4+\sin 2x}\approx\sqrt{4+2x}$ $=(4+2x)^{\frac{1}{2}}$ $=2\left(1+\frac{x}{2}\right)^{\frac{1}{2}}$

	$\sqrt{4+\sin 2x} = 2\left[1+\frac{1}{2}\left(\frac{x}{2}\right)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\left(\frac{x}{2}\right)^2+\cdots\right]$ $= 2\left(1+\frac{x}{4}-\frac{x^2}{32}+\cdots\right)$ $= 2+\frac{x}{2}-\frac{x^2}{16}+\cdots$
(iii)	$\int_0^{0.1}\sqrt{4-\sin 2x}\,\mathrm{d}x = \int_0^{0.1}\sqrt{4+\sin(-2x)}\,\mathrm{d}x$ $= \int_0^{0.1}\left(2-\frac{x}{2}-\frac{x^2}{16}\right)\mathrm{d}x$ (Replace x with $-x$) ≈ 0.19748 (to 5 s.f.)
7 (i)	$\mathbf{a}\cdot(\alpha\mathbf{a}+\beta\mathbf{b}) = 0$ $\alpha\lvert\mathbf{a}\rvert^2+\beta\mathbf{a}\cdot\mathbf{b} = 0$ $\alpha+\beta\mathbf{a}\cdot\mathbf{b} = 0$ $\mathbf{a}\cdot\mathbf{b} = -\frac{\alpha}{\beta}$ Since angle between **a** and **b** is $\frac{5\pi}{6}$, $\cos\left(\frac{5\pi}{6}\right) = \frac{\mathbf{a}\cdot\mathbf{b}}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}$ $-\frac{\sqrt{3}}{2} = \frac{-\frac{\alpha}{\beta}}{\lvert\mathbf{b}\rvert}$ $\lvert\mathbf{b}\rvert = \frac{2\sqrt{3}}{3}\left(\frac{\alpha}{\beta}\right)$ (shown) Or



$$\cos\left(\frac{\pi}{6}\right)=\frac{\alpha|\mathbf{a}|}{\beta|\mathbf{b}|}$$

$$|\mathbf{b}|=\frac{\alpha|\mathbf{a}|}{\beta\cos\left(\frac{\pi}{6}\right)}$$

$$|\mathbf{b}|=\frac{\alpha}{\beta\frac{\sqrt{3}}{2}}$$

$$=\frac{2\alpha}{\sqrt{3}\beta}$$

$$=\frac{2\sqrt{3}}{3}\left(\frac{\alpha}{\beta}\right)$$

(ii) $|\mathbf{a}\square\mathbf{b}|$ is the length of projection of $\mathbf{b}$ onto $\mathbf{a}$

$$|\mathbf{a}\square\mathbf{b}|=\left||\mathbf{a}||\mathbf{b}|\cos\left(\frac{5\pi}{6}\right)\right|$$

$$=|\mathbf{b}|\frac{\sqrt{3}}{2}$$

$$=\frac{2\sqrt{3}}{3}\left(\frac{\alpha}{\beta}\right)\frac{\sqrt{3}}{2}$$

$$=\left(\frac{\alpha}{\beta}\right)$$

(iii) By Ratio theorem,

$$\overrightarrow{OM}=\lambda\mathbf{b}+(1-\lambda)\mathbf{a}$$

$$\overrightarrow{ON} = \overrightarrow{OM} + \overrightarrow{MN}$$
$$= \left[\lambda\mathbf{b} + (1-\lambda)\mathbf{a}\right] + \overrightarrow{OB}$$
$$= \left[\lambda\mathbf{b} + (1-\lambda)\mathbf{a}\right] + \mathbf{b}$$
$$= (\lambda+1)\mathbf{b} + (1-\lambda)\mathbf{a}$$

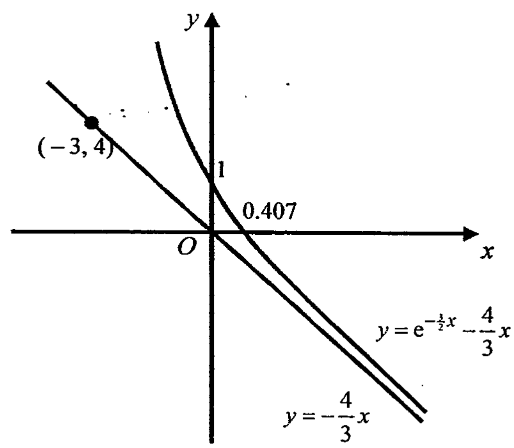
Area of triangle OAN

$$= \frac{1}{2}\left|\overrightarrow{OA} \times \overrightarrow{ON}\right|$$
$$= \frac{1}{2}\left|\mathbf{a} \times \left[(\lambda+1)\mathbf{b} + (1-\lambda)\mathbf{a}\right]\right|$$
$$= \frac{1}{2}\left|\,(\lambda+1)\mathbf{a}\times\mathbf{b} + (1-\lambda)\mathbf{a}\times\mathbf{a}\right|$$
$$= \frac{1}{2}(\lambda+1)\left|\mathbf{a}\times\mathbf{b}\right| \qquad \text{since } |\lambda+1| = \lambda+1 \text{ as } 0<\lambda<1$$
$$= \frac{1}{2}(\lambda+1)|\mathbf{a}||\mathbf{b}|\left|\sin\left(\frac{5\pi}{6}\right)\right|$$
$$= \frac{(\lambda+1)}{2}\left(\frac{2\sqrt{3}}{3}\right)\left(\frac{\alpha}{\beta}\right)\left(\frac{1}{2}\right)$$
$$= \frac{(\lambda+1)\sqrt{3}}{6}\left(\frac{\alpha}{\beta}\right)$$

8(i)	$\frac{dw}{dx}=-(\frac{3}{2}w+2)$ $\int \frac{1}{\frac{3}{2}w+2}\,dw=\int -1\,dx$ $\frac{2}{3}\int \frac{\frac{3}{2}}{\frac{3}{2}w+2}\,dw=\int -1\,dx$ $\frac{2}{3}\ln\left\|\frac{3}{2}w+2\right\|=-x+A$ where A is an arbitrary constant $\ln\left\|\frac{3}{2}w+2\right\|=-\frac{3}{2}x+\frac{3}{2}A$ $\left\|\frac{3}{2}w+2\right\|=e^{-\frac{3}{2}x+\frac{3}{2}A}=e^{\frac{3}{2}A}.e^{-\frac{3}{2}x}$ $\frac{3}{2}w+2=\pm e^{\frac{3}{2}A}.e^{-\frac{3}{2}x}=Be^{-\frac{3}{2}x}$ where $B=\pm e^{\frac{3}{2}A}$ $w=\frac{2}{3}Be^{-\frac{3}{2}x}-\frac{4}{3}$ $w=Ce^{-\frac{3}{2}x}-\frac{4}{3}$ where $C=\frac{2}{3}B$
8(ii)	$\frac{dy}{dx}=Ce^{-\frac{3}{2}x}-\frac{4}{3}$ $\int \frac{dy}{dx}\,dx=\int\left(Ce^{-\frac{3}{2}x}-\frac{4}{3}\right)dx$ $y=-\frac{2}{3}Ce^{-\frac{3}{2}x}-\frac{4}{3}x+E$ where E is an arbitrary constant $y=De^{-\frac{3}{2}x}-\frac{4}{3}x+E$ where $D=-\frac{2}{3}C$
8(iii)	When $D=0$ (or $C=0$) and $E=0$, $y=-\frac{4}{3}x$

When $D = 1$ (or $C = -\frac{3}{2}$) [or any value of D or C] and $E = 0$ [or any value of E corresponding to the choice of E above],

$$y = e^{-\frac{3}{2}x} - \frac{4}{3}x$$



9 (a) (i)

$$z^3 = 1 - i\sqrt{3}$$

$$= 2e^{i\left(\frac{\pi}{3}+2k\pi\right)}, \; k \in \mathbb{Z}$$

$$z = 2^{\frac{1}{3}} e^{i\frac{1}{3}\left(\frac{\pi}{3}+2k\pi\right)}, \; k = 0, \pm 1$$

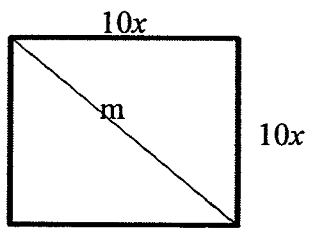
$$= 2^{\frac{1}{3}} e^{i\left(\frac{\pi}{9}+\frac{2k\pi}{3}\right)}, \; k = 0, \pm 1$$

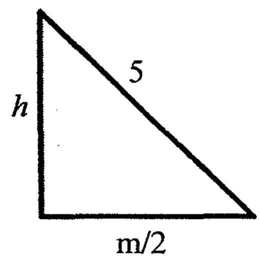
$$z = 2^{\frac{1}{3}} e^{i\frac{5\pi}{9}}, 2^{\frac{1}{3}} e^{-i\frac{\pi}{9}}, 2^{\frac{1}{3}} e^{-i\frac{7\pi}{9}}$$

(ii)

$$\left(z^n - 2e^{i\theta}\right)\left(z^n - 2e^{-i\theta}\right)$$
$$= z^{2n} - 2z^n\left(e^{i\theta} + e^{-i\theta}\right) + 4$$
$$= z^{2n} - 2z^n\left(\cos\theta + i\sin\theta + \cos(-\theta) + i\sin(-\theta)\right) + 4$$
$$= z^{2n} - 2z^n\left(\cos\theta + i\sin\theta + \cos\theta - i\sin\theta\right) + 4$$
$$= z^{2n} - 4z^n\cos\theta + 4 \quad \text{(shown)}$$

	Hence, $z^6 - 2z^3 + 4 = 0$ $z^6 - 4\left(\frac{1}{2}\right)z^3 + 4 = 0$ $z^6 - 4\left(\cos\frac{\pi}{3}\right)z^3 + 4 = 0$ $\left(z^3 - 2e^{i\frac{\pi}{3}}\right)\left(z^3 - 2e^{-i\frac{\pi}{3}}\right) = 0$ The roots of are $z^3 = 2e^{-i\frac{\pi}{3}}$ $z = 2^{\frac{1}{3}}e^{i\frac{5\pi}{9}}, 2^{\frac{1}{3}}e^{-i\frac{\pi}{9}}, 2^{\frac{1}{3}}e^{-i\frac{7\pi}{9}}$ (from (i)) Since the coefficients of the equation are all real, complex roots occur in conjugate pairs. Therefore, for $z^3 = 2e^{i\frac{\pi}{3}}$, the roots are $z = 2^{\frac{1}{3}}e^{-i\frac{5\pi}{9}}, 2^{\frac{1}{3}}e^{i\frac{\pi}{9}}, 2^{\frac{1}{3}}e^{i\frac{7\pi}{9}}$
9(b)	$1 - z^2 = 1 - (\cos 2\theta + i\sin 2\theta)$ $= 1 - \cos 2\theta - i(2\sin\theta\cos\theta)$ $= 2\sin^2\theta - i(2\sin\theta\cos\theta)$ $= (-2i\sin\theta)(\cos\theta + i\sin\theta)$ $= (-2i\sin\theta)z$ (shown) Alternatively :

	$1-z^2=1-(e^{i2\theta})$ $=e^{i\theta}(e^{-i\theta}-e^{i\theta})$ $=e^{i\theta}(\cos\theta-i\sin\theta-\cos\theta-i\sin\theta)$ $=z(-2i\sin\theta)$ (Shown) $\left\|1-z^2\right\|=\left\|-2i\sin\theta\right\|\left\|z\right\|=2\sin\theta$ $\arg(1-z^2)=\arg(-2i\sin\theta)+\arg(z)$ $=\arg(2\sin\theta)+\arg(-i)+\arg(z)$ $=\theta-\dfrac{\pi}{2}$
10(i)	Let V be the total volume of the solid and h be the height of the pyramid. Height of trapezium, l $=\sqrt{25x^2-9x^2}$ $=\sqrt{16x^2}$ $=4x$ $V_{\text{base}}=\dfrac{1}{2}(4x+10x)(4x)(10x)$ $=280x^3$  $m=\sqrt{2(10x)^2}$ $=10\sqrt{2}x$ $\dfrac{m}{2}=5\sqrt{2}x$



$$h=\sqrt{25-\left(5\sqrt{2}x\right)^2}$$
$$=\sqrt{25-50x^2}$$

$$V_{pyramid}=\frac{1}{3}(10x)^2\left(\sqrt{25-50x^2}\right)$$
$$=\frac{100x^2}{3}\left(\sqrt{25-50x^2}\right)$$

$$V=280x^3+\frac{100x^2}{3}\sqrt{25-50x^2}$$

$$\frac{\mathrm{d}V}{\mathrm{d}x}=840x^2+\frac{100}{3}\left\{x^2\left(\frac{1}{2\sqrt{25-50x^2}}\right)(-100x)+2x\sqrt{25-50x^2}\right\}$$
$$=840x^2+\frac{100}{3}\left\{2x\sqrt{25-50x^2}-\frac{50x^3}{\sqrt{25-50x^2}}\right\}$$
$$=840x^2+\frac{100}{3}(2x)\left\{\sqrt{25-50x^2}-\frac{25x^2}{\sqrt{25-50x^2}}\right\}$$
$$=\frac{20}{3}x\left\{126x+10\left[\sqrt{25-50x^2}-\frac{25x^2}{\sqrt{25-50x^2}}\right]\right\}$$

For stationary values of V,

$$\frac{\mathrm{d}V}{\mathrm{d}x}=0$$

Since $x>0$,

$$126x+10\left[\sqrt{25-50x^2}-\frac{25x^2}{\sqrt{25-50x^2}}\right]=0$$

	$126x=-10\left[\sqrt{25-50x^2}-\dfrac{25x^2}{\sqrt{25-50x^2}}\right]$ $126x\sqrt{25-50x^2}=-10\left[\left(25-50x^2\right)-25x^2\right]$ $126x\sqrt{25-50x^2}=-10\left[25-75x^2\right]$ $=750x^2-250$ Squaring both sides, $\left(126x\sqrt{25-50x^2}\right)^2=\left(750x^2-250\right)^2$ $15876x^2\left(25-50x^2\right)=562500x^4-375000x^2+62500$ $396900x^2-793800x^4=562500x^4-375000x^2+62500$ $3969x^2-7938x^4=5625x^4-3750x^2+625$ $13563x^4-7719x^2+625=0$
(ii)	Using G.C., since $x>0$, the two values of x are $0.6865562\approx0.68656$ or $0.3126699\approx0.31267$ (to 5 dp).
(iii)	$\dfrac{dV}{dx}=\dfrac{20}{3}x\left\{126x+10\left[\sqrt{25-50x^2}-\dfrac{25x^2}{\sqrt{25-50x^2}}\right]\right\}$ For $x=0.6865562$, $\dfrac{dV}{dx}=\dfrac{20}{3}(0.6865562)\{126(0.6865562)$ $+10\left[\sqrt{25-50(0.6865562)^2}-\dfrac{25(0.6865562)^2}{\sqrt{25-50(0.6865562)^2}}\right]\}$ $=0.03789$ ≈0 For $x=0.3126699$, $\dfrac{dV}{dx}=\dfrac{20}{3}(0.3126699)\{126(0.3126699)$ $+10\left[\sqrt{25-50(0.3126699)^2}-\dfrac{25(0.3126699)^2}{\sqrt{25-50(0.3126699)^2}}\right]\}$ $=164.24$ $\neq0$

	Hence $x_1 = 0.68656$
11 (i)	$l_1: \mathbf{r} = \begin{pmatrix}5\\-1\\4\end{pmatrix} + \gamma\begin{pmatrix}1\\-1\\0\end{pmatrix}$ $p_1: \mathbf{r} \cdot \begin{pmatrix}a\\0\\1\end{pmatrix} = 5a+4$ Acute $\angle$ between l_1 and $p_1 = \dfrac{\pi}{6}$ $\Rightarrow$ Acute $\angle$ between $\begin{pmatrix}1\\-1\\0\end{pmatrix}$ and $\begin{pmatrix}a\\0\\1\end{pmatrix} = \dfrac{\pi}{3}$ $\cos\left(\dfrac{\pi}{3}\right) = \dfrac{\left\lvert\begin{pmatrix}a\\0\\1\end{pmatrix} \bullet \begin{pmatrix}1\\-1\\0\end{pmatrix}\right\rvert}{\sqrt{a^2+1}\sqrt{2}}$ $\dfrac{1}{2} = \dfrac{a}{\sqrt{2(a^2+1)}}$ $2(a^2+1) = 4a^2$ $2a^2 = 2$ $a = \pm 1$ Since $a > 0$, $a = 1$. (proven)
(ii)	$p_2: \quad \mathbf{r} = \lambda\begin{pmatrix}1\\1\\0\end{pmatrix} + \mu\begin{pmatrix}0\\1\\1\end{pmatrix}$ Normal vector of $p_2 = \begin{pmatrix}1\\1\\0\end{pmatrix} \times \begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}1\\-1\\1\end{pmatrix}$ $p_2: \mathbf{r} \cdot \begin{pmatrix}1\\-1\\1\end{pmatrix} = 0$ Using GC, solving $x+z=9$ and $x-y+z=0$,

$x = 9 - z$

$y = 9$

$z = z$

Therefore equation of l_2,

$$\underset{\sim}{r} = \begin{pmatrix} 9 \\ 9 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, s \in \mathbb{R}$$

(iii)

$x + z = 9 \cdots (1)$

$x + z = y \cdots (2)$

$5x + 4y + \alpha z = \beta \cdots (3)$

(1) and (2) are the cartesian equations of planes p_1 and p_2 respectively.

Let $p_3 : \mathbf{r} \cdot \begin{pmatrix} 5 \\ 4 \\ \alpha \end{pmatrix} = \beta$

Since the system of equations is known to have more than one solution, p_1, p_2 and p_3 intersect at l_2. Therefore, l_2 lies in p_3.

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 4 \\ \alpha \end{pmatrix} = 0$$

$-5 + \alpha = 0$

$\alpha = 5$

$$\begin{pmatrix} 9 \\ 9 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 4 \\ 5 \end{pmatrix} = \beta$$

$\beta = 81$

Since the system of equations is known to have exactly one solution, p_1, p_2 and p_3 intersect at a point. Therefore, l_2 intersects p_3 at a point.

$$\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 4 \\ \alpha \end{pmatrix} \neq 0$$

$-5 + \alpha \neq 0$

$\alpha \neq 5$

	$\beta \in$ □
	Since $\alpha = 5$ and $\beta \neq 81$, the planes do not intersect at a common point. Since the 3 planes are not parallel to each other, the 3 planes form a triangular prism.

Preliminary Examination

MATHEMATICS
Higher 2 **9740/02**

Paper 2

Thursday **15 September 2016** **3 hours**

Additional materials : Answer paper
List of Formulae (MF15)
Cover Sheet

READ THESE INSTRUCTIONS FIRST

Write your name, civics group and index number on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.

Answer **all** the questions. Total marks : **100**

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
At the end of the examination, fasten all your work securely together.

Section A: Pure Mathematics [40 marks]

1 Prove by the method of mathematical induction that $\sum_{r=2}^{n}(r-1)\ln\left(\frac{r}{r-1}\right)=\ln\left[\frac{n^{n-1}}{(n-1)!}\right]$. **[5]**

2 Solve the inequality $\frac{x^2+6x+8}{x-1}\geq 0$.

Hence, by completing the square, solve the inequality $\frac{y^2+2y+15}{|y+1|-1}\geq -6$. **[6]**

3 It is given that $\mathrm{f}(x)=\frac{5-ax^2}{1+x^2}$ where $a>1,\ a\in\square^{+}$.

(i) Sketch $y=\mathrm{f}(x)$, showing clearly the coordinates of the turning point, any intersections with the axes and the equation(s) of any asymptote(s). **[3]**

(ii) By drawing a sketch of another suitable curve on the same diagram, find the number of real roots of the equation

$$x^4+(a+1)x^2-5=0 \ .$$ **[2]**

(iii) Let $\mathrm{g}(x)=x^4+(a+1)x^2-5$. Show that $\mathrm{g}(x)=\mathrm{g}(-x)$. What can be said about the four roots of the equation $\mathrm{g}(x)=0$? **[3]**

4 A curve C has parametric equations

$$x=a\sin 2t\,,\ y=a\sin 3t$$

where $0\leq t\leq\frac{\pi}{2}$ and a is a positive constant.

(i) Find the gradient of C at the point $\left(a\sin 2\theta\,,a\sin 3\theta\right)$ where $0\leq\theta\leq\frac{\pi}{2}$. Hence, what can be said about the tangent to C as $\theta\to\frac{\pi}{4}$? **[3]**

(ii) Find the equation of the normal, in exact form, at the point where $t=\frac{\pi}{12}$. **[3]**

(iii) With the aid of a sketch, show that the area bounded by the curve C, the y-axis and the line $y=\frac{a\sqrt{2}}{2}$ can be written as

$$3a^2\int_0^{\frac{\pi}{12}}(\cos 3t\sin 2t)\,dt\ .$$

Hence, find the exact area of the region bounded by the curve C, the y-axis and the normal to the curve at $t=\frac{\pi}{12}$ in the form $ka^2\left[b\cos\frac{\pi}{12}+c\sin\frac{\pi}{12}\right]+da^2$, where k, b, c and d are constants to be determined. **[7]**

5 The complex number z is given by $z=re^{i\theta}$, where $1\le r\le 2$ and $\frac{1}{6}\pi\le\theta\le\frac{3}{4}\pi$.

(i) State $|z|$ and arg(z) in terms of r and θ. Hence, draw an Argand diagram to show the locus of z as r and θ varies. You should identify the modulus and argument of the end-points of the locus. **[3]**

(ii) Find the exact minimum value of $|z+5-6i|$ and the corresponding complex number z representing the point at which this minimum value occurs, giving your answer in the form $x+iy$, where x and y are real numbers. **[3]**

Another complex number w satisfies the equation $\arg(w-2\sqrt{3})=\frac{5\pi}{6}$.

(iii) On the same diagram as part **(i)**, sketch the locus of w and indicate the set of points that satisfies both the locus of z and w. **[2]**

Section B: Statistics [60 marks]

6 A survey is to be carried out to obtain feedback from the members of a new female-only fitness club regarding its various facilities and fitness classes. The membership of this fitness club comprises 5000 female members and the number of members belonging to the various age groups are given in the table below:

Age group	18 - 25	26-30	31 – 40	41 – 50	51 & above
Number of members	500	1000	1500	1500	500

It is proposed to carry out the survey by interviewing members who visit the club on a particular weekday in the morning.

(i) Explain why this proposed method is inappropriate. **[1]**

(ii) Suggest an appropriate method of carrying out the survey and describe how you intend to implement the sampling method to obtain a representative sample of 200 members. **[3]**

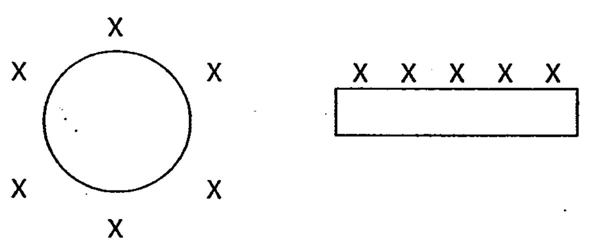
7 **(a)** For events X and Y, it is given that $P(X \cup Y) = \frac{5}{8}, P(X \cap Y') = \frac{7}{24}$ and $P(X' | Y) = \frac{9}{16}$.

(i) Find $P(X' \cap Y)$, **[3]**

(ii) Find $P(X)$ and determine if the events X and Y' are independent. **[3]**

(b) The Mathematics department consists of 5 female teachers and 6 male teachers. After a meeting, the department went to a nearby food court for lunch. Due to the lunch crowd, they only managed to find a circular table for 6 and a long table with a row of 5 seats as shown below. The seats at both tables are fixed and cannot be rearranged.

Ms Koh was among the Mathematics teachers who attended the lunch.



(i) Find the probability that Ms Koh is seated between 2 male teachers. **[4]**

(ii) Given that Ms Koh is seated between 2 male teachers, find the probability that the male and female teachers alternate at both tables. **[3]**

8 **(i)** Given that $X \sim N(\mu, \sigma^2)$ and $P(X < 28.4) = P(X > 77.6) = 0.012$, find the value of μ and σ. **[3]**

(ii) The mass, in grams, of a randomly chosen packet of sweets is normally distributed with mean μ and variance σ^2 obtained from part **(i)**. Every packet of sweets is priced at $1.20 per 100 grams. Find the probability that the sum of 4 packets of sweets cost at most $2.60. **[3]**

9 The hens on a farm lay either white or brown eggs. The eggs are randomly put into boxes of six. The farmer claims that the number of brown eggs in a box can be modelled by a binomial distribution $\mathrm{B}(6, p)$.

(i) State, in context, two assumptions to support the farmer's claim. **[2]**

(ii) Given that the probability a box contains at least 5 brown eggs is 0.04096, find the value of p. **[2]**

A supermarket orders 100 boxes of eggs daily.

(iii) By using a suitable approximation, find the probability that there are at least 90 boxes that contains at most 4 brown eggs in a particular day. **[3]**

(iv) The supermarket places a daily order of 100 boxes of eggs for 8 weeks. Estimate the probability that the mean number of boxes that contains at least 5 brown eggs in a day is between 4 and 7.
[You may assume that there are 7 days in a week.] **[3]**

10 A researcher wishes to investigate the length of time that patients spend with a doctor at a particular clinic. The time a patient spends with the doctor is denoted by X minutes. Based on past records, the clinic claims that the mean length of time for the doctor to see a patient is at most 10 minutes. To test this claim, the researcher recorded the actual times spent by the doctor to see a random sample of 12 patients.

$$\sum x = 147, \quad \sum x^2 = 1927.91$$

(i) Stating a necessary assumption, carry out an appropriate test, at the 5% significance level, to determine whether there is any evidence to doubt the clinic's claim. **[5]**

(ii) Suppose now that the population standard deviation of X is 15 and that the assumption made in part **(i)** is still valid. A new sample of n patients is obtained and the sample mean length of time is found to be unchanged. Using this sample, the researcher conducts another test and found that the null hypothesis is not rejected at the 5% significance level. Obtain an inequality involving n and find the set of values that n can take. **[3]**

11 An experiment is conducted to calibrate an anemometer*. In this calibration process, the wind speed X is fixed precisely and the resulting anemometer speed Y is recorded.

For a particular anemometer, this process produced the following set of measurements:

Wind speed (m/s), X	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0
Anemometer (revs/min), Y	24	28	47	92	164	236	312	360

(i) Calculate the product moment correlation coefficient between X and Y. **[1]**

(ii) Sketch a scatter diagram for the data. **[1]**
Explain why it is advisable to sketch the scatter diagram in addition to calculating the product moment correlation coefficient before interpreting this set of bivariate data. **[1]**

A proposed model for the above data is $Y = a + bX^2$.

(iii) Calculate the product moment correlation coefficient and the equation of the least squares regression line for the proposed model.
Explain whether the model for Y on X^2 or Y on X is a better model for the data set. **[3]**

(iv) Use an appropriate regression line to estimate the value of X when the value of Y is 120. Give a reason for the choice of your regression line. **[2]**

[* *An anemometer is a device commonly used in a weather station for measuring wind speed.*]

12 The managers of 2 branches of a travel agency were discussing whether the number of customers who bought the Luxury Cruise Package per week could be modelled by a Poisson distribution. One of the managers said, "It must be assumed that the number of customers who bought the package per week is a constant."

(i) Give a corrected version of the manager's statement, and explain why the correction is necessary. **[1]**

It is given that the number of customers who bought the Luxury Cruise Package per week can be modelled by a Poisson distribution. The average number of customers who bought the Luxury Cruise Package per week at Branch A and Branch B is 3.5 and 4.5 respectively. Assume that the number of customers who bought the package at Branch A and Branch B are independent.

(ii) Find the probability that, in a randomly chosen week, the total number of customers who bought the Luxury Package at both branches is between 5 and 10. **[2]**

(iii) Given that the probability that at most one customer bought the Luxury Cruise Package at Branch A in n weeks is less than 0.1, find the value of the least n. **[3]**

(iv) Using a suitable approximation, find the probability that, in one month, the number of customers who bought the Luxury Cruise Package at Branch B exceeds the number of customers at Branch A by not more than 5. **[4]**
[You may assume that there are 4 weeks in 1 month.]

Explain why the Poisson distribution may not be a good model for the number of customers who bought the Luxury Cruise Package in a year. **[1]**

End of Paper

2016 Prelim Paper 2 Solution

1	Let P(n) be the statement, $\sum_{r=2}^{n}(r-1)\ln\left(\frac{r}{r-1}\right)=\ln\left[\frac{n^{n-1}}{(n-1)!}\right]$, $n\in\mathbb{Z}^{+}, n\geq 2$.. When $n=2$, LHS $=(2-1)\ln\left(\frac{2}{2-1}\right)=\ln 2$ RHS $=\ln\left[\frac{2^{2-1}}{(2-1)!}\right]=\ln 2$ $\therefore$ LHS = RHS P(2) is true. Assume that P(k) is true for some positive integer k, $k\in\mathbb{Z}^{+}$, $k\geq 2$ i.e. Assume $\sum_{r=2}^{k}(r-1)\ln\left(\frac{r}{r-1}\right)=\ln\left[\frac{k^{k-1}}{(k-1)!}\right]$. To prove P($k$ + 1) is true. i.e. to prove $\sum_{r=2}^{k+1}(r-1)\ln\left(\frac{r}{r-1}\right)=\ln\left[\frac{(k+1)^{k}}{k!}\right]$ LHS $=\sum_{r=2}^{k+1}(r-1)\ln\left(\frac{r}{r-1}\right)$ $=\sum_{r=2}^{k}(r-1)\ln\left(\frac{r}{r-1}\right)+k\ln\left(\frac{k+1}{k}\right)$ $=\ln\left[\frac{k^{k-1}}{(k-1)!}\right]+k\ln\left(\frac{k+1}{k}\right)$ $=\ln\left[\frac{k^{k-1}}{(k-1)!}\right]+\ln\left(\frac{k+1}{k}\right)^{k}$ $=\ln\left[\frac{k^{k-1}}{(k-1)!}\times\frac{(k+1)^{k}}{k^{k}}\right]$ $=\ln\left[\frac{k^{-1}(k+1)^{k}}{(k-1)!}\right]$ $=\ln\left[\frac{(k+1)^{k}}{k(k-1)!}\right]$ $=\ln\left[\frac{(k+1)^{k}}{k!}\right]$ = RHS Since P(2) is true and if P(k) is true, it implies that P(k+1) is true. By Mathematical Induction, P(n) is true for all positive integers $n\geq 2$.

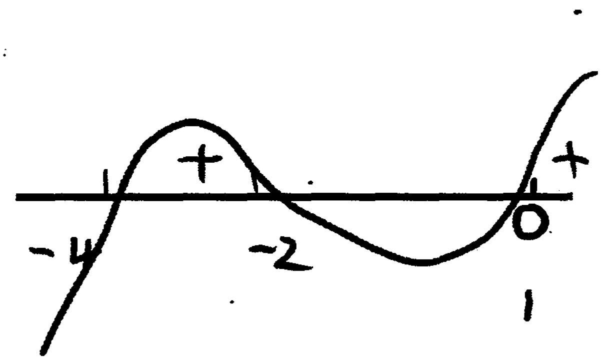
2

$$\frac{x^2+6x+8}{x-1} \geq 0,\ x \neq 1$$

$$\frac{(x+2)(x+4)}{(x-1)} \geq 0$$

Multiply by $(x-1)^2$,

$(x-1)(x+2)(x+4) \geq 0$,



$-4 \leq x \leq -2$ or $x > 1$. (Ans)

$$\frac{y^2+2y+15}{|y+1|-1} \geq -6$$

$$\frac{y^2+2y+15+6(|y+1|-1)}{|y+1|-1} \geq 0$$

$$\frac{(y+1)^2+14+6|y+1|-6}{|y+1|-1} \geq 0$$

$$\frac{(y+1)^2+6|y+1|+8}{|y+1|-1} \geq 0$$

Since $(y+1)^2 = |y+1|^2$

We use the substitution $x = |y+1|$,

then we obtain $\frac{x^2+6x+8}{x-1} \geq 0$

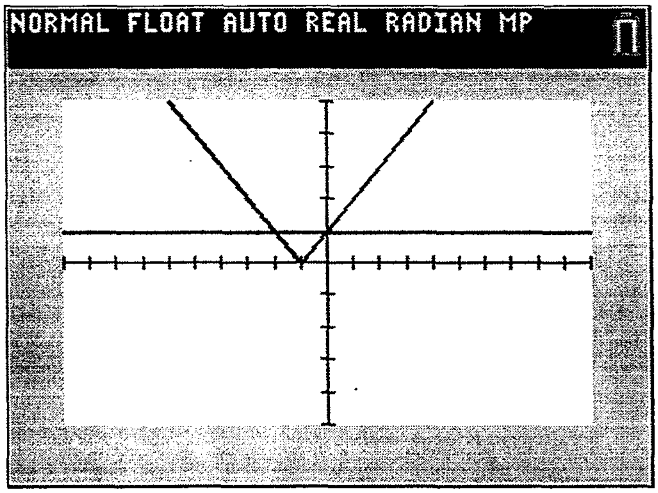
and from the answer from (i),

$-4 < |y+1| < -2$ (no solution since $|y+1| \geq 0$ for all $y \in$ □)

or $|y+1|>1$

$\therefore \quad y+1>1$ or $y+1<-1$

(or use of the graphical method)

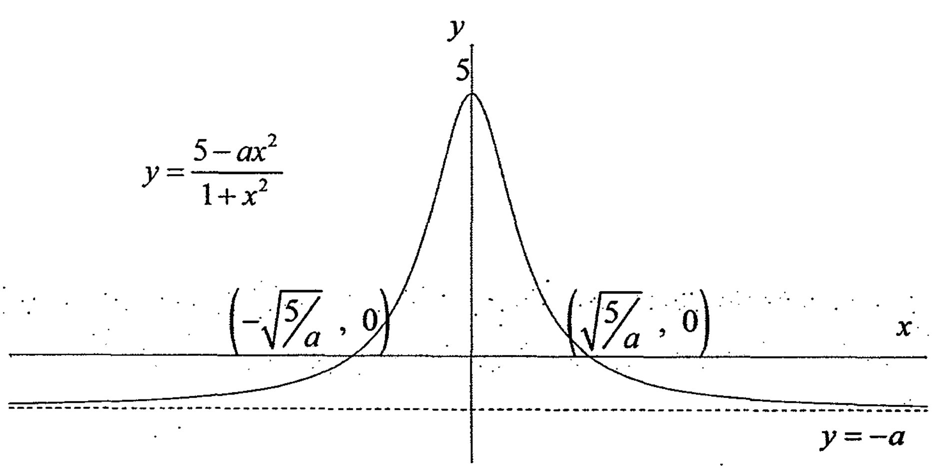


From GC, the intersection points are $(-2,1)$ and $(0,1)$

$y>0$ or $y<-2$ (Ans)

3(i)

$$\mathrm{f}(x)=\frac{5-ax^2}{1+x^2}=\frac{5-a(1+x^2)+a}{1+x^2}=-a+\frac{5+a}{1+x^2} \qquad , a>1$$



(ii)

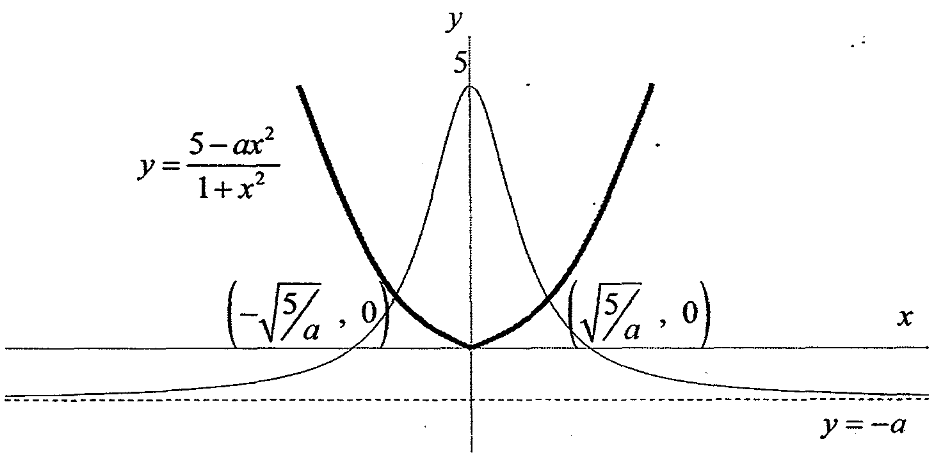
$x^4 + (a+1)x^2 - 5 = 0$

$x^4 + x^2 = 5 - ax^2$

$x^2(1+x^2) = 5 - ax^2$

$x^2 = \dfrac{5-ax^2}{1+x^2}$

Hence we should sketch the curve $y = x^2$.



From the graph we can see that there are only two real roots.

To show: $g(-x) = (-x)^4 + (a+1)(-x)^2 - 5 = x^4 + (a+1)x^2 - 5 = g(x)$

As there are only two real roots, the other two roots should be complex roots.

As the coefficients of equation are all real, the remaining two roots must be complex roots that form a pair of complex conjugates.

As there is only one pair the complex conjugates, and $g(x) = g(-x) = 0$, then the complex conjugates must be purely imaginary.

Explanation:

If $z = x + \mathrm{i}y$ is a root where $x, y \in □$, so $g(x+\mathrm{i}y) = 0$. Since $g(z) = g(-z)$ for all $z \in □$ and $g(z) = 0$, then $g(-z) = 0$ and hence $z = -(x+\mathrm{i}y) = -x - \mathrm{i}y$ is also a root. As the complex roots need to be in conjugate pairs, then

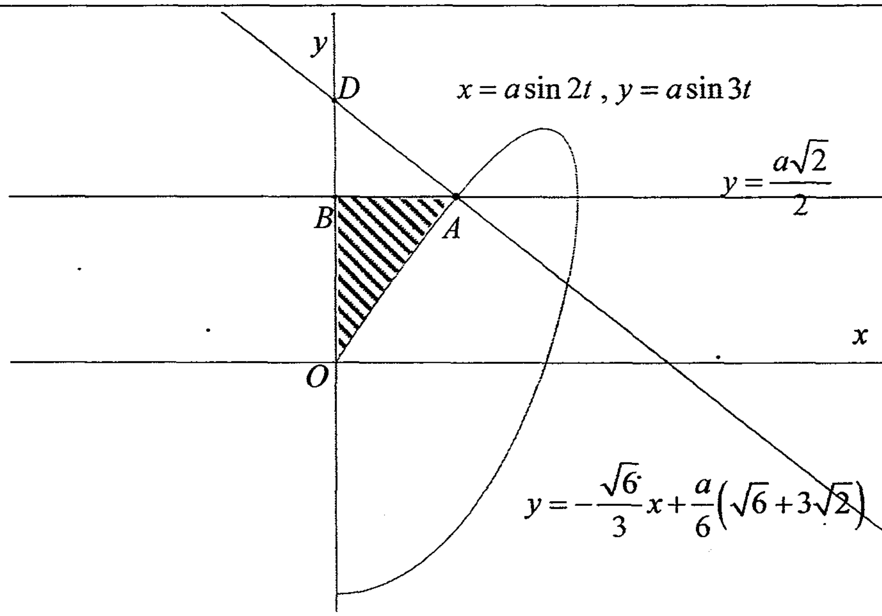
	$(x+\mathrm{i}y)^* = -x-\mathrm{i}y$ $x-\mathrm{i}y = -x-\mathrm{i}y$ Comparing the Real Part, $x=-x$ $2x=0$ $x=0$ [Note that the imaginary part is not necessary, as it yields $-y=-y$ which is trivial.] Hence the complex roots must be purely imaginary. Alternative explanation: $x^4+(a+1)x^2-5=0$ $x^2=\frac{(a+1)\pm\sqrt{(a+1)^2-4(1)(-5)}}{2}$ $x^2=\frac{(a+1)\pm\sqrt{(a+1)^2+20}}{2}$ $x^2=\frac{(a+1)+\sqrt{(a+1)^2+20}}{2}$ or $x^2=\frac{(a+1)-\sqrt{(a+1)^2+20}}{2}$ $x=\pm\sqrt{\frac{(a+1)+\sqrt{(a+1)^2+20}}{2}}$ or $x=\pm\sqrt{\frac{(a+1)-\sqrt{(a+1)^2+20}}{2}}$ Since $\frac{(a+1)+\sqrt{(a+1)^2+20}}{2}>0$ and $\frac{(a+1)-\sqrt{(a+1)^2+20}}{2}<0$ then the complex roots must be purely imaginary.
4(i)	Given $x=a\sin 2t$, $y=a\sin 3t$, $\frac{\mathrm{d}x}{\mathrm{d}t}=2a\cos 2t$, $\frac{\mathrm{d}y}{\mathrm{d}t}=3a\cos 3t$ Hence, At the point $(a\sin 2\theta, a\sin 3\theta)$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3\cos 3\theta}{2\cos 2\theta}$ As $\theta\to\frac{\pi}{4}$, $\cos 2\theta\to\cos\frac{\pi}{2}=0$, $\frac{\mathrm{d}y}{\mathrm{d}x}\to-\infty$.

	i.e. $\frac{dy}{dx}$ will be undefined. Hence, the tangent to C as $\theta \to \frac{\pi}{4}$ will become a vertical line.
(ii)	At $t=\frac{\pi}{12}$, $\frac{dy}{dx}=\frac{3\cos\left(3\left(\frac{\pi}{12}\right)\right)}{2\cos\left(2\left(\frac{\pi}{12}\right)\right)}$ $=\frac{3\left(\frac{1}{\sqrt{2}}\right)}{2\left(\frac{\sqrt{3}}{2}\right)}$ $=\frac{\sqrt{6}}{2}$ Gradient of normal $=-\frac{2}{\sqrt{6}}$ Coordinates of point at $t=\frac{\pi}{12}$: $x=a\sin\left(2\left(\frac{\pi}{12}\right)\right)$ $=\frac{a}{2}$ $y=a\sin\left(3\left(\frac{\pi}{12}\right)\right)$ $=\frac{a\sqrt{2}}{2}$ Equation of normal: $y-\frac{a\sqrt{2}}{2}=-\frac{2}{\sqrt{6}}\left(x-\frac{a}{2}\right)$ $=-\frac{\sqrt{6}}{3}\left(x-\frac{a}{2}\right)$

$$y=-\frac{\sqrt{6}}{3}x+\frac{a\sqrt{6}}{6}+\frac{a\sqrt{2}}{2}$$

$$=-\frac{\sqrt{6}}{3}x+\frac{a}{6}\left(\sqrt{6}+3\sqrt{2}\right)$$

(iii)



$y=0, t=0$

$y=\frac{a\sqrt{2}}{2}, t=\frac{\pi}{12}$ (from (ii))

Area bounded by curve, y-axis and the line $y=\frac{\sqrt{2}a}{2}$, region OAB

$$=\int_0^{\frac{\sqrt{2}a}{2}} x\,\mathrm{d}y$$

$$=\int_0^{\frac{\pi}{12}}(a\sin 2t)\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)\mathrm{d}t$$

$$=\int_0^{\frac{\pi}{12}}(a\sin 2t)(3a\cos 3t)\,\mathrm{d}t$$

$$=3a^2\int_0^{\frac{\pi}{12}}(\cos 3t\sin 2t)\,\mathrm{d}t \text{ (Shown)}$$

y-intercept of normal is:

$$y=\frac{a}{6}\left(\sqrt{6}+3\sqrt{2}\right)$$

Hence,

Required area, region *OADB*

$$= 3a^2\int_0^{\frac{\pi}{12}}(\cos 3t\sin 2t)\,\mathrm{d}t + \frac{1}{2}\left(\frac{a}{2}\right)\left[\frac{a}{6}\left(\sqrt{6}+3\sqrt{2}\right)-\frac{a\sqrt{2}}{2}\right]$$

$$= \frac{3a^2}{2}\int_0^{\frac{\pi}{12}}(\sin 5t - \sin t)\,\mathrm{d}t + \frac{a^2}{4}\left[\frac{\left(\sqrt{6}+3\sqrt{2}\right)}{6}-\frac{\sqrt{2}}{2}\right]$$

$$= \frac{3a^2}{2}\left[-\frac{1}{5}\cos 5t + \cos t\right]_0^{\frac{\pi}{12}} + \frac{a^2}{4}\left(\frac{\sqrt{6}}{6}\right)$$

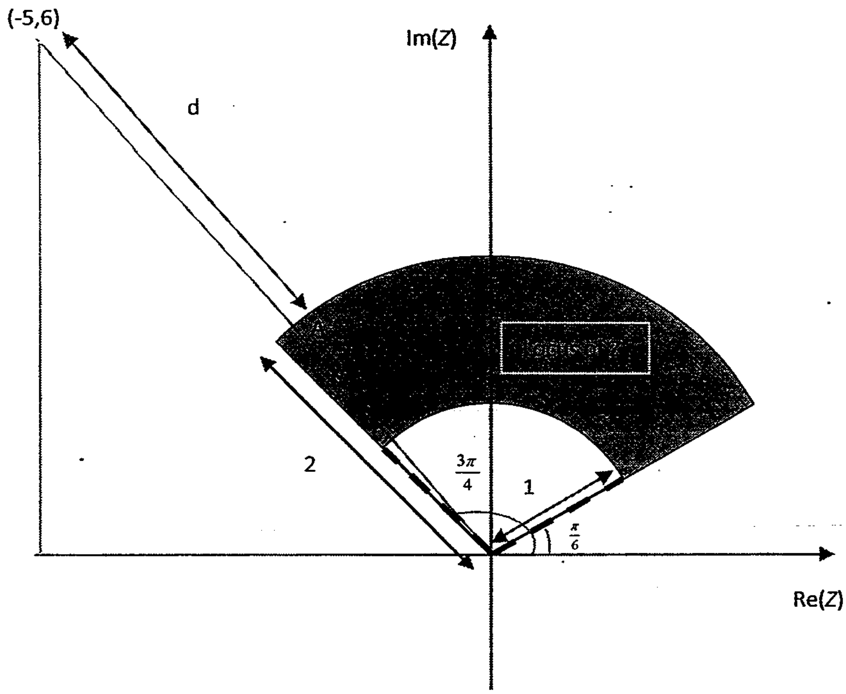
$$= \frac{3a^2}{2}\left[-\frac{1}{5}\cos\frac{5\pi}{12} + \cos\frac{\pi}{12} + \frac{1}{5}\cos 0 - \cos 0\right] + \frac{a^2\sqrt{6}}{24}$$

$$= \frac{3a^2}{2}\left[-\frac{1}{5}\cos\frac{5\pi}{12} + \cos\frac{\pi}{12} - \frac{4}{5}\right] + \frac{a^2\sqrt{6}}{24}$$

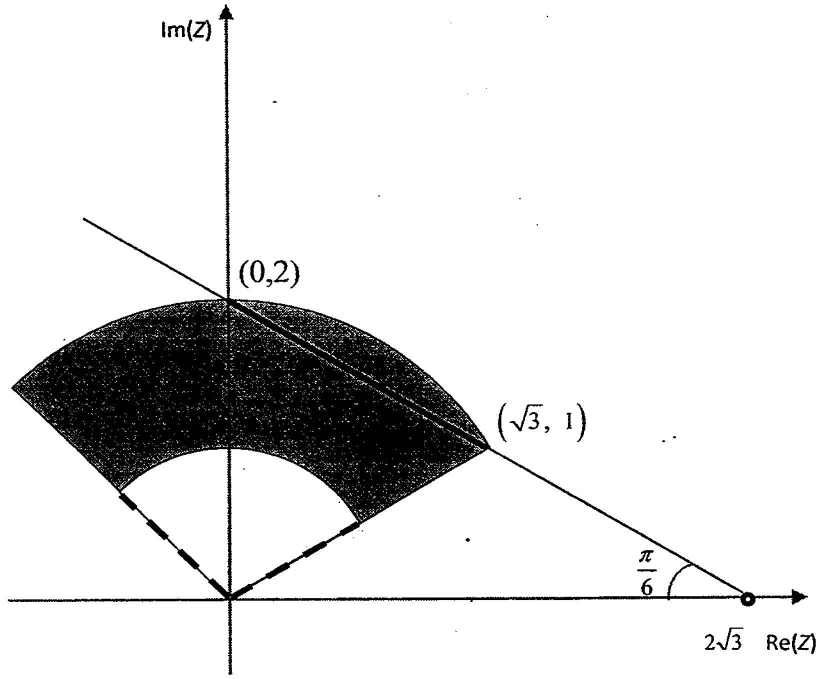
$$= \frac{3a^2}{2}\left[\cos\frac{\pi}{12} - \frac{1}{5}\cos\left(\frac{\pi}{2}-\frac{\pi}{12}\right)\right] + a^2\left(\frac{\sqrt{6}}{24}-\frac{6}{5}\right)$$

$$= \frac{3a^2}{2}\left[\cos\frac{\pi}{12} - \frac{1}{5}\sin\frac{\pi}{12}\right] + a^2\left(\frac{\sqrt{6}}{24}-\frac{6}{5}\right) \text{ units}^2$$

where $k = \frac{3}{2}, b = 1, c = -\frac{1}{5}, d = \frac{\sqrt{6}}{24} - \frac{6}{5}$

5(i), (ii)	$\arg(z) = \theta$ $\lvert z \rvert = r$ 
(ii)	The minimum distance, $d = \sqrt{(-5)^2 + 6^2} - 2 = \sqrt{61} - 2$ Let θ be the basic angle of the point $(-5, 6)$. From the diagram, $\tan\theta = \frac{6}{5}$ so $\sin\theta = \frac{6}{\sqrt{61}}$ and $\cos\theta = \frac{5}{\sqrt{61}}$. Hence complex number that corresponds to point A, $z = 2(-\cos\theta + i\sin\theta) = -2\left(\frac{5}{\sqrt{61}}\right) + 2i\left(\frac{6}{\sqrt{61}}\right) = \frac{1}{\sqrt{61}}(-10 + 12i)$ $= -1.28 + 1.54i$

(iii)



6(i)

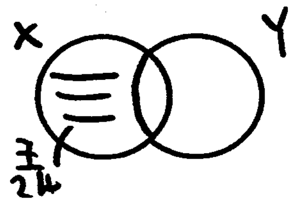
The people who visits the fitness centre on a particular weekday may not be representative of the members of the club as some members may be working or attending school and only go to club later in the day. Thus the sample collected would contain mainly women who are not working/do not attend school thereby under representing women of some age groups.

(ii)

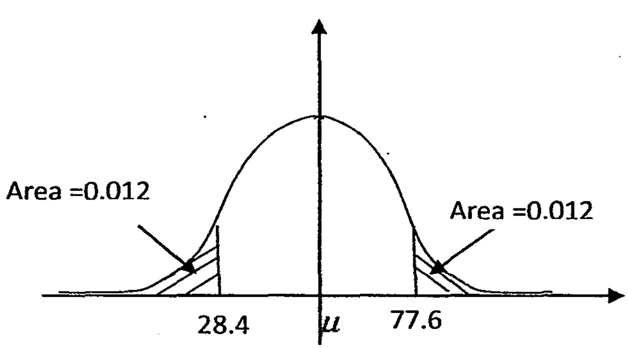
Stratified sampling method should be used to obtain a representative sample of 200 members.

1. Draw up a list of the members of the fitness centre and calculate the proportion for the different corresponding age group.

18 – 25	26-30	31 – 40	41 – 50	51 & above
500	1000	1500	1500	500
$\frac{500}{5000}\times 200 = 20$	$\frac{1000}{5000}\times 200 = 40$	$\frac{1500}{5000}\times 200 = 60$	$\frac{1500}{5000}\times 200 = 60$	$\frac{500}{5000}\times 200 = 20$

	2. Within each age group, use simple random sampling method to select the required sample size for survey.
7 (a)(i)	$P(X \cup Y) = P(Y) + P(X \cap Y')$  $\frac{5}{8} = P(Y) + \frac{7}{24}$ $P(Y) = \frac{8}{24} = \frac{1}{3}$ Given $P(X' \mid Y) = \frac{9}{16}$, $\frac{P(X' \cap Y)}{P(Y)} = \frac{9}{16}$ $P(X' \cap Y) = \frac{9}{16}.\frac{1}{3} = \frac{3}{16}$
(ii)	$P(X) = 1 - [P(X \cup Y)' + P(X' \cap Y)] = 1 - \frac{3}{8} - \frac{9}{48} = \frac{7}{16}$ $P(X).P(Y') = \frac{7}{16}.\frac{2}{3} = \frac{7}{24} = P(X \cap Y')$ X and Y' are independent.
7(b) **(i)**	Without restriction $= \binom{11}{6}(6-1)!5! = 6652800$ Case 1: Ms Koh sits at round table with two male teachers No. of ways $= \binom{8}{3}\binom{6}{2}2!(4-1)!5! = 1209600$ Case 2: Ms Koh sits at long table with two male teachers No. of ways $= \binom{8}{2}\binom{6}{2}2!3!(6-1)! = 604800$ **Required Probability**

	$= \dfrac{\binom{8}{3}\binom{6}{2}2!(4-1)!5! + \binom{8}{2}\binom{6}{2}2!3!(6-1)!}{\binom{11}{6}(6-1)!5!}$ $= \dfrac{3}{11}$ or 0.273 (to 3.s.f)
(ii)	Required Probability $= \dfrac{P(\text{Ms koh sits between 2 male teachers and male and female teachers alternate})}{P(\text{Ms koh sits between 2 male teachers})}$ $= \dfrac{P(\text{male and female teachers alternate})}{P(\text{ms koh sits between 2 male teachers})}$ $= \dfrac{\dfrac{\binom{6}{3}\binom{5}{3}(3-1)!3!3!2!}{6652800}}{\dfrac{3}{11}}$ OR $\dfrac{\dfrac{6\times5\times4\times5\times4\times2\times3\times2}{6652800}}{\dfrac{3}{11}}$ $= \dfrac{\dfrac{1}{231}}{\dfrac{3}{11}}$ $= \dfrac{1}{63}$ or 0.159 (to 3.s.f)

	Alternative Solution: Required Probability $=\dfrac{P(\text{ms koh sits between 2 male teachers and male and female teachers alternate})}{P(\text{ms koh sits between 2 male teachers})}$ Case 1: Ms Koh sits at the long table and the male and female teachers alternate no. of ways $=\binom{6}{3}\binom{4}{1}3!2!3!(3-1)!=11520$ Case 2: Ms Koh sits at the circular table and the male and female teachers alternate no. of ways $=\binom{6}{3}\binom{4}{2}(3-1)!3!3!2!=17280$ Total no. of ways ms koh sits between 2 male teachers and male and female teachers alternate $=\binom{6}{3}\binom{4}{1}3!2!3!(3-1)!+\binom{6}{3}\binom{4}{2}(3-1)!3!3!2!$ $=28800$ Required Probaility $=\dfrac{\frac{28800}{6652800}}{\frac{3}{11}}$ $=\dfrac{1}{63}$ [Note that for the males and females to seat on alternate seats, on the round table there must be 3 males and 3 females and the long table there must be 3 males and 2 females]
8 (i)	 <u>Method 1 (recognise μ is the midpoint)</u> By symmetry, $\mu=\dfrac{28.4+77.6}{2}=53$ $P(X<28.4)=0.012$ $P(Z<\dfrac{28.4-53}{\sigma})=0.012$ Using GC,

$\frac{-24.6}{\sigma} = -2.25712924$

$\sigma = 10.8988$

$\sigma = 10.9$ (3 s.f.)

Method 2 (simultaneous equations- not recommended)

$P(X < 28.4) = 0.012$

$P(Z < \frac{28.4-\mu}{\sigma}) = 0.012$

$\frac{28.4-\mu}{\sigma} = -2.25712924$

$28.4-\mu = -2.25712924\sigma$

$P(X < 77.6) = 0.988$

$P(Z < \frac{77.6-\mu}{\sigma}) = 0.012$

$\frac{77.6-\mu}{\sigma} = 2.25712924$

$77.6-\mu = 2.25712924\sigma$

Solve simultaneously, $\mu = 53, \quad \sigma = 10.8988$

(ii)

Let X be the weight of a packet of sweets in grams.

$X \sim N(53, 10.8988^2)$

Method 1 (expression in terms of mass)

$X_1 + X_2 + \ldots X_4 \sim N(212, 475.13536)$

$\$1.20 \rightarrow 100g$

$\$2.60 \rightarrow \frac{2.6\times100}{1.2} = 216.667g$

$P(X_1 + X_2 + \ldots X_4 \le 216.667) = 0.58476$

$= 0.585$ (3 s.f.)

$P(X_1 + X_2 + \ldots X_4 \le \frac{2.6\times100}{1.2}) = 0.58476$

$= 0.585$ (3 s.f.)

Method 2 (expression in terms of cost)

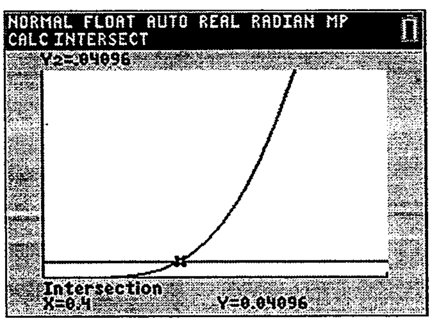
Let C denote the cost of 4 packets of sweets.

$C = \frac{1.20}{100}(X_1 + X_2 + \ldots X_4) = 0.012(X_1 + X_2 + \ldots X_4)$

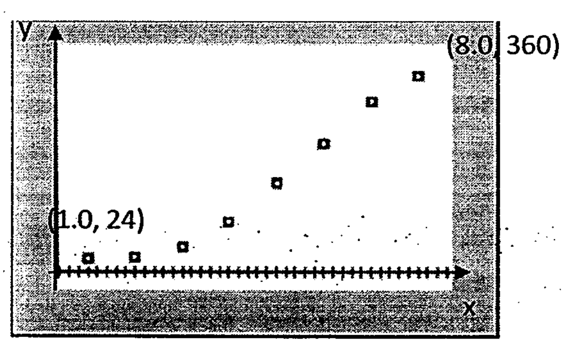
Then

$E(C) = 0.012(4\times53) = 2.544$

$Var(C) = 0.012^2(4\times10.8988^2) = 0.0684195$

	$P(C \le 2.60) = 0.5847618 = 0.585$ (3 s.f.)
9 (i)	The probability of picking a brown egg from a box is constant. The colour of an egg is independent of other eggs.
(ii)	Let X be the r.v. "number of brown eggs in a box of 6 eggs" $X \sim B(6, p)$ $P(X \ge 5) = 0.04096$ $P(X = 5) + P(X = 6) = 0.04096$ $\binom{6}{5} p^5 (1-p) + \binom{6}{6} p^6 (1-p)^0 = 0.04096$  Using GC, $p = 0.4$
(iii)	Let A be the r.v. "number of boxes that contain at most 4 brown eggs in a box out of 100 boxes" $A \sim B(100,\ p_1)$, where $p_1 = 1 - 0.04096 = 0.95904$. Let Y be the r.v. "number of boxes that contain at least 5 brown eggs in a box out of 100 boxes" $Y \sim B(100, 0.04096)$ Note that $A + Y = 100$ Since $n = 100$ is large, $np = 100(0.04096) = 4.096 < 5$, $Y \sim P_0(4.096)$ approximately $P(A \ge 90) = P(100 - Y \ge 90) = P(Y \le 10) = 0.997$
(iv)	$Y \sim B(100, 0.04096)$

	$E(Y)=100(0.04096)=4.096$ $Var(Y)=100(0.04096)(0.95904)=3.9282$ In 8 weeks, there are 56 days altogether. Mean number of boxes with at least 5 brown eggs is $\overline{Y}=\dfrac{Y_1+Y_2+Y_3+\cdots+Y_{56}}{56}$ Since sample size = 56 is large, by Central Limit Theorem, $\overline{Y} \sim \mathrm{N}\left(4.096, \dfrac{3.9282}{56}\right)$ approximately i.e. $\overline{Y} \sim \mathrm{N}(4.096, 0.0701469)$ $\mathrm{P}\left(4<\overline{Y}<7\right)=0.641$ (correct to 3 sig fig)
10(i)	Let X be the random variable ‘ length of time a patient spent with the doctor’ Given $\sum x = 147, \quad \sum x^2 = 1927.91$ The unbiased estimates of population mean μ and population variance σ^2 are $\overline{x}=12.25, \quad s^2=\dfrac{1}{n-1}[\sum x^2-\dfrac{(\sum x)^2}{n}]=11.56$ **Assumption:** The length of time, X, a patient spent with the doctor follows a normal distribution. Test H_0: $\mu=10$ H_1: $\mu>10$ Under H_0, $T=\dfrac{\overline{X}-\mu}{S/\sqrt{n}} \sim t(11)$ Use a right tailed t-test at the 5% level of significance. From GC, p-value = 0.0213

	Since p-value $= 0.0213 < 0.05$, we reject H_0. There is sufficient evidence to conclude at the 5% level of significance, the mean time spent with a patient is more than 10 minutes.
ii	Since population variance is given, z-test should be used. Test H_0: $\mu = 10$ H_1: $\mu > 10$ Under H_0, $Z = \dfrac{\overline{X} - \mu}{\sigma / \sqrt{n}} \sim N(0,1)$ Since H_0 is not rejected, $z = \dfrac{12.25 - 10}{15 / \sqrt{n}} < 1.64485$ $0.15\sqrt{n} < 1.64485$ $n < 120.25$ $\therefore$ Maximum $n = 120$. Set of values of n is $\{ n \in \square : 0 < n \leq 120 \}$
11(i)	$r = 0.974$
(ii)	 Although the value of product moment correlation coefficient indicates a strong positive linear correlation however from the scatter diagram, X and Y follows a non-linear relationship, hence it is advisable to interpret the data using both the scatter diagram and the value of product moment correlation coefficient.

(iii)	Using GC, $Y = 9.67 + 5.81X^2$ $r = 0.993$ Since the value of product moment correlation coefficient for Y on X^2 is closer to 1, compared to the value of product moment correlation coefficient for Y on X, the new proposed model is a better model for the data set.
(iv)	$Y = 9.6714285 + 5.81190X^2$ When $y = 120$, $x = 4.36$ (3 s.f.) The line of Y on X^2 is used because the value of X is fixed precisely and hence X is the independent variable.
12(i)	It must be assumed that the <u>average</u> number of customers who bought the package per week is a constant. The number of customers who bought the package per week varies, and cannot be a constant.
(ii)	Let X and Y be the number of customers who bought the Luxury Cruise Package in a week at Branch A and Branch B respectively. $X \sim \text{Po}(3.5)$ $Y \sim \text{Po}(4.5)$ $X + Y \sim \text{Po}(8)$ $\text{P}(5 < X + Y < 10) = \text{P}(X + Y \le 9) - \text{P}(X + Y \le 5) \approx 0.525$
(iii)	Let U be the number of customers who bought the Luxury Cruise Package in n weeks at Branch A. $U \sim \text{Po}(3.5n)$ Given $\text{P}(U \le 1) < 0.1$ From GC, when $n = 1$, $\text{P}(U \le 1) = 0.136 > 0.1$ when $n = 2$, $\text{P}(U \le 1) = 0.0073 < 0.1$ Least $n = 2$
(iv)	Let S and T be the number of customers who bought the Luxury Cruise Package in one month at Branch A and Branch B respectively. $S \sim \text{Po}(14)$

	Since $\lambda > 10$, S~N(14, 14) approximately. T~Po(18) Since $\lambda > 10$, T~N(18, 18) approximately. T-S ~ N(4, 32) $\text{P}(0 < T - S \le 5) \xrightarrow{C.C.} \text{P}(0.5 < T - S \le 5.5) \approx 0.337$
	The mean number of customers who bought the Luxury Package might not be a constant from one week to another because of fluatuations such as sales, holidays, the economic climate etc. Hence the Poisson distribution may not be a good model for the number of customers who bought the package in a year.