H2

JC2 PRELIMINARY EXAMINATION

MATHEMATICS 9740/01

Paper 1

Tuesday, 13 Sep 2016

3 hours

Additional Materials: Answer paper
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your name and civics group on all the work you hand in, including the Cover Page.

Write in dark blue or black pen on both sides of the paper.

You may use an HB pencil for any diagrams or graphs.

Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use an approved graphing calculator.

Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

1 A curve C has equation $$e^{x+y}+e=(3y+1)^2.$$

(i) By considering $\frac{dy}{dx}$, show that C has no stationary points. [5]

(ii) Write down an equation relating x and y at which the tangent is parallel to the y-axis. [1]

2 Referred to the origin O, the points A and B have position vectors given by $\mathbf{a}=\begin{pmatrix}\cos t\\ -\sin t\\ 0.5\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}\cos 2t\\ \sin 2t\\ -1\end{pmatrix}$ respectively, where t is a real parameter such that $0\le t<\pi$.

(i) Show that $\mathbf{a}\cdot\mathbf{b}=p+\cos(qt)$, where p and q are constants to be determined. [2]

(ii) Hence find the exact value of t for which $\angle AOB$ is a maximum. [3]

3 (i) Describe a sequence of transformations that will transform the curve with equation $y=\frac{1}{x^2}$ on to the curve with equation $y=\frac{4}{(x-1)^2}$. [2]

(ii) It is given that

$$f(x)=\begin{cases} x+2 & \text{for } 0<x\le 2,\\ \dfrac{4}{(x-1)^2} & \text{for } 2<x\le 3,\end{cases}$$

and that $f(x)=f(x+3)$ for all real values of x.

Sketch the graph of $y=f(x)$ for $-2\le x\le 6$. [3]

4 **Do not use a calculator in answering this question.**

One root of the equation $z^3+az^2+bz+15=0$, where a and b are real, is $z=1+2i$.

(i) Write down the other complex root. [1]

(ii) Explain why the cubic equation must have one real root. [1]

(iii) Find the value of the real root and the values of a and b. [5]

5 Functions f and g are defined by

$$\mathrm{f}: x \mapsto \frac{3x-1}{3x-3}, \quad x \in \square,\ x < 1,$$
$$\mathrm{g}: x \mapsto \sqrt{x-2}, \quad x \in \square,\ 2 \le x < 3.$$

(i) Find $\mathrm{f}^{-1}(x)$. [2]

(ii) Show that $\mathrm{f}^2(x) = x$. Hence find the exact value of $\mathrm{f}^{2017}(0)$. [2]

(iii) Show that the composite function fg exists. Find an expression for $\mathrm{fg}(x)$ and state the domain and range of fg. [5]

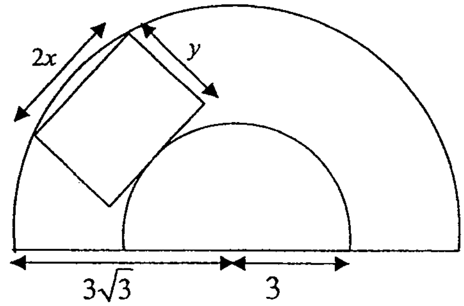
6 It is given that $\mathrm{f}(x) = \dfrac{x+3}{(1-x)^n}$, where $-1 < x < 1$ and n is a positive integer.

(i) Find the binomial expansion of $\mathrm{f}(x)$, in ascending powers of x, up to and including the term in x^2. [4]

(ii) Given that the coefficient of x^2 in the above expansion is 21, find the value of n. [3]

(iii) Given now that $n = 2$, by substituting a suitable value of x into the expansion in part (i), find the exact value of $\displaystyle\sum_{r=1}^{\infty} \frac{4r-1}{4^{r-1}}$. [3]

7



The figure shows a rectangular sheet of length $2x$ metres and breadth y metres to be placed in a horizontal position along a garden walkway bounded by low vertical fence of which a horizontal cross-section is two concentric semicircles of radii 3 metres and $3\sqrt{3}$ metres. One side of the sheet of length $2x$ metres must be tangential to the inner fence, and the two ends of the opposite side must touch the outer fence, as shown in the figure. The rectangular sheet is assumed to have negligible thickness.

(i) By finding x^2 in terms of y, show that the area A square metres of the sheet, is given by $A = 2\sqrt{18y^2 - 6y^3 - y^4}$. [3]

(ii) Use differentiation to find, the maximum value of A, proving that it is a maximum. [7]

8 The plane p_1 passes through the points $A(4,1,1)$ and $B(2,1,0)$ and is parallel to the vector $4\mathbf{i}-\mathbf{j}+2\mathbf{k}$. A line l has equation $\dfrac{x-2}{-2}=y+1=z+4$.

(i) Show that a vector perpendicular to the plane p_1 is parallel to $\mathbf{i}-2\mathbf{k}$. Find the equation of p_1 in scalar product form. [3]

(ii) Find the coordinates of the point C at which l intersects p_1. [3]

(iii) The point D with coordinates $(2, -1, -4)$ lies on l. Find the position vector of the foot of the perpendicular from D to p_1. Find the coordinates of the point E which is the mirror image of D in p_1. [4]

The plane p_2 contains the line l and the point A.

(iv) The planes p_1 and p_2 meet in a line m. Find a vector equation for m. [2]

9 A sequence $u_1, u_2, u_3, \ldots$ is given by

$$u_1 = 1 \quad \text{and} \quad 3u_{n+1} = 2u_n - 1 \text{ for } n \geq 1.$$

(i) Use the method of mathematical induction to prove that

$$u_n = 3\left(\frac{2}{3}\right)^n - 1.$$ [5]

(ii) Find $\sum_{r=1}^{n} u_r$. [3]

(iii) Give a reason why the series $\sum_{r=1}^{n}(u_r+1)$ converges, and write down the value of the sum to infinity. [2]

(iv) Explain, with the aid of a sketch, whether the value of $\sum_{r=1}^{\infty}(u_r+1)$ is an overestimation or underestimation of the value of $\int_0^{\infty} 3\left(\frac{2}{3}\right)^x \mathrm{d}x$. [2]

10 The mass, x grams, of a certain substance present in a chemical reaction at time t minutes satisfies the differential equation

$$\frac{dx}{dt} = k(4 + 2x - x^2),$$

where $0 \le x \le 1$ and k is a constant. It is given that $x = 1$ and $\frac{dx}{dt} = -\frac{1}{2}$ when $t = 0$.

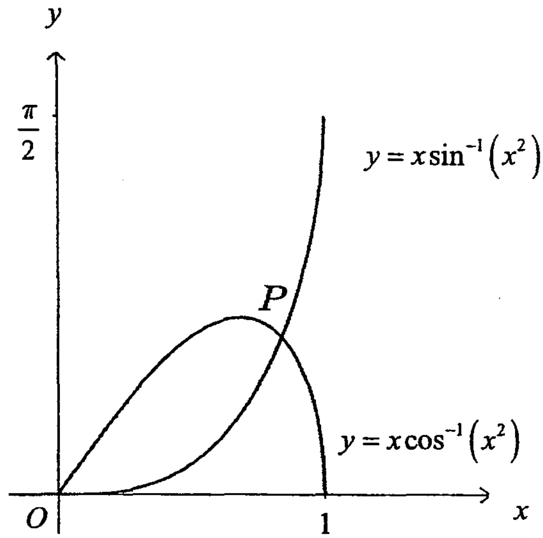
(i) Show that $k = -\frac{1}{10}$. [1]

(ii) By first expressing $4 + 2x - x^2$ in completed square form, find t in terms of x. [5]

(iii) Hence find the time taken for there to be none of the substance present in the chemical reaction, giving your answer correct to 3 decimal places. [1]

(iv) Express the solution of the differential equation in the form $x = f(t)$ and sketch the part of the curve with this equation which is relevant in this context. [5]

11



The diagram shows the curves with equations $y = x\sin^{-1}(x^2)$ and $y = x\cos^{-1}(x^2)$, where $0 \le x \le 1$. The curves meet at the point P with coordinates $\left(\frac{1}{\sqrt[4]{2}}, \frac{\pi}{4}\left(\frac{1}{\sqrt[4]{2}}\right)\right)$.

(i) Find the derivative of $\sqrt{1-x^4}$. [1]

(ii) Find the exact value of the area of the region bounded by the two curves. [5]

(iii) The region bounded by the curve $y = x\sin^{-1}(x^2)$, the line $y = \frac{\pi}{4}\left(\frac{1}{\sqrt[4]{2}}\right)$ and the y-axis is rotated about the y-axis through $360°$. By considering the parametric equations

$$x = t \quad \text{and} \quad y = t\sin^{-1}(t^2),$$

show that the volume of the solid formed is given by

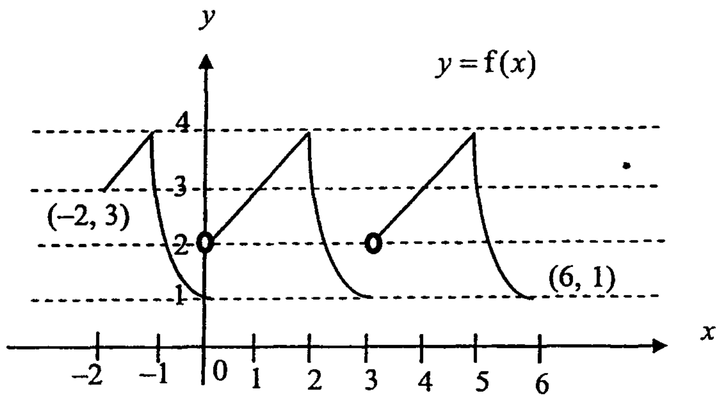
$$\pi\int_0^{\frac{1}{\sqrt[4]{2}}} \left[\frac{2t^4}{\sqrt{1-t^4}} + t^2\sin^{-1}(t^2)\right] \mathrm{d}t.$$ [3]

(iv) Hence find the volume of the solid formed when the region bounded by the curve $y = x\sin^{-1}(x^2)$ and the line $y = \frac{\pi}{4}x$ is rotated through $360°$ about the y-axis. Give your answer correct to 5 significant figures. [3]

End of Paper

2016 JC2 Preliminary Examination
H2 Mathematics Paper 1
Solution

1(i)	$e^{x+y}+e=(3y+1)^2$ Differentiate w.r.t x: $e^{x+y}\left(1+\frac{dy}{dx}\right)=2(3y+1)\left(3\frac{dy}{dx}\right)$ $e^{x+y}+e^{x+y}\frac{dy}{dx}=(18y+6)\frac{dy}{dx}$ $\frac{dy}{dx}\left(18y+6-e^{x+y}\right)=e^{x+y}$ $\therefore\frac{dy}{dx}=\frac{e^{x+y}}{18y+6-e^{x+y}}$ Since $e^{x+y}>0\ \forall x\in□, \forall y\in□$, $\therefore\frac{dy}{dx}\neq 0$ $\therefore$ Curve C has no stationary points.	
(ii)	Since tangent is parallel to the y-axis, $\frac{dy}{dx}$ is undefined $\therefore 18y+6-e^{x+y}=0$ $6(3y+1)=e^{x+y}$	
2(i)	$\mathbf{a}□\mathbf{b}=\begin{pmatrix}\cos t\\ -\sin t\\ 0.5\end{pmatrix}\cdot\begin{pmatrix}\cos 2t\\ \sin 2t\\ -1\end{pmatrix}=\cos t\cos 2t-\sin t\sin 2t-\frac{1}{2}$ $=\cos(t+2t)-\frac{1}{2}=\cos 3t-\frac{1}{2}$ $p=-\frac{1}{2}, q=3$	
(ii)	$\lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{b}\rvert\cos\angle AOB=\cos(3t)-\frac{1}{2}$ For maximum $\angle \boldsymbol{AOB}$, since $0\le\angle \boldsymbol{AOB}\le\pi$ and $\cos\theta$ is a decreasing function over $[0,\pi]$, we aim to minimize $\cos\angle \boldsymbol{AOB}$, ie. To minimize $\cos 3t-\frac{1}{2}$. Thus, $\cos 3t=-1$ ie. $t=\frac{\pi}{3}$(since $0\le t<\pi$).	

3(i)	Translation $\begin{pmatrix}1\\0\end{pmatrix}$ / Translate 1 unit in the positive x direction Stretching factor 4 parallel to the y-axis.	
(ii)		
4(i)	The other roots is 1 – 2i	
(ii)	Since the polynomial has <u>all real coefficients</u>, complex roots must occur in <u>conjugate pair</u>. The polynomial is of <u>degree 3</u> so there must be a pair of complex conjugate roots and <u>one real root</u>.	
(iii)	$z^3+az^2+bz+15=0$ $(1+2i)^3+a(1+2i)^2+b(1+2i)+15=0$ $-11-2i+a(-3+4i)+b(1+2i)+15=0$ $(4-3a+b)+i(-2+4a+2b)=0$ Equating real and imaginary parts, $4-3a+b=0$ or $-2+4a+2b=0$ Solving the equations, $a=1,\ b=-1$ $z^3+z^2-z+15=(z^2-2z+5)(z+3)=0$ $\alpha=-3$ <u>Alternative method:</u> Let the real root be α. Sum of roots = $(1+2i)+(1-2i)+\alpha=-a$ $2+\alpha=-a$ (1) Sum of product of 2 roots = $(1+2i)(1-2i)+(1+2i)\alpha+(1-2i)\alpha=b$ $5+2\alpha=b$ (2) Product of roots = $(1+2i)(1-2i)\alpha=-15$ $5\alpha=-15$	

	$\alpha = -3$ Substitute $\alpha = -3$ into (1) and (2), $a = 1$ and $b = -1$	
5(i)	Let $y = \dfrac{3x-1}{3x-3}$ $y(3x-3) = 3x-1$ $3x(y-1) = 3y-1$ $x = \dfrac{3y-1}{3y-3}$ $f^{-1}(x) = \dfrac{3x-1}{3x-3}, x < 1$	
(ii)	$f^2(x) = ff^{-1}(x) = x$ $f^{2017}(0) = f(0)$ $= \dfrac{3(0)-1}{3(0)-3} = \dfrac{1}{3}$	
(iii)	Since $R_g = [0,1) \subseteq (-\infty,1) = D_f$, fg exists. $fg(x) = \dfrac{(3\sqrt{x-2})-1}{(3\sqrt{x-2})-3}$ $D_{fg} = D_g = [2, 3)$ $R_{fg} = \left(-\infty, \dfrac{1}{3}\right]$	

6(i)	$f(x)=\dfrac{x+3}{(1-x)^n}$ $=(x+3)(1-x)^{-n}$ $=(x+3)\left(1+(-n)(-x)+\dfrac{(-n)(-n-1)}{2!}(-x)^2+\ldots\right)$ $=(x+3)\left(1+nx+\left(\dfrac{n(n+1)}{2}\right)x^2+\ldots\right)$ $=x+nx^2+3+3nx+\left(\dfrac{3n(n+1)}{2}\right)x^2+\ldots$ $=3+(3n+1)x+\left(n+\dfrac{3n(n+1)}{2}\right)x^2+\ldots$	
(ii)	$n+\dfrac{3n(n+1)}{2}=21$ $3n^2+5n-42=0$ $n=-\dfrac{14}{3}\left(\text{reject as } n\in\mathbb{Z}^+\right)$ or $n=3$ $\therefore n=3$	
(iii)	When $n=2$, $\dfrac{x+3}{(1-x)^2}=3+7x+11x^2+\ldots$ $\sum_{r=1}^{\infty}\dfrac{4r-1}{4^{r-1}}=3+\dfrac{7}{4}+\dfrac{11}{4^2}+\ldots=3+7\left(\dfrac{1}{4}\right)+11\left(\dfrac{1}{4}\right)^2+\ldots$ By substituting $x=\dfrac{1}{4}$ into $\dfrac{x+3}{(1-x)^2}$ in part **(i)**, $\sum_{r=1}^{\infty}\dfrac{4r-1}{4^{r-1}}=\dfrac{\frac{1}{4}+3}{\left(1-\frac{1}{4}\right)^2}=\dfrac{52}{9}$	
7(i)	$(3+y)^2+x^2=\left(3\sqrt{3}\right)^2$ $9+6y+y^2+x^2=27$ $x^2=18-6y-y^2$ Area of the sheet, $A=2xy$ $=2y\sqrt{18-6y-y^2}$ $=2\sqrt{18y^2-6y^3-y^4}$	

(ii)	$\frac{dA}{dy}=2\frac{1}{2\sqrt{18y^2-6y^3-y^4}}\left(36y-18y^2-4y^3\right)$ $=\frac{2y\left(18-9y-2y^2\right)}{\sqrt{18y^2-6y^3-y^4}}$ [OR $A^2=4\left(18y^2-6y^3-y^4\right)$ $2A\frac{dA}{dy}=4\left(36y-18y^2-4y^3\right)$ $A\frac{dA}{dy}=4y\left(18-9y-2y^2\right)$] For maximum A, $\frac{dA}{dy}=0$ $4y\left(18-9y-2y^2\right)=0$ $2y^2+9y-18=0$ or $y=0$ (reject as $y>0$) $(2y-3)(y+6)=0$ $y=\frac{3}{2}$ or $y=-6$ (reject as $y>0$)	
	(table below)	
	A is maximum when $y=\frac{3}{2}$ When $y=\frac{3}{2}$, Maximum $A=2\sqrt{18\left(\frac{3}{2}\right)^2-6\left(\frac{3}{2}\right)^3-\left(\frac{3}{2}\right)^4}=\frac{9\sqrt{3}}{2}=7.79\text{m}^2$	
8(i)	$\left[\begin{pmatrix}4\\1\\1\end{pmatrix}-\begin{pmatrix}2\\1\\0\end{pmatrix}\right]\times\begin{pmatrix}4\\-1\\2\end{pmatrix}=\begin{pmatrix}2\\0\\1\end{pmatrix}\times\begin{pmatrix}4\\-1\\2\end{pmatrix}=\begin{pmatrix}1\\0\\-2\end{pmatrix}$ Hence a vector perpendicular to the plane p_1 is parallel to $\mathbf{i}-2\mathbf{k}$.	

y	$\left(\frac{3}{2}\right)^-$	$\left(\frac{3}{2}\right)$	$\left(\frac{3}{2}\right)^+$
$\frac{dA}{dy}$	>0	0	<0

	Equation of the plane $p_1 : \mathbf{r} \cdot \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = 2$	
(ii)	Equation of l : $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$ At C, $\begin{pmatrix} 2-2\lambda \\ -1+\lambda \\ -4+\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = 2$ $2-2\lambda+8-2\lambda=2$ $\Rightarrow \lambda = 2$ $\therefore \overrightarrow{OC} = \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix}$ Coordinates $C(-2,1,-2)$	
(iii)	Let F be the foot of perpendicular from D to the plane p_1 . Equation of FD: $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \beta \in \mathbb{R}$ At F, $\begin{pmatrix} 2+\beta \\ -1 \\ -4-2\beta \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = 2$ $2+\beta+8+4\beta=2$ $\Rightarrow \beta = -\frac{8}{5}$ $\therefore \overrightarrow{OF} = \begin{pmatrix} 2-\frac{8}{5} \\ -1 \\ -4-2(-\frac{8}{5}) \end{pmatrix} = \begin{pmatrix} \frac{2}{5} \\ -1 \\ -\frac{4}{5} \end{pmatrix}$ Let E be the image of$D$ under a reflection in the plane p_1 . $\therefore \overrightarrow{OE} = 2\begin{pmatrix} \frac{2}{5} \\ -1 \\ -\frac{4}{5} \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix} = \begin{pmatrix} -\frac{6}{5} \\ -1 \\ \frac{12}{5} \end{pmatrix}$ Coordinates $E\left(-\frac{6}{5}, -1, \frac{12}{5}\right)$	

(iv)	The line of intersection of the planes p_1 and p_2 is AC, $\overline{AC} = \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \\ -3 \end{pmatrix} = -3\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ Equation of line m: $\mathbf{r} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \ \lambda \in \mathbb{R}$	
9(i)	Let P_n be the statement $u_n = 3\left(\frac{2}{3}\right)^n - 1$ for all $n \in \mathbb{Z}^+$. When $n = 1$, LHS $= u_1 = 1$ RHS $= 3\left(\frac{2}{3}\right)^1 - 1 = 1$ LHS = RHS $\therefore P_1$ is true. Assume P_k is true for some $k \in \mathbb{Z}^+$. i.e. $u_k = 3\left(\frac{2}{3}\right)^k - 1$ Prove that P_{k+1} is true. i.e. $u_{k+1} = 3\left(\frac{2}{3}\right)^{k+1} - 1$ $u_{k+1} = \frac{2u_k - 1}{3}$ $= \frac{2\left(3\left(\frac{2}{3}\right)^k - 1\right) - 1}{3}$ $= 2\left(\frac{2}{3}\right)^k - 1$ $= 3\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)^k - 1$ $= 3\left(\frac{2}{3}\right)^{k+1} - 1$ $\therefore P_k$ is true $\Rightarrow P_{k+1}$ is true. Since P_1 is true, P_k is true $\Rightarrow P_{k+1}$ is true, by mathematical induction, P_n is true for all $n \in \mathbb{Z}^+$.	

(ii)

$$\sum_{r=1}^{n} u_r = \sum_{r=1}^{n}\left[3\left(\frac{2}{3}\right)^r - 1\right]$$

$$= 3\sum_{r=1}^{n}\left(\frac{2}{3}\right)^r - n$$

$$= 3\left(\frac{\frac{2}{3}\left(1-\left(\frac{2}{3}\right)^n\right)}{1-\frac{2}{3}}\right) - n$$

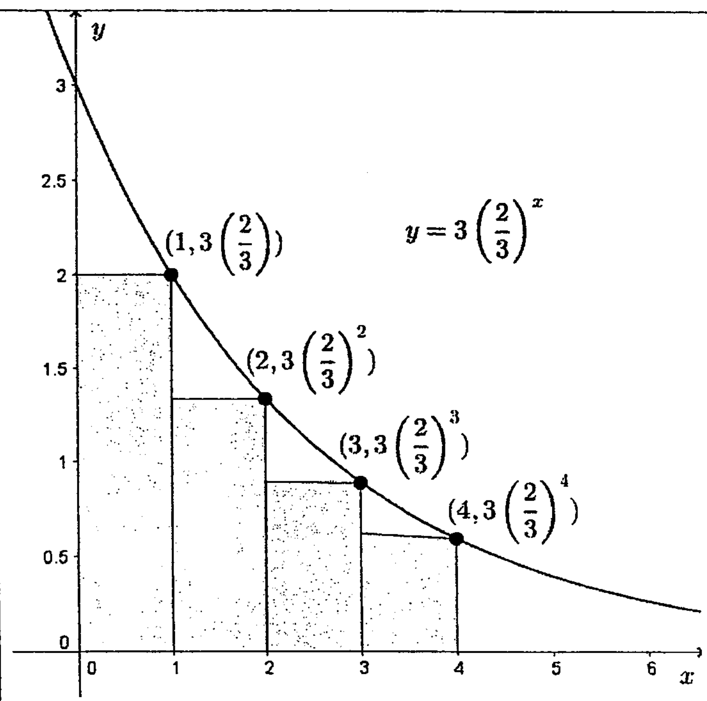
$$= 6\left(1-\left(\frac{2}{3}\right)^n\right) - n$$

(iii)

As $n \to \infty, \left(\frac{2}{3}\right)^n \to 0.$ $\therefore \sum_{r=1}^{n}(u_r+1) \to 6.$

The sum to infinity is 6 OR $S_\infty = 6$.

(iv)



$$\sum_{r=1}^{\infty}(u_r+1) = \sum_{r=1}^{\infty}3\left(\frac{2}{3}\right)^r = 3\left(\frac{2}{3}\right)^1 + 3\left(\frac{2}{3}\right)^2 + 3\left(\frac{2}{3}\right)^3 + \ldots$$

The value of $\sum_{r=1}^{\infty}(u_r+1)$ can be obtained by adding up the areas of the rectangles of width 1 as seen in the sketch. Since the total area of all the rectangles is lesser than the total area under the curve for $x \geq 0$,

$\sum_{r=1}^{\infty}(u_r+1)$ is an underestimation of $\int_0^{\infty} 3\left(\frac{2}{3}\right)^x \, dx$.

10(i)	$\frac{dx}{dt}=k(4+2x-x^2)$ when $t=0, x=1$ and $\frac{dx}{dt}=-\frac{1}{2}$, $-\frac{1}{2}=k(4+2-1)$ $k=-\frac{1}{10}$	
(ii)	$4+2x-x^2 = 4-(x^2-2x)$ $= 4-[(x-1)^2-1]$ $= 5-(x-1)^2$ $\frac{dx}{dt}=k(4+2x-x^2)$ $\int\frac{1}{4+2x-x^2}\,dx=\int k\,dt$ $\int\frac{1}{5-(x-1)^2}\,dx=\int -\frac{1}{10}dt$ $\frac{1}{2(\sqrt{5})}\ln\left\|\frac{\sqrt{5}+(x-1)}{\sqrt{5}-(x-1)}\right\|=-\frac{1}{10}t+c$ When $t=0$, $x=1$ $c=\ln 1=0$ $t=\frac{-10}{2(\sqrt{5})}\ln\left\|\frac{\sqrt{5}+(x-1)}{\sqrt{5}-(x-1)}\right\|$ $t=\frac{5}{\sqrt{5}}\ln\left\|\frac{\sqrt{5}+(x-1)}{\sqrt{5}-(x-1)}\right\|^{-1}$ $t=\sqrt{5}\ln\left\|\frac{\sqrt{5}-(x-1)}{\sqrt{5}+(x-1)}\right\|$ $t=\sqrt{5}\ln\left\|\frac{\sqrt{5}-x+1}{\sqrt{5}+x-1}\right\|$	
(iii)	When $x=0$, $t=\sqrt{5}\ln\left\|\frac{\sqrt{5}+1}{\sqrt{5}-1}\right\|$ Time taken is 2.152 minutes	

<table>
<tr><td>(iv)</td><td>

$t=\sqrt{5}\ln\left|\frac{\sqrt{5}-x+1}{\sqrt{5}+x-1}\right|$

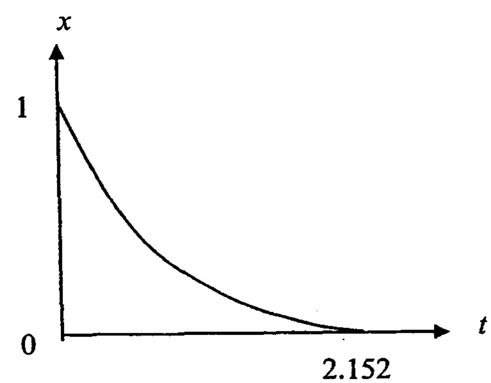
$\ln\left|\frac{\sqrt{5}-x+1}{\sqrt{5}+x-1}\right|=\frac{t}{\sqrt{5}}$

$\frac{\sqrt{5}-x+1}{\sqrt{5}+x-1}=e^{\frac{t}{\sqrt{5}}}$

$(\sqrt{5}+x-1)=e^{-\frac{t}{\sqrt{5}}}(\sqrt{5}-x+1)$

$e^{-\frac{t}{\sqrt{5}}}x+x=e^{-\frac{t}{\sqrt{5}}}(\sqrt{5}+1)-\sqrt{5}+1$

$x=\frac{e^{-\frac{t}{\sqrt{5}}}(\sqrt{5}+1)-\sqrt{5}+1}{1+e^{-\frac{t}{\sqrt{5}}}}$



</td><td></td></tr>
<tr><td></td><td></td><td></td></tr>
<tr><td>11 (i)</td><td>

$\frac{d}{dx}\left(\sqrt{1-x^4}\right)=\frac{-2x^3}{\sqrt{1-x^4}}$

</td><td></td></tr>
<tr><td>(ii)</td><td>

Area of region

$=\int_0^{\frac{1}{\sqrt[4]{2}}}x\left(\cos^{-1}\left(x^2\right)-\sin^{-1}\left(x^2\right)\right)dx$

$=\left[\left(\frac{x^2}{2}\right)\left(\cos^{-1}\left(x^2\right)-\sin^{-1}\left(x^2\right)\right)\right]_0^{\frac{1}{\sqrt[4]{2}}}-\int_0^{\frac{1}{\sqrt[4]{2}}}\left(\frac{x^2}{2}\right)\left(\frac{-4x}{\sqrt{1-x^4}}\right)dx$

$=\frac{1}{2\sqrt{2}}\left(\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)-\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)-\int_0^{\frac{1}{\sqrt[4]{2}}}\frac{-2x^3}{\sqrt{1-x^4}}dx$

$=\frac{1}{2\sqrt{2}}\left(\frac{\pi}{4}-\frac{\pi}{4}\right)-\left[\sqrt{1-x^4}\right]_0^{\frac{1}{\sqrt[4]{2}}}$

$=1-\frac{\sqrt{2}}{2}$

</td><td></td></tr>
</table>

(iii)	Volume of solid formed $= \pi \int_0^{\frac{\pi}{4}\left(\frac{1}{\sqrt[4]{2}}\right)} x^2 \, \mathrm{d}y$ $= \pi \int_0^{\frac{1}{\sqrt[4]{2}}} t^2 \left[\frac{2t^2}{\sqrt{1-t^4}} + \sin^{-1}\left(t^2\right) \right] \mathrm{d}t$ $= \pi \int_0^{\frac{1}{\sqrt[4]{2}}} \left[\frac{2t^4}{\sqrt{1-t^4}} + t^2 \sin^{-1}\left(t^2\right) \right] \mathrm{d}t$	
(iv)	Required volume $= \pi \int_0^{\frac{1}{\sqrt[4]{2}}} \left[\frac{2t^4}{\sqrt{1-t^4}} + t^2 \sin^{-1}\left(t^2\right) \right] \mathrm{d}t - \frac{1}{3}\pi \left(\frac{1}{\sqrt[4]{2}}\right)^2 \left(\frac{\pi}{4}\left(\frac{1}{\sqrt[4]{2}}\right)\right)$ $= 0.909285 - 0.489042$ $= 0.42024$ (5 sig. fig.)	

H2

JC2 PRELIMINARY EXAMINATION

MATHEMATICS — 9740/02

Paper 2

Thursday, 15 Sep 2016

3 hours

Additional Materials: Answer paper

List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your name and civics group on all the work you hand in, including the Cover Page.

Write in dark blue or black pen on both sides of the paper.

You may use an HB pencil for any diagrams or graphs.

Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use an approved graphing calculator.

Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.

The number of marks is given in brackets [ ] at the end of each question or part question.

Section A: Pure Mathematics [40 marks]

1 The curve C has parametric equations

$$x = 2\cos t\,, \quad y = 3\sin t\,.$$

(i) Find the equation of the normal to C at the point P with parameter θ, leaving your answer in terms of θ. [3]

(ii) This normal to C at the point P meets the x- and y-axes at points A and B respectively. Find the cartesian equation of the locus of the midpoint of AB as θ varies. [4]

2 Using partial fractions, find the exact value of

$$\int_0^1 \frac{15x^2 - x + 17}{(2x+1)(x^2+4)}\,dx\,.$$ [8]

3 Mrs X wants to sew handmade gifts for the guests attending her daughter's wedding. On the first day of gift preparation, she spends 270 minutes of her time. Subsequently, she will decrease the amount of time spent each day on gift preparation by a certain amount. The total time taken to complete her gift preparation is 9000 minutes.

(i) Mrs X spends, on each subsequent day, 2.5% less time on gift preparation than on the previous day. Find, to the nearest minute, the time Mrs X spends on the 10th day, and find the minimum number of days Mrs X requires to complete her gift preparation. [6]

(ii) It takes 20 minutes for Mrs X to complete one gift. If Mrs X has the opportunity to make more gifts using this model, in theory, find the maximum number of gifts she can complete. [2]

(iii) After doing some calculations, Mrs X realises that she has to decrease the number of days spent on gift preparation. She still spends 270 minutes on the first day, but on each subsequent day, the amount of time spent is 4 minutes less than on the previous day. Find the minimum number of days Mrs X requires to complete her gift preparation. [3]

4 (a) The complex number w is given by $3+3(\sqrt{3})\text{i}$.

(i) Find the modulus and argument of w, giving your answers in exact form. [2]

(ii) Without using a calculator, find the smallest positive integer value of n for which $\left(\dfrac{w^3}{w^*}\right)^n$ is a real number. [4]

(b) The complex number z is such that $z^5 = -4\sqrt{2}$.

(i) Find the values of z in the form $re^{\text{i}\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. [4]

(ii) Show the roots on an Argand diagram. [2]

(iii) The roots represented by z_1 and z_2 are such that $0 < \arg(z_1) < \arg(z_2) < \pi$. The locus of all points z such that $|z - z_1| = |z - z_2|$ intersects the line segment joining points representing z_1 and z_2 at the point P. P represents the complex number p. Find, in exact form, the modulus and argument of p. [2]

Section B: Statistics [60 marks]

5 It is desired to conduct a survey among university students regarding the use of the university's facilities. A random sample of 100 students is obtained.

(i) Explain what is meant in this context by the term 'a random sample'. [2]

(ii) Now it is necessary to obtain a representative range by faculties. Name a more appropriate sampling method and explain how it can be carried out. [2]

6 75% of the employees in a factory own a cell phone.

(i) A random sample of 8 employees is taken. Find the probability that the number of employees who own a cell phone is between 4 and 6 inclusive. [2]

In an industrial park, every factory has 160 employees.

(ii) Use a suitable approximation to find the probability that in a randomly selected factory, at least 115 employees own a cell phone. [3]

(iii) A random sample of 15 factories in the industrial park is taken. Find the probability that at most 11 of these factories each have at least 115 employees who own a cell phone. [2]

7 The average number of parking tickets that a traffic warden issues per day is being investigated.

(i) State, in context, two assumptions that need to be made for the number of parking tickets issued per day to be well modelled by a Poisson distribution. [2]

Assume that the number of parking tickets issued per day has the distribution Po (3.6).

(ii) Find the probability that, in seven days, the traffic warden issues more than 22 parking tickets altogether. [3]

(iii) The probability that the traffic warden issues more than N parking tickets altogether in 10 days is less than 0.05. Using a suitable approximation, find the least possible value of N. [2]

8 A fruit stall sells papayas. The mass of papayas is denoted by X kg. The stall owner claims that the mean mass of the papayas is at least 1.2 kg. The masses of a random sample of 8 papayas are summarised by

$$\sum x = 8.84, \quad \sum x^2 = 9.95.$$

(i) Find unbiased estimates of the population mean and variance. [2]

(ii) Stating a necessary assumption, test at the 5% level of significance whether there is any evidence to doubt the stall owner's claim. [5]

9 Two players A and B decide to play two consecutive card games. A fair coin is tossed to decide which player has the first move in the first game. The loser of the first game has the first move in the second game. A player must win both games to be declared the overall winner. If each player wins a game, the result is a draw.

When A has the first move in a game, the probability that A wins that game is $\frac{2}{3}$. When B has the first move in a game, the probability that B wins that game is $\frac{3}{5}$. Every game ends with either A or B as the winner.

(i) Show that the probability that two consecutive card games end in a draw is $\frac{142}{225}$. [2]

(ii) Given that A wins the second game, find the probability that two consecutive card games end in a draw. [3]

(iii) To make their games more enjoyable, A and B agree to change the procedure for deciding who has the first move in the first game. As a result of their new procedure, the probability of A having the first move in the first game is p. Find the exact value of p which gives A and B equal chances of winning both games. [3]

10(a) Sketch a scatter diagram that might be expected when x and y are related approximately by $y = px^2 + q$ in each of the cases **(i)** and **(ii)** below. In each case your diagram should include 5 points, approximately equally spaced with respect to x, and with all x- and y-values positive.

(i) p is positive and q is positive,

(ii) p is negative and q is positive. [2]

(b) A car website gives the following information on the ages in months (m) and resale price in dollars (P) of used passenger cars of a particular model.

m	10	20	28	35	40	45	56	62	70	74
P	110600	79900	78700	69200	66100	60200	53800	50600	46700	43800

It is thought that the price after m months can be modelled by one of the formulae

$$P = am + b\ , \quad P = c\ln m + d\ ,$$

where a, b, c and d are constants.

(i) Find, correct to 4 decimal places, the value of the product moment correlation coefficient between

(A) m and P,

(B) $\ln m$ and P. [2]

(ii) Explain which of $P = am + b$ and $P = c\ln m + d$ is the better model and find the equation of a suitable line for this model. [3]

(iii) Use the equation of your regression line to estimate the price of a used passenger car that is 80 months old. Comment on the reliability of your estimate. [2]

11 A group of 5 girls and 7 boys play ice breaker games to get to know each other.

(i) The group stands in a line.

(a) Find the number of different possible arrangements. [1]

(b) The girls have names that start with different letters. Find the number of different possible arrangements in which all the girls are separated, with the girls' names in alphabetical order. [2]

(ii) The group forms two circles of 6, with one circle inside the other, such that each person in the inner circle stands facing a person in the outer circle. Find the number of different possible arrangements. [2]

(iii) The group has to split into a group of 3, a group of 4, and a group of 5. Find the number of possible ways in which the groups can be chosen if there is no girl in at least one of the groups. [4]

12 A supermarket sells two types of guava, A and B. The masses, in grams, of the guava of each type have independent normal distributions. The means and standard deviations of these distributions are shown in the following table.

	Mean (g)	Standard deviation (g)
Type A	200	12
Type B	175	12

Find the probability that

(i) the total mass of 4 randomly chosen guava of type A is more than 810 g, [2]

(ii) the mean mass of 4 randomly chosen guava of type A differs from the mean mass of 3 randomly chosen guava of type B by at least 30 g. [3]

Mr Tan buys 20 guavas, m of them are guava of type A and the rest are guava of type B.

(iii) Find the least value of m such that the probability that the total mass of these 20 guavas exceeding 3500 g is more than 0.95. (Answers obtained by trial and improvement from a calculator will obtain no marks.) [4]

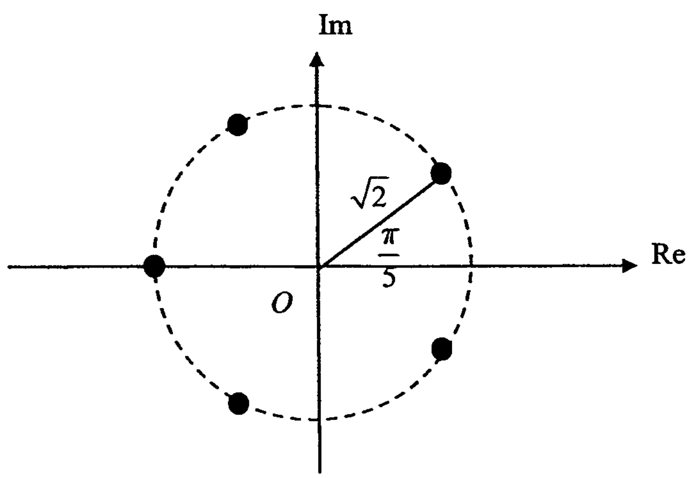
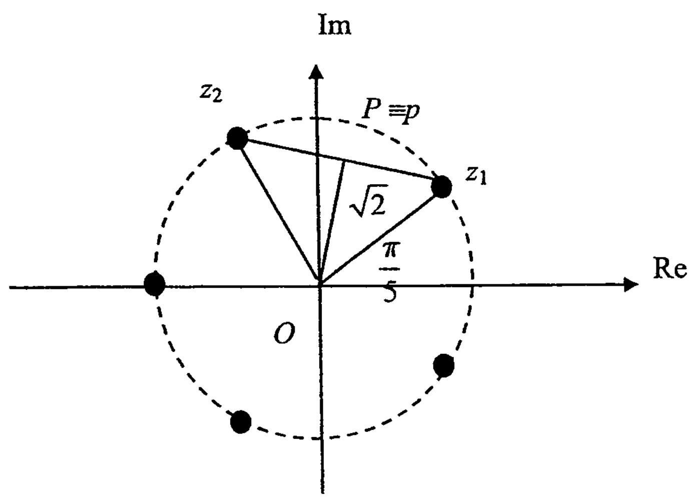
End of Papèr

2016 JC2 Preliminary Examination
H2 Mathematics Paper 2
Solution

1(i)	$x=2\cos t,\quad y=3\sin t$ $\frac{dx}{dt}=-2\sin t\quad \frac{dy}{dt}=3\cos t$ $\frac{dy}{dx}=\frac{3\cos t}{-2\sin t}$ $=-\frac{3}{2}\cot t$ Gradient of normal to C at $P(2\cos\theta, 3\sin\theta)$ is $\frac{2}{3}\tan\theta$. Equation of normal to C at P is $y-3\sin\theta=\left(\frac{2}{3}\tan\theta\right)(x-2\cos\theta)$ $y=\left(\frac{2}{3}\tan\theta\right)(x-2\cos\theta)+3\sin\theta$ $y=\left(\frac{2}{3}\tan\theta\right)x+\frac{5}{3}\sin\theta$	
(ii)	When $y=0$, $-3\sin\theta=\left(\frac{2}{3}\frac{\sin\theta}{\cos\theta}\right)(x-2\cos\theta)$ $x=-\frac{9}{2}\cos\theta+2\cos\theta$ $=-\frac{5}{2}\cos\theta$ $A\left(-\frac{5}{2}\cos\theta, 0\right)$ When $x=0,\ y=\left(\frac{2}{3}\frac{\sin\theta}{\cos\theta}\right)(-2\cos\theta)+3\sin\theta$ $=\frac{5}{3}\sin\theta$ $B\left(0,\frac{5}{3}\sin\theta\right)$ Midpoint of $AB=\left(-\frac{5}{4}\cos\theta, \frac{5}{6}\sin\theta\right)$. $x=-\frac{5}{4}\cos\theta,\quad y=\frac{5}{6}\sin\theta$ $\sin^2\theta+\cos^2\theta=1$	

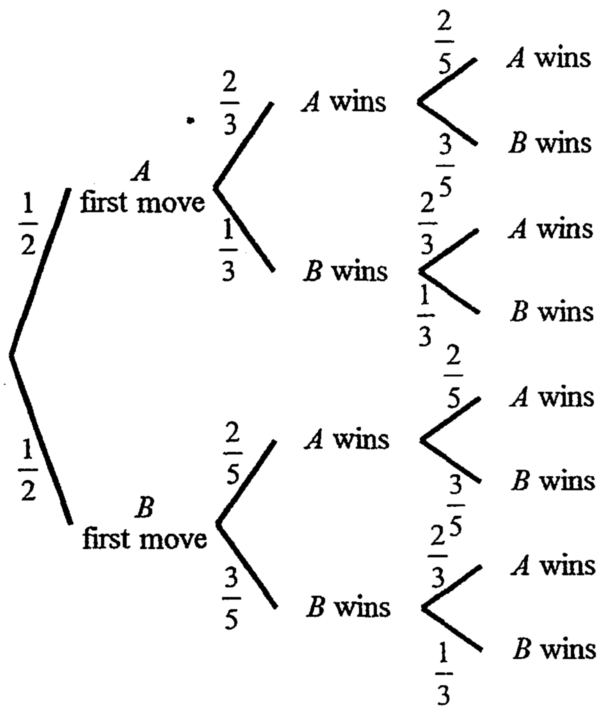
	$\left(\frac{6}{5}y\right)^2+\left(\frac{4}{5}x\right)^2=1$ $16x^2+36y^2=25$	
2	Let $\frac{15x^2-x+17}{(2x+1)(x^2+4)}=\frac{A}{2x+1}+\frac{Bx+C}{x^2+4}$ $15x^2-x+17=A(x^2+4)+(Bx+C)(2x+1)$ Sub $x=-\frac{1}{2}$, $A=5$ Sub $x=0$, $17=4A+C$ $C=-3$ Sub $x=1$, $31=25+3(B-3)$ $B=5$ $\frac{15x^2-x+17}{(2x+1)(x^2+4)}=\frac{5}{2x+1}+\frac{5x-3}{x^2+4}$ $\int_0^1 \frac{15x^2-x+17}{(2x+1)(x^2+4)}\,dx$ $=\int_0^1 \frac{5}{2x+1}\,dx+5\int_0^1 \frac{x}{x^2+4}\,dx-3\int_0^1 \frac{1}{x^2+4}\,dx$ $=\frac{5}{2}\int_0^1 \frac{2}{2x+1}\,dx+\frac{5}{2}\int_0^1 \frac{2x}{x^2+4}\,dx-3\int_0^1 \frac{1}{x^2+2^2}\,dx$ $=\left[\frac{5}{2}\ln\lvert 2x+1\rvert+\frac{5}{2}\ln\lvert x^2+4\rvert-\frac{3}{2}\tan^{-1}(\frac{x}{2})\right]_0^1$ $=\frac{5}{2}\ln 3+\frac{5}{2}\ln 5-\frac{3}{2}\tan^{-1}\left(\frac{1}{2}\right)-\frac{5}{2}\ln 1-\frac{5}{2}\ln 4$ $=\frac{5}{2}\ln\left(\frac{15}{4}\right)-\frac{3}{2}\tan^{-1}\left(\frac{1}{2}\right)$	
3(i)	Let r be the common ratio. Let n be the number of days required. $r=0.975$ Time spent on 10th day $=(270)(0.975)^{10-1}$ $=214.98$ ≈ 215 minutes	

	$\dfrac{270\left(1-0.975^n\right)}{1-0.975} \geq 9000$ $\Rightarrow 10800\left(1-0.975^n\right) \geq 9000$ $\Rightarrow 1-0.975^n \geq \dfrac{5}{6}$ $\Rightarrow 0.975^n \leq \dfrac{1}{6}$ $\Rightarrow n \geq \dfrac{\ln\frac{1}{6}}{\ln 0.975} = 70.771$ Hence, $n = 71$	
(ii)	Theoretical maximum total time $= \dfrac{270}{1-0.975} = 10800$ Maximum number of gifts $= \dfrac{10800}{20} = 540$	
(iii)	Total number of minutes $= \dfrac{n}{2}\left(2(270)+(n-1)(-4)\right) \geq 9000$ $\Rightarrow 2n^2 - 272n + 9000 \leq 0$ From GC, $56.864 \leq n \leq 79.136$. Hence, $n = 57$ (time spent each day on task must be positive)	
4(a) (i)	$\lvert w \rvert = \sqrt{3^2 + (3\sqrt{3})^2} = 6$ $\arg(w) = \tan^{-1}\left(\dfrac{3\sqrt{3}}{3}\right) = \tan^{-1}(\sqrt{3}) = \dfrac{\pi}{3}$	
(ii)	$\left(\dfrac{w^3}{w^*}\right)^n = \left(\dfrac{\left(6e^{i\frac{\pi}{3}}\right)^3}{6e^{-i\frac{\pi}{3}}}\right)^n = \left(\dfrac{216e^{i\pi}}{6e^{-i\frac{\pi}{3}}}\right)^n = 36^n e^{i\frac{4\pi n}{3}}$ Since $\left(\dfrac{w^3}{w^*}\right)^n$ is real, $\sin\left(\dfrac{4\pi n}{3}\right) = 0$. Hence, $\dfrac{4\pi n}{3} = 0, \pi, 2\pi, 3\pi, 4\pi, \ldots$ $n = 0, \dfrac{3}{4}, \dfrac{3}{2}, \dfrac{9}{4}, 3, \ldots$ Smallest positive integer value of n is 3.	

(b) (i)	$z^5 = -4\sqrt{2}$ $= 4\sqrt{2}e^{i\pi}$ $z = \left(4\sqrt{2}e^{i(\pi+2k\pi)}\right)^{1/5}, \quad k = -2, -1, 0, 1, 2$ $= \sqrt{2}e^{i\left(\frac{\pi}{5}+\frac{2k\pi}{5}\right)}$ When $k = 0$, $z_1 = \sqrt{2}e^{i\left(\frac{\pi}{5}\right)}$ When $k = 1$, $z_2 = \sqrt{2}e^{i\left(\frac{3\pi}{5}\right)}$ When $k = 2$, $z_3 = \sqrt{2}e^{i\pi}$ When $k = -2$, $z_4 = \sqrt{2}e^{i\left(-\frac{3\pi}{5}\right)}$ When $k = -1$, $z_5 = \sqrt{2}e^{i\left(-\frac{\pi}{5}\right)}$	
(ii)		
(iii)		

5(i)	A random sample is obtained by selecting 100 students from the population of university students such that every student has an equal chance of being selected and the selection of students is made independently.	
(ii)	A more appropriate sampling method is stratified sampling. Divide the students into mutually exclusive strata (groups) such as the faculties the students belong toand then randomly select the students with the sample size proportional to the relative size of each stratum to form the sample of size 100. The students to be included in the sample can then be selected from each stratum using random sampling or systematic sampling.	
6(i)	Let X be the number of employees who own a cell phone out of 8 employees. Then $X \sim B(8, 0.75)$ $P(4 \le X \le 6) = P(X \le 6) - P(X \le 3)$ $= 0.60562$ ≈ 0.606 (3 s.f.)	
(ii)	Let Y be the number of employees who own a cell phone out of 160 employees Then $Y \sim B(160, 0.75)$ Since $n = 160 > 50$ is large, $np = 160(0.75) = 120 > 5$, $n(1-p) = 160(0.25) = 40 > 5$, $np(1-p) = 30$, $Y \sim N(120, 30)$ approximately $P(Y \ge 115) \overset{c.c}{=} P(Y > 114.5)$ $= 0.84235$ ≈ 0.842 (3 s.f.)	
(iii)	Let W be the number of factories with more than 115 employees who own a cell phone Then $W \sim B(15, 0.84235)$ $P(W \le 11) = 0.202$ (3 s.f.)	
7(i)	The possible assumptions are: (a) The average / mean number of parking tickets issued per day is constant. (b) The parking tickets are issued independently / randomly.	
(ii)	Let Y be the number of parking tickets issued in seven days. Then $Y \sim Po(3.6 \times 7)$ i.e. $Y \sim Po(25.2)$	

	$P(Y > 22) = 1 - P(Y \le 22)$ $= 0.69634$ ≈ 0.696 (3 s.f.)	
(iii)	Let W be the number of parking tickets issued in 10 days. Then $W \sim Po(36)$ Since $\lambda = 36 > 10$ is large, $W \sim N(36, 36)$ approximately $P(W > N) < 0.05 \Rightarrow P(W > N + 0.5) < 0.05$ (using c.c) Using G.C., When $N = 45$, $P(W > N + 0.5) = 0.05667 > 0.05$ When $N = 46$, $P(W > N + 0.5) = 0.04006 < 0.05$ Hence, the least possible value of N is 46 Alternative solution $P(W < N + 0.5) > 0.95$ $\frac{N + 0.5 - 36}{6} > 1.64485$ $N > 45.369$ Hence, the least possible value of N is 46	
8(i)	Unbiased estimate of population mean: $\bar{x} = \frac{\sum x}{n} = \frac{8.84}{8} = 1.105$ Unbiased estimate of population variance: $s^2 = \frac{1}{7}\left(\sum x^2 - \frac{(\sum x)^2}{8}\right) = \frac{1}{7}\left(9.95 - \frac{8.84^2}{8}\right) = 0.02597$	
(ii)	Let X be the mass of papayas To test $H_0: \mu = 1.2$ against $H_1: \mu < 1.2$ at 5% level of significance Since $n = 8$ is small, population variance is unknown, Use t-test. Assume that mass of papayas follows a normal distribution. Test Statistics: $T = \frac{\bar{X} - 1.2}{\sqrt{\frac{s^2}{8}}} \sim t(7)$ under H_0 $t_{test} = \frac{1.105 - 1.2}{\sqrt{\frac{0.02597}{8}}} = -1.67$ Using t-test, by GC, $\mu = 1.2$, $n = 8$, $\bar{x} = 1.105$, $s^2 = 0.02597$, $t_{test} = -1.67$,	

	p-value = 0.0697 Since p-value = 0.0697 >0.05, we do not reject H_0 and conclude that there is insufficient evidence, at the 5% significance level, to reject the stall owner's claim.	
9(i)	 $\mathrm{P}(\text{games end in a draw}) = \frac{1}{2}\left(\frac{2}{3}\times\frac{3}{5}+\frac{1}{3}\times\frac{2}{3}+\frac{2}{5}\times\frac{3}{5}+\frac{3}{5}\times\frac{2}{3}\right)$ $= \frac{142}{225}$	
(ii)	$\mathrm{P}(A \text{ wins the second game and games end in a draw})$ $= \frac{1}{2}\left(\frac{1}{3}\times\frac{2}{3}+\frac{3}{5}\times\frac{2}{3}\right)$ $= \frac{14}{45}$ $\mathrm{P}(A \text{ wins the second game}) = \frac{1}{2}\left(\frac{2}{3}\times\frac{2}{5}+\frac{1}{3}\times\frac{2}{3}+\frac{2}{5}\times\frac{2}{5}+\frac{3}{5}\times\frac{2}{3}\right)$ $= \frac{118}{225}$	

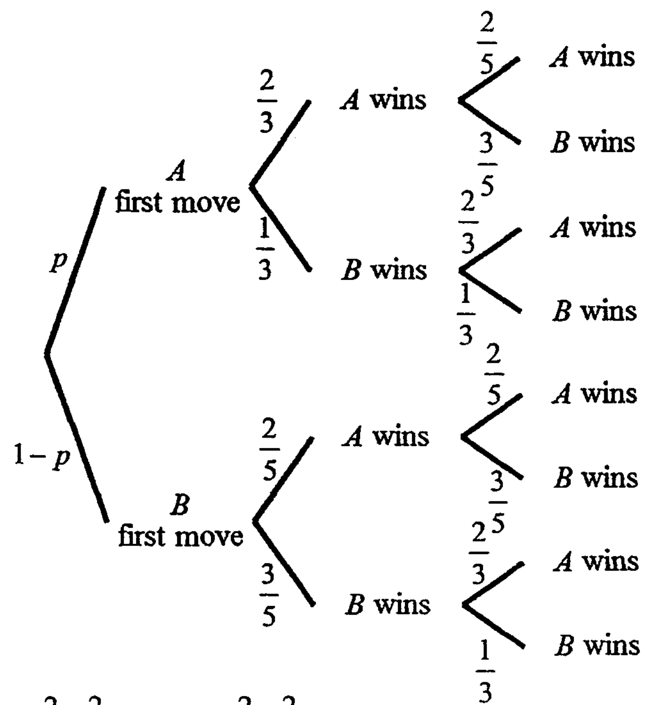
$$\text{P}(\text{games end in a draw} \mid A \text{ wins the second game})$$

$$= \frac{\text{P}(A \text{ wins the second game and games end in a draw})}{\text{P}(A \text{ wins the second game})}$$

$$= \frac{\frac{14}{45}}{\frac{118}{225}}$$

$$= \frac{35}{59}$$

(iii)



$$\text{P}(A \text{ wins}) = p \times \frac{2}{3} \times \frac{2}{5} + (1-p) \times \frac{2}{5} \times \frac{2}{5}$$

$$= \frac{8}{75}p + \frac{4}{25}$$

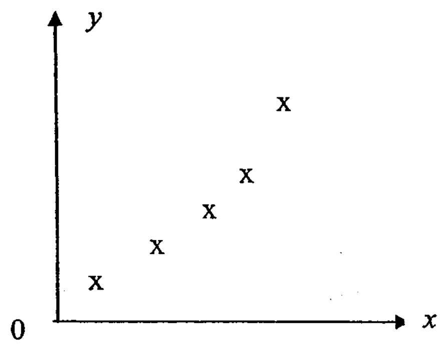
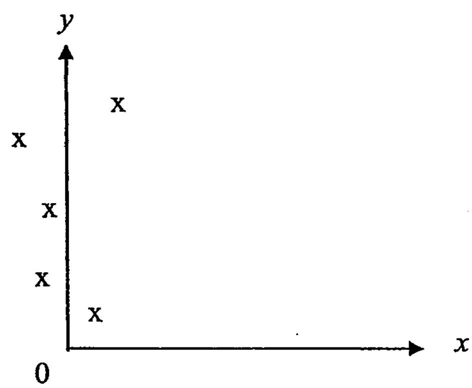
$$\text{P}(B \text{ wins}) = p \times \frac{1}{3} \times \frac{1}{3} + (1-p) \times \frac{3}{5} \times \frac{1}{3}$$

$$= -\frac{4}{45}p + \frac{1}{5}$$

$$\text{P}(A \text{ wins}) = \text{P}(B \text{ wins})$$

$$\Rightarrow \frac{8}{75}p + \frac{4}{25} = -\frac{4}{45}p + \frac{1}{5}$$

$$\Rightarrow p = \frac{9}{44}$$

10(a) (i)		
(ii)		
(b)(i)	(A) r = −0.9492 (B) r = − 0.9916	
(ii)	$P = c\ln m + d$ is a better model since r is closer to −1. As m increases, $P = c\ln m + d$ decreases but cannot be less than zero. / Pdecreases at a decreasing rate. Equation of regression line is $P = -31800\ln m + 182000$	
(iii)	When $m = 80$, $P = -31825.69267\ln 80 + 181762.4239$ $P = 42300$ Estimate is not reliable. $m = 80$ is outside data range. Extrapolation is not a good practice.	
11(i) (a)	Number of different possible orders $= 12! = 479001600$	
(b)	Number of different possible orders $= 7! \times {}^{8}C_{5}$ $= 282240$	
(ii)	Number of different possible arrangements $= {}^{12}C_{6} \times \frac{6!}{6} \times 6!$ $= 79833600$	

(iii)	<u>Case 1: No girls in group of 3</u> Number of ways = ${}^{7}C_3 \times {}^{9}C_4 \times {}^{5}C_5 = 4410$ <u>Case 2: No girls in group of 4</u> Number of ways = ${}^{7}C_4 \times {}^{8}C_3 \times {}^{5}C_5 = 1960$ <u>Case 3: No girls in group of 5</u> Number of ways = ${}^{7}C_5 \times {}^{7}C_3 \times {}^{4}C_4 = 735$ <u>Case 4: No girls in group of 3 and group of 4</u> Number of ways = ${}^{7}C_3 \times {}^{4}C_4 \times {}^{5}C_5 = 35$ Total number of ways $= 4410 + 1960 + 735 - 35 = 7070$ <u>Alternatively, we can use the complement method (not recommended)</u> (There are 6 cases for every group to have at least one girl.) Number of ways $= {}^{12}C_3 \times {}^{9}C_4 \times {}^{5}C_5 - {}^{5}C_2 \times {}^{7}C_1 \times {}^{3}C_2 \times {}^{6}C_2 \times {}^{1}C_1 \times {}^{4}C_4$ $- {}^{5}C_2 \times {}^{7}C_1 \times {}^{3}C_1 \times {}^{6}C_3 \times {}^{2}C_2 \times {}^{3}C_3 - {}^{5}C_1 \times {}^{7}C_2 \times {}^{4}C_2 \times {}^{5}C_2 \times {}^{2}C_2 \times {}^{4}C_4$ $- {}^{5}C_3 \times {}^{2}C_1 \times {}^{7}C_3 \times {}^{1}C_1 \times {}^{4}C_4 - {}^{5}C_1 \times {}^{7}C_2 \times {}^{4}C_1 \times {}^{5}C_3 \times {}^{3}C_3 \times {}^{2}C_2$ $- {}^{5}C_1 \times {}^{7}C_2 \times {}^{4}C_3 \times {}^{5}C_1 \times {}^{1}C_1 \times {}^{4}C_4$ $= 7070$	
12(i)	Let X be the mass of guava of type A Let Y be the mass of guava of type B $X \sim \mathrm{N}(200, 12^2)$ and $Y \sim \mathrm{N}(175, 12^2)$ Let T be the total mass of 4 guava of type A. $T \sim \mathrm{N}(4 \times 200, 4 \times 12^2)$ $T \sim \mathrm{N}(800, 576)$ $\mathrm{P}(T > 810) = 0.33846 \approx 0.338$	
(ii)	$\overline{X} \sim \mathrm{N}(200, 36)$, $\overline{Y} \sim \mathrm{N}(175, 48)$, $\overline{X} - \overline{Y} \sim \mathrm{N}(25, 84)$ $\mathrm{P}\left(\lvert\overline{X} - \overline{Y}\rvert \geq 30\right) = 1 - \mathrm{P}\left(\lvert\overline{X} - \overline{Y}\rvert \leq 30\right)$ $= 0.29269$ $= 0.293$	
(iii)	Let V be the total mass of m guava of type A Let W be the total mass of $(20 - m)$ guava of type B $V \sim \mathrm{N}(200m, 144m)$, $W \sim \mathrm{N}(175(20 - m), 144(20 - m))$ $V + W \sim \mathrm{N}(25m + 3500, 2880)$ $\mathrm{P}(V + W > 3500) > 0.95$	

	$\Rightarrow P\left(Z > \frac{-25m}{\sqrt{2880}}\right) > 0.95$ $\Rightarrow P\left(Z < \frac{25m}{\sqrt{2880}}\right) > 0.95$ $\Rightarrow P\left(Z < \frac{25m}{\sqrt{2880}}\right) > P(Z < 1.64485)$ $\frac{25m}{\sqrt{2880}} > 1.64485$ $m > 3.53$ Least value of $m = 4$	