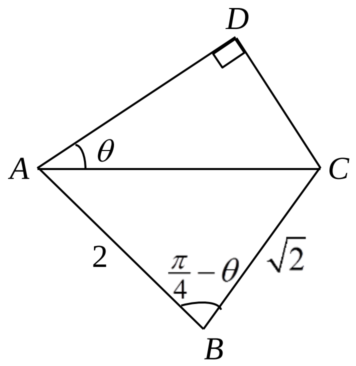
ANDERSON JUNIOR COLLEGE
2017 Preliminary Examination
H2 Mathematics Paper 1 (9758/01)

Duration: 3 hours

1	Mr Tan invested a total of \$25,000 in a structured deposit account, bonds and an estate fund. He invested \$7,000 more in bonds than in estate fund. The projected annual interest rates for structured deposit account, bonds and estate fund are 2%, 3% and 4.5% respectively. Money that is not drawn out at the end of the year will be re-invested for the following year. Mr Tan plans to draw out his money from all investments at the end of the second year and estimates that he will receive a total of \$26,300. Find the amount of money Mr Tan invested in each investment, giving your answer to the nearest dollar. [5]
2	Show that the differential equation $$\frac{dy}{dx}+\frac{3xy}{1-3x^2}-x+1=0$$ may be reduced by means of the substitution $y=u\sqrt{1-3x^2}$ to $$\frac{du}{dx}=\frac{x-1}{\sqrt{1-3x^2}}.$$ Hence find the general solution for y in terms of x. [5]
3	 The diagram above shows a quadrilateral $ABCD$, where AB = 2, $BC=\sqrt{2}$, angle $ABC=\frac{\pi}{4}-\theta$ radians and angle $CAD=\theta$ radians. Show that $$AC=\sqrt{6-4\cos\theta-4\sin\theta}.$$ [2] Given that θ is small enough for θ^3 and higher powers of θ to be neglected, show that $$AD\approx a+b\theta+c\theta^2,$$ where a, b and c are constants to be determined. [5]

4

(a) Given that $\sum_{n=1}^{N} \frac{1}{4n^2-1} = \frac{1}{2} - \frac{1}{2(2N+1)}$, find $\sum_{n=1}^{2N} \frac{1}{4(n+1)^2-1}$.

Deduce that $\sum_{n=1}^{2N} \frac{1}{(2n+3)^2}$ is less than $\frac{1}{6}$. [5]

(b) The sum to n terms of a series is given by $S_n = n\ln 2 - \frac{n^2-1}{\mathrm{e}}$.

Find an expression for the n^{th} term of the series, in terms of n.
Show that the terms of the series follow an arithmetic progression. [4]

5

A curve C has equation $y = \mathrm{f}(x)$. The equation of the tangent to the curve C at the point where $x = 0$ is given by $2x - ay = 3$ where a is a positive constant.

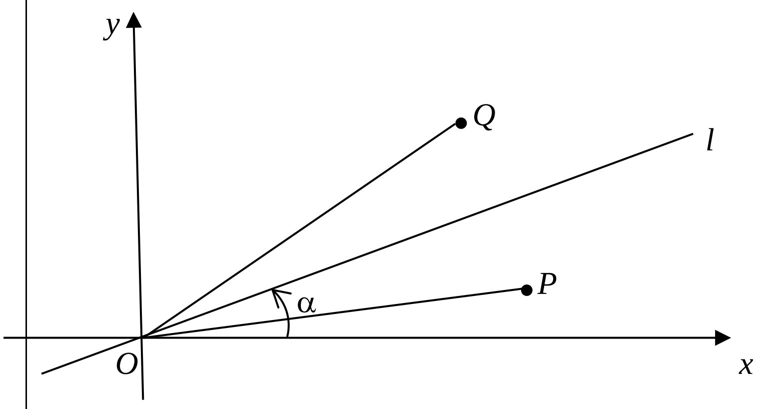
It is also given that $y = \mathrm{f}(x)$ satisfies the equation $(1+2x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and that the third term in the Maclaurin's expansion of $\mathrm{f}(x)$ is $\frac{1}{3}x^2$.

Find the value of a. Hence, find the Maclaurin's series for $\mathrm{f}(x)$ in ascending powers of x, up to and including the term in x^3. [7]

6

The diagram below shows the line l that passes through the origin and makes an angle α with the positive real axis, where $0 < \alpha < \frac{\pi}{2}$.

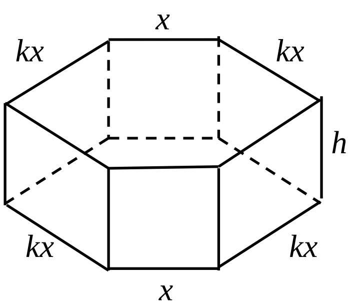
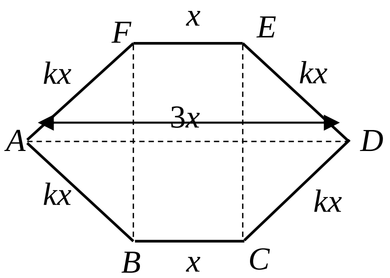
Point P represents the complex number z_1 where $0 < \arg z_1 < \alpha$ and length of OP is r units. Point P is reflected in line l to produce point Q, which represents the complex number z_2.



Prove that $\arg z_1 + \arg z_2 = 2\alpha$. [2]

Deduce that $z_1 z_2 = r^2(\cos 2\alpha + i\sin 2\alpha)$. [1]

Let R be the point that represents the complex number $z_1 z_2$. Given that $\alpha = \frac{\pi}{4}$, write down the cartesian equation of the locus of R as z_1 varies. [2]

7	 Fig 1  Fig 2 Figure 1 shows a solid metal hexagonal prism of height h cm. Figure 2 shows the hexagonal cross-section $ABCDEF$ of the prism where $AD = 3x$ cm, $BC = FE = x$ cm and the remaining 4 sides are of length kx cm each, where k is a constant. Show that $S = 8x^2\sqrt{k^2 - 1} + 2xh(1 + 2k)$, [3] where S is the surface area of this solid hexagonal prism. **(a)** If the volume of the prism is fixed at 400 cm^3, use differentiation to find, in terms of k, the exact value of x that gives a stationary value of S. [3] Let $k = 2$. **(b)** The prism is heated and it expands in such a way that, at time t seconds, the rate of increase of x is the same as the rate of increase of its height h. At the instant when $x = 3$, the prism's height is 8 cm and its surface area is increasing at a constant rate of 0.5 cm^2/s. Find the rate of change of the volume of the prism at this instant. [6]
8	The curve C has equation $y = \dfrac{4x^2 - kx + 2}{x - 2}$, where k is a constant. **(i)** Show that curve C has stationary points when $k < 9$. [3] **(ii)** Sketch the graph of C for the case where $6 < k < 9$, clearly indicating any asymptotes and points of intersection with the axes. [4] **(iii)** Describe a sequence of transformations which transforms the graph of $y = 2x + \dfrac{1}{x}$ to the graph of $y = \dfrac{4x^2 - 8x + 2}{x - 2}$. [3] **(iv)** By drawing a suitable graph on the same diagram as the graph of C, solve the inequality $\dfrac{4x^2 - 8x + 2}{x - 2} > \dfrac{1}{x^2}$. [3]

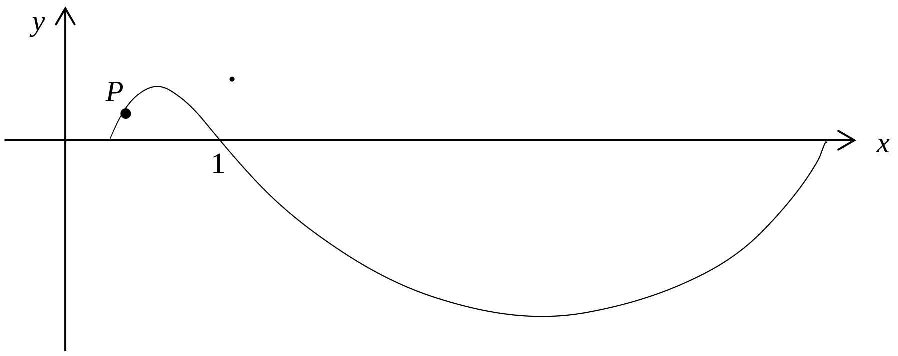
9	The position vectors of A, B and C with respect to the origin O are **a, b** and **c** respectively. It is given that $\overrightarrow{AC} = 4\overrightarrow{CB}$ and $\lvert\mathbf{a}+\mathbf{b}\rvert^2 = \lvert\mathbf{a}\rvert^2 + \lvert\mathbf{b}\rvert^2$. **(i)** By considering $(\mathbf{a}+\mathbf{b}).(\mathbf{a}+\mathbf{b})$, show that **a** and **b** are perpendicular. [2] **(ii)** Find the length of the projection of **c** on **a** in terms of $\lvert\mathbf{a}\rvert$. [3] **(iii)** Given that F is the foot of the perpendicular from C to OA and **f** denotes the position vector $\overrightarrow{OF}$, state the geometrical meaning of $\lvert\mathbf{c}\times\mathbf{f}\rvert$. [1] **(iv)** Two points X and Y move along line segments OA and AB respectively such that $\overrightarrow{OX} = (\cos 3t)\mathbf{i} + (\sin 3t)\mathbf{j} + \frac{1}{2}\mathbf{k},$ $\overrightarrow{OY} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} - 2\mathbf{k},$ where t is a real parameter, $0 \le t \le 2\pi$. By expressing the scalar product of $\overrightarrow{OX}$ and $\overrightarrow{OY}$ in the form of $p\sin(qt)+r$ where p, q and r are real values to be determined, find the greatest value of the angle XOY. [5]
10	There are 25 toll stations, represented by T_1, T_2, T_3,……, T_{25} along a 2000 km stretch of highway. T_1 is located at the start of the highway and T_2 is located x km from T_1. Subsequently, the distance between two consecutive toll stations is 2 km more than the previous distance. Find the range of values x can take. [3] Use $x = 60$ for the rest of this question. Each toll station charges a fee based on the distance travelled from the previous toll station. The fee structure at each toll station is as follows: For the first 60 km, the fee per km will be 5 cents. For every additional 2 km, the fee per km will be 2% less than the previous fee per km. **(i)** Find, in terms of n, the amount of fees a driver will need to pay at T_n. [3] **(ii)** Find the total amount of fees a driver will need to pay, if he drives from T_1 to T_n. Leave your answer in terms of n. [4] More toll stations are built along the highway in the same manner, represented by T_{26}, T_{27}, T_{28},…….. beyond the 2000 km stretch. **(iii)** If a driver starts driving from T_1 and only has \$200, at which toll station will he not have sufficient money for the fees? [2]
11	**(i)** Show by integration that

$$\int \mathrm{e}^{-2x} \sin x \ \mathrm{d}x = -\frac{2}{5}\mathrm{e}^{-2x} \sin x - \frac{1}{5}\mathrm{e}^{-2x} \cos x + A$$

where A is an arbitrary constant. [3]

The diagram below shows a sketch of curve C, with parametric equations

$$x = \mathrm{e}^{-t} \ , \ y = \mathrm{e}^{-t} \sin t \ , \ -\pi \le t \le \pi .$$



Point P lies on C where $t = \dfrac{\pi}{2}$.

(ii) Find the equation of the normal at P. [3]

(iii) Find the exact area bounded by the curve C for $0 \le t \le \pi$, the line $x = 1$ and the normal at P. [5]

(iv) The normal at P cuts the curve C again at two points where $t = q$ and $t = r$. Find the values of q and r. [3]

End of paper

ANNEX B

AJC H2 Math JC2 Preliminary Examination Paper 1

QN	Topic Set	Answers
1	Equations and Inequalities	x = 13937.6 = 13938 (nearest dollars), y = 9031.2 ≈9031, z = 2031.2 ≈ 2031
2	Differential Equations	$y=-\frac{1}{3}\left(1-3x^2\right)-\frac{\sqrt{1-3x^2}}{\sqrt{3}}\sin^{-1}\left(\sqrt{3}x\right)+C\sqrt{1-3x^2}$
3	Binomial Expansion	$a=\sqrt{2}, b=-\sqrt{2}, c=-\frac{\sqrt{2}}{2}$
4	Sigma Notation and Method of Difference	(a) $\frac{1}{6}-\frac{1}{2\left(4N+3\right)}$ (b) $\ln 2-\frac{1}{e}\left(2n-1\right)$
5	Maclaurin series	$a=3;\ -1+\frac{2}{3}x+\frac{1}{3}x^2-\frac{5}{27}x^3+\ldots$
6	Complex numbers	$x=0, y>0$
7	Differentiation & Applications	(a) $x=\sqrt[3]{\frac{25\left(1+2k\right)}{2(k^2-1)}}$ (b) 1.02
8	Graphs and Transformation	(iii) A – Translate the graph by 2 units in the direction of x-axis B - Scaling, parallel to the y-axis by a scale factor of 2. C - Translate the graph by 8 units in the direction of y-axis Alternately: A – Translate the graph by 2 units in the direction of x-axis B - Translate the graph by 4 units in the direction of y-axis C - Scaling, parallel to the y-axis by a scale factor of 2 (iv) $0.805<x<1.69$ or $x>2$
9	Vectors	(ii) $\frac{1}{5}\lvert\mathbf{a}\rvert$ (iii) twice the area of the triangle COF (iv) 143.1°
10	AP and GP	(i) $7.9-4.9\left(0.98^{n-2}\right)$ (ii) $7.9n+245\left(0.98^{n-1}\right)-252.9$

		(iii) 45th toll station
11	Application of Integration	(ii) $y=-x+2e^{-\pi/2}$ (iii) $\frac{11}{10}e^{-\pi}-2e^{-\pi/2}+\frac{7}{10}$ (iv) $q=-1.92$ and $r=-1.01$
12	Q12 Topic	
13	Q13 Topic	

Anderson Junior College
Preliminary Examination 2017
H2 Mathematics Paper 1 (9758/01) solutions with comments

1	Let x, y and z be the amounts Mr Tan invested in structured deposit account, bonds and an estate fund respectively. $x+y+z = 25000$ --- (1) $y = z + 7000$ --- (2) $\left[(1.02x)\times 1.02\right]+\left[(1.03y)\times 1.03\right]+\left[(1.045z)\times 1.045\right]=26300$ --- (3) Solving the 3 simultaneous equations : $x = 13937.6 = 13938$ (nearest dollars), $y = 9031.2 \approx 9031$, $z = 2031.2 \approx 2031$
2	Let $y=u\sqrt{1-3x^2}$ $\Rightarrow \quad \frac{dy}{dx}=\frac{du}{dx}\sqrt{1-3x^2}+u\left(\frac{1}{2}\right)\frac{-6x}{\sqrt{1-3x^2}}$ DE : $\frac{dy}{dx}+\frac{3xy}{1-3x^2}-x+1=0$ $\Rightarrow \quad \frac{du}{dx}\sqrt{1-3x^2}-\frac{-3xu}{\sqrt{1-3x^2}}+\frac{3x}{1-3x^2}\left(u\sqrt{1-3x^2}\right)-x+1=0$ $\Rightarrow \quad \frac{du}{dx}\sqrt{1-3x^2}-\frac{3xu}{\sqrt{1-3x^2}}+\frac{3xu}{\sqrt{1-3x^2}}=x-1$ $\Rightarrow \quad \frac{du}{dx}\sqrt{1-3x^2}=x-1$ $\Rightarrow \quad \frac{du}{dx}=\frac{x}{\sqrt{1-3x^2}}-\frac{1}{\sqrt{1-3x^2}}$ $\Rightarrow \quad u=-\frac{1}{6}\int\frac{-6x}{\sqrt{1-3x^2}}dx-\int\frac{1}{\sqrt{1-3x^2}}\,dx$ $\Rightarrow \quad \frac{y}{\sqrt{1-3x^2}}=-\frac{1}{6}\left[2\sqrt{1-3x^2}\right]-\frac{\sin^{-1}\left(\sqrt{3}x\right)}{\sqrt{3}}+C$ $\Rightarrow \quad y=-\frac{1}{3}\left(1-3x^2\right)-\frac{\sqrt{1-3x^2}}{\sqrt{3}}\sin^{-1}\left(\sqrt{3}x\right)+C\sqrt{1-3x^2}$
3	Consider triangle ABC, $AC^2=4+2-2(2)\sqrt{2}\cos\left(\frac{\pi}{4}-\theta\right)$ $=6-4\sqrt{2}\left(\cos\frac{\pi}{4}\cos\theta+\sin\frac{\pi}{4}\sin\theta\right)=6-4\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta\right)$ $AC=\sqrt{6-4\cos\theta-4\sin\theta}$ (shown) Consider triangle ACD, $\cos\theta=\frac{AD}{AC}$ $AD=\cos\theta\sqrt{6-4\cos\theta-4\sin\theta}$

	Since θ is small, $\sin\theta\approx\theta,\ \cos\theta\approx 1-\dfrac{\theta^2}{2}$, $AD\approx\left(1-\dfrac{\theta^2}{2}\right)\sqrt{6-4\left(1-\dfrac{\theta^2}{2}\right)-4\theta}$ $=\left(1-\dfrac{\theta^2}{2}\right)\left(2+2\theta^2-4\theta\right)^{\frac{1}{2}}$ $=\left(1-\dfrac{\theta^2}{2}\right)\sqrt{2}\left(1+\theta^2-2\theta\right)^{\frac{1}{2}}$ $=\sqrt{2}\left(1-\dfrac{\theta^2}{2}\right)\left(1+\dfrac{1}{2}(\theta^2-2\theta)+\dfrac{(\frac{1}{2})(-\frac{1}{2})}{2}(\theta^2-2\theta)^2+\ldots\right)$ $=\sqrt{2}\left(1-\dfrac{\theta^2}{2}\right)\left(1+\dfrac{1}{2}\theta^2-\theta-\dfrac{1}{2}\theta^2+\ldots\right)$ $=\sqrt{2}\left(1-\dfrac{\theta^2}{2}\right)\left(1-\theta+\ldots\right)$ $=\sqrt{2}\left(1-\theta-\dfrac{\theta^2}{2}+\ldots\right)$ $\approx\sqrt{2}-\sqrt{2}\theta-\dfrac{\sqrt{2}}{2}\theta^2$
4(a)	$\displaystyle\sum_{n=1}^{2N}\frac{1}{4\left(n+1\right)^2-1}=\sum_{n=2}^{2N+1}\frac{1}{4n^2-1}$ $\displaystyle=\sum_{n=1}^{2N+1}\frac{1}{4n^2-1}-\frac{1}{3}$ $\displaystyle=\frac{1}{2}-\frac{1}{2\left[2\left(2N+1\right)+1\right]}-\frac{1}{3}$ $\displaystyle=\frac{1}{6}-\frac{1}{2\left(4N+3\right)}$ $\dfrac{1}{(2n+3)^2}=\dfrac{1}{4n^2+12n+9}$ and $\dfrac{1}{4(n+1)^2-1}=\dfrac{1}{4n^2+8n+3}$ $\therefore\quad\dfrac{1}{(2n+3)^2}<\dfrac{1}{4\left(n+1\right)^2-1}$ Alternative: $\dfrac{1}{\left(2n+3\right)^2}<\dfrac{1}{\left(2n+1\right)\left(2n+3\right)}=\dfrac{1}{4\left(n+1\right)^2-1}$ Hence $\displaystyle\sum_{n=1}^{2N}\frac{1}{\left(2n+3\right)\left(2n+3\right)}<\sum_{n=1}^{2N}\frac{1}{4\left(n+1\right)^2-1}$ $\displaystyle\sum_{n=1}^{2N}\frac{1}{\left(2n+3\right)\left(2n+3\right)}<\frac{1}{6}-\frac{1}{2\left(4N+3\right)}$ [since N>0 & $\dfrac{1}{2\left(4N+3\right)}>0$] $<\dfrac{1}{6}$

4b	$T_n = S_n - S_{n-1} = n\ln 2 - \frac{n^2-1}{e} - \left[(n-1)\ln 2 - \frac{(n-1)^2-1}{e}\right]$ $= \left[n-(n-1)\right]\ln 2 - \frac{1}{e}\left[\left(n^2-1\right)-(n-1)^2+1\right]$ $= \ln 2 - \frac{1}{e}\left[n^2-1-n^2+2n-1+1\right]$ $= \ln 2 - \frac{1}{e}(2n-1)$ $T_n - T_{n-1} = \ln 2 - \frac{1}{e}(2n-1) - \left[\ln 2 - \frac{1}{e}\left(2(n-1)-1\right)\right]$ $= -\frac{2}{e}$ Since $-\frac{2}{e}$ is a constant, the terms follow an AP.
5	Curve C : $y = f(x)$ Tangent to C at x = 0 is $2x - ay = 3 \Rightarrow y = -\frac{3}{a} + \frac{2}{a}x$ Since the tangent to C at x = 0 is $y = f(0) + f'(0)x$, $\therefore f(0) = -\frac{3}{a}$ and $f'(0) = \frac{2}{a}$ The 3rd term of the series for $f(x)$ is $\frac{1}{3}x^2$ $\Rightarrow \frac{f''(0)}{2!}x^2 = \frac{1}{3}x^2$ $\Rightarrow f''(0) = \frac{2}{3}$ From $(1+2x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, When x = 0, we have $\frac{2}{3} + \left(-\frac{3}{a}\right)\left(\frac{2}{a}\right) = 0$ $\Rightarrow a^2 = 9$ $\Rightarrow a = 3$ (since $a > 0$)

$$(1+2x)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+y\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

Differentiate w.r.t. x:

$$(1+2x)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)=0$$

When $x=0$, $y=-\frac{3}{3}=-1$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2}{3}$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{6}{9}=\frac{2}{3}$,

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+(2-1)\left(\frac{2}{3}\right)+\left(\frac{2}{3}\right)^2=0$$

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=-\frac{2}{3}-\frac{4}{9}=-\frac{10}{9}$$

$$\therefore\ y=-1+\frac{2}{3}x+\frac{1}{3}x^2-\frac{10}{9(3!)}x^3+\ldots$$

$$=-1+\frac{2}{3}x+\frac{1}{3}x^2-\frac{5}{27}x^3+\ldots$$

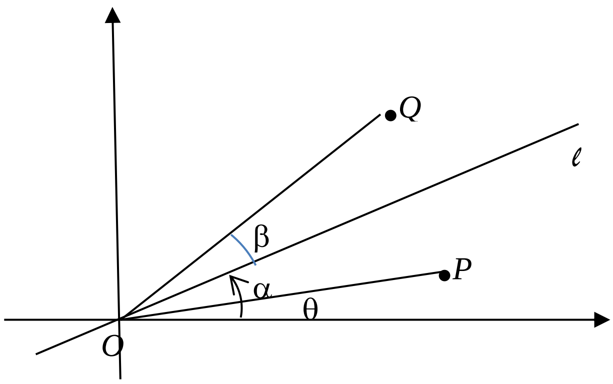
6

$P\equiv z_1=re^{i\theta}$,

$|z_1|=r$ & $\arg(z_1)=\theta$

Let β be angle between lines OQ & l,

$\beta=(\alpha-\theta)$ since line l **bisects** $\angle$POQ



$\arg z_1+\arg z_2$

$=\theta+(\alpha+\beta)$

$=\theta+\alpha+(\alpha-\theta)$

$=2\alpha$

$|z_1z_2|=|z_1||z_2|=r^2$ AND $\arg(z_1z_2)=\arg z_1+\arg z_2=2\alpha$

Hence $z_1z_2=r^2(\cos 2\alpha+i\sin 2\alpha)$.

$\alpha=\frac{\pi}{4}\quad\Rightarrow\quad z_1z_2=r^2\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=r^2i$ (Purely imaginary).

Cartesian equation of the locus of R is $x=0, y>0$

7

$FB=EC=2\sqrt{(kx)^2-x^2}=2x\sqrt{k^2-1}$

Area of cross-section of prism

$=$ Area of $ABCD$ + Area of $AFED$

$=2$(Area of trapezium $ABCD$)

$$=2\left[\frac{1}{2}(x+3x)\sqrt{(kx)^2-x^2}\right]$$

$$=4x^2\sqrt{k^2-1}$$

Surface area of prism, $S=2\left(4x^2\sqrt{k^2-1}\right)+2xh+4kxh$

Hence $S=8x^2\sqrt{k^2-1}+2xh(1+2k)$ (shown) --- (2)

7a	Volume of prism $= 400 = \left(4x^2\sqrt{k^2-1}\right)h$ $h = \dfrac{100}{x^2\sqrt{k^2-1}}$ --- (1) (1) in (2): $S = 8x^2\sqrt{k^2-1} + 2(1+2k)\left(\dfrac{100}{x\sqrt{k^2-1}}\right)$ $\dfrac{dS}{dx} = 16x\sqrt{k^2-1} - \dfrac{200(1+2k)}{x^2\sqrt{k^2-1}}$ When $\dfrac{dS}{dx} = 0$, $x^3 = \dfrac{200(1+2k)}{16(k^2-1)} \Rightarrow x = \sqrt[3]{\dfrac{25(1+2k)}{2(k^2-1)}}$
7b	When k = 2, $S = 8x^2\sqrt{k^2-1} + 2xh(1+2k) = 8\sqrt{3}x^2 + 10xh$ and $V = \left(4x^2\sqrt{k^2-1}\right)h = 4\sqrt{3}x^2h$ Given that $\dfrac{dx}{dt} = \dfrac{dh}{dt}$ $\Rightarrow \dfrac{dh}{dx} = \dfrac{dh}{dt} \times \dfrac{dt}{dx} = 1$ <u>Method 1</u> $\dfrac{dS}{dx} = 8\sqrt{3}(2x) + 10h + 10x\dfrac{dh}{dx} = 16\sqrt{3}x + 10h + 10x$ --- (1) $\dfrac{dV}{dx} = 4\sqrt{3}\left(h.2x + x^2\dfrac{dh}{dx}\right) = 4\sqrt{3}\left(2xh + x^2\right)$ --- (2) When $x = 3,\ h = 8,\ \dfrac{dS}{dt} = 0.5,$ $\dfrac{dS}{dx} = 16\sqrt{3}(3) + 10(8+3) = 48\sqrt{3} + 110$ $\dfrac{dV}{dx} = 4\sqrt{3}\left(2.3.8 + 3^2\right) = 228\sqrt{3}$ $\dfrac{dV}{dt} = \dfrac{dV}{dx} \times \dfrac{dx}{dS} \times \dfrac{dS}{dt}$ $= 228\sqrt{3} \times \dfrac{1}{48\sqrt{3}+110} \times 0.5$ $= 1.02$ (to 3 s.f.) <u>Method 2</u> $\dfrac{dS}{dt} = 8\sqrt{3}\left(2x\dfrac{dx}{dt}\right) + 10\left(h\dfrac{dx}{dt} + x\dfrac{dh}{dt}\right) = \left(16\sqrt{3}x + 10h + 10x\right)\dfrac{dx}{dt}$ --- (1) And $\dfrac{dV}{dt} = 4\sqrt{3}\left(h.2x\dfrac{dx}{dt} + x^2\dfrac{dh}{dt}\right) = 4\sqrt{3}\left(2xh + x^2\right)\dfrac{dx}{dt}$ --- (2)

	When $x=3,\ h=8,\ \dfrac{\mathrm{d}S}{\mathrm{d}t}=0.5,$ using eqn (1) to find $\dfrac{\mathrm{d}x}{\mathrm{d}t}$ $0.5=\left(16\sqrt{3}(3)+10(8)+10(3)\right)\dfrac{\mathrm{d}x}{\mathrm{d}t}$ $\dfrac{\mathrm{d}x}{\mathrm{d}t}=\dfrac{0.5}{48\sqrt{3}+110}\approx 0.0025888$ Sub into (2) to get $\dfrac{\mathrm{d}V}{\mathrm{d}t}$ $\dfrac{\mathrm{d}V}{\mathrm{d}t}=4\sqrt{3}\left(2.3.8+3^2\right)(0.0025888)=1.022343317\approx 1.02$ (to 3 sf)
8i)	$y=\dfrac{4x^2-kx+2}{x-2}$ By long division, $y=4x+8-k+\dfrac{18-2k}{x-2}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{(x-2)(8x-k)-(4x^2-kx+2)(1)}{(x-2)^2}$ $=\dfrac{4x^2-16x+2k-2}{(x-2)^2}$ Let $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0 \Rightarrow 4x^2-16x+2k-2=0$ $\Rightarrow 2x^2-8x+k-1=0$ $\Rightarrow x=\dfrac{8\pm\sqrt{64-4(2)(k-1)}}{4}=2\pm\sqrt{\dfrac{9-k}{2}}$ C has stationary point when $k\le 9$ However, when k = 9, the value x=2 is undefined on the curve. In fact, the curve C is a straight line, y = 4x - 1. Hence C has stationary point when k < 9. <u>Alternative Presentation 1:</u> Let $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0 \Rightarrow 4x^2-16x+2k-2=0$ $\Rightarrow 2x^2-8x+k-1=0$ For $\dfrac{dy}{dx}$=0 to have real roots, "$\mathrm{b}^2-4ac\ge 0$" $\Rightarrow 8^2-4(2)(k-1)\ge 0$ $\Rightarrow 64-8k+8\ge 0$ $\Rightarrow 8k\le 72$ $\Rightarrow k\le 9$ <u>Alternative Presentation 2:</u> $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0 \Rightarrow 4x^2-16x+2k-2=0$ $\Rightarrow 2x^2-8x+k-1=0$ $\Rightarrow 2(x-2)^2+k-9=0$ $\Rightarrow 2(x-2)^2=9-k$ For $\dfrac{dy}{dx}=0$ to have roots x, $9-k\ge 0\Rightarrow k\le 9$ However, when k = 9, the value x=2 is undefined on the curve. In fact, the curve C is a straight line, y = 4x - 1. Hence C has stationary point when k < 9.

<table>
<tr>
<td>(ii)</td>
<td>

$$y=\frac{4x^2-kx+2}{x-2}=4x+(8-k)+\frac{18-2k}{x-2}$$

Asymptotes of C are $y = 4x + 8 - k$ and $x = 2$
When $x=0$, $y==-1$.
When $y=0$, $4x^2-kx+2=0$

$$\Rightarrow x=\frac{k\pm\sqrt{k^2-32}}{8}$$

The axial intercepts are $(0,-1)$, $\left(\frac{k-\sqrt{k^2-32}}{8},0\right)$ and $\left(\frac{k+\sqrt{k^2-32}}{8},0\right)$.

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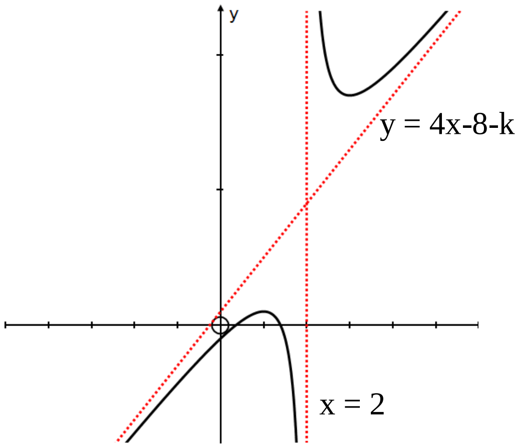
<tr>
<td>ii)</td>
<td>



</td>
</tr>
<tr>
<td>(iii)</td>
<td>

When k = 8, $y=4x+(8-8)+\frac{18-2(8)}{x-2}=4x+\frac{2}{x-2}$

$$y=2x+\frac{1}{x}\xrightarrow{A}y=2\left(2x+\frac{1}{x}\right)=4x+\frac{2}{x}$$

$$y=4x+\frac{2}{x}\xrightarrow{B}y=4(x-2)+\frac{2}{(x-2)}=y=4x-8+\frac{2}{(x-2)}$$

$$y=4x-8+\frac{2}{(x-2)}\xrightarrow{C}y=\left(4x-8+\frac{2}{(x-2)}\right)+8=4x+\frac{2}{(x-2)}$$

A – Translate the graph by 2 units in the direction of x-axis
B - Scaling, parallel to the y-axis by a scale factor of 2.
C - Translate the graph by 8 units in the direction of y-axis

<u>Alternate Sequence of Transformations:</u>
A – Translate the graph by 2 units in the direction of x-axis
B - Translate the graph by 4 units in the direction of y-axis
C - Scaling, parallel to the y-axis by a scale factor of 2.

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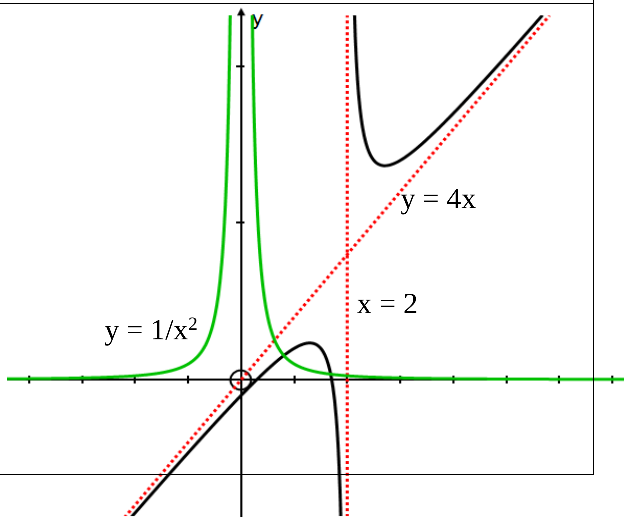
<tr>
<td>iv)</td>
<td>

When $k=8$, $\frac{4x^2-kx+2}{x-2}>\frac{1}{x^2}$

$$\Rightarrow \frac{4x^2-8x+2}{x-2}>\frac{1}{x^2}$$

From G.C,
$0.805<x<1.69$ or $x>2$.



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</table>

9(i)	$(\mathbf{a}+\mathbf{b}).(\mathbf{a}+\mathbf{b})=\mathbf{a.a}+2\mathbf{a.b}+\mathbf{b.b}$ Since $(\mathbf{a}+\mathbf{b}).(\mathbf{a}+\mathbf{b})=\lvert\mathbf{a}+\mathbf{b}\rvert^2$ and given $\lvert\mathbf{a}+\mathbf{b}\rvert^2=\lvert\mathbf{a}\rvert^2+\lvert\mathbf{b}\rvert^2$ $\therefore\ \mathbf{a.a}+2\mathbf{a.b}+\mathbf{b.b}=\lvert\mathbf{a}\rvert^2+\lvert\mathbf{b}\rvert^2$ $\lvert\mathbf{a}\rvert^2+2\mathbf{a.b}+\lvert\mathbf{b}\rvert^2=\lvert\mathbf{a}\rvert^2+\lvert\mathbf{b}\rvert^2$ $2\mathbf{a.b}=0$ $\mathbf{a.b}=0$ $\therefore\ \mathbf{a}\perp\mathbf{b}$
ii)	Using ratio theorem, $\overrightarrow{OC}=\frac{4\mathbf{b}+\mathbf{a}}{5}=\frac{1}{5}\mathbf{a}+\frac{4}{5}\mathbf{b}$. **Length of projection of $\overrightarrow{\mathbf{OC}}$ onto $\overrightarrow{\mathbf{OA}}$** $=\frac{\lvert\overrightarrow{OC}\bullet\overrightarrow{OA}\rvert}{\lvert\overrightarrow{OA}\rvert}$ $=\frac{\left\lvert\left(\frac{1}{5}\mathbf{a}+\frac{4}{5}\mathbf{b}\right)\bullet\mathbf{a}\right\rvert}{\lvert\mathbf{a}\rvert}=\frac{\left\lvert\frac{1}{5}\mathbf{a}\bullet\mathbf{a}+\frac{4}{5}\mathbf{b}\bullet\mathbf{a}\right\rvert}{\lvert\mathbf{a}\rvert}$ $=\frac{\left\lvert\frac{1}{5}\lvert\mathbf{a}\rvert^2+\frac{4}{5}\mathbf{b}\bullet\mathbf{a}\right\rvert}{\lvert\mathbf{a}\rvert}\quad=\frac{1}{5}\lvert\mathbf{a}\rvert$ **(since $\mathbf{a}\perp\mathbf{b}$)**
iii)	$\lvert\mathbf{c}\times\mathbf{f}\rvert$ denotes twice the area of the triangle COF.
(iv)	$\overrightarrow{OX}\bullet\overrightarrow{OY}=\begin{pmatrix}\cos 3t\\ \sin 3t\\ \frac{1}{2}\end{pmatrix}\bullet\begin{pmatrix}\sin t\\ \cos t\\ -2\end{pmatrix}=\cos 3t\sin t+\sin 3t\cos t-1=\sin(4t)-1$ $\cos\measuredangle XOY=\frac{\overrightarrow{OX}\bullet\overrightarrow{OY}}{\lvert\overrightarrow{OX}\rvert\lvert\overrightarrow{OY}\rvert}=\frac{\sin 4t-1}{\sqrt{\cos^2 3t+\sin^2 3t+\frac{1}{4}}\sqrt{\sin^2 t+\cos^2 t+4}}$ $=\frac{\sin 4t-1}{\sqrt{\frac{5}{4}}\sqrt{5}}$ $=\frac{2}{5}(\sin 4t-1)$ Maximum $\measuredangle XOY$ occurs when is most negative. i.e. when $\sin 4t=-1$. At that value of *t*, $\cos\measuredangle XOY=\frac{2}{5}(-1-1)=-\frac{4}{5}$ $\therefore\measuredangle XOY=\cos^{-1}\left(-\frac{4}{5}\right)=143.1°$
10	$x+(x+2)+(x+2(2))\ +\ \ldots\ +(x+\ 23(2))\le 2000$ This is an AP with first term= *x*, common difference = 2, number of terms = 24 $\frac{24}{2}\left[2x+23(2)\right]\le 2000$ $0<x\le\frac{181}{3}$

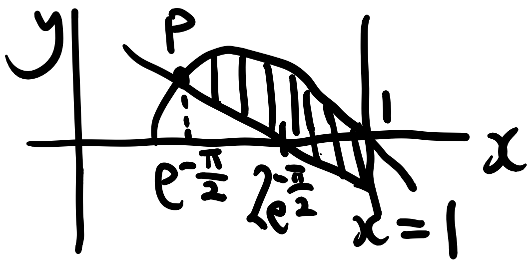
10(i)	<table><tr><td>n</td><td>Amount paid at T_n</td></tr><tr><td>2</td><td>$60(0.05)$</td></tr><tr><td>3</td><td>$60(0.05)+2(0.05)(0.98)$</td></tr><tr><td>4</td><td>$60(0.05)+2(0.05)(0.98)+2(0.05)(0.98)^2$</td></tr><tr><td></td><td>. .</td></tr><tr><td>n</td><td>$60(0.05)+2(0.05)(0.98)+2(0.05)(0.98)^2+...+2(0.05)(0.98)^{n-2}$</td></tr></table> Amount of fees at T_n $=3+\dfrac{0.098\left(1-0.98^{n-2}\right)}{1-0.98}$ $=3+4.9\left(1-0.98^{n-2}\right)$ $=7.9-4.9\left(0.98^{n-2}\right)$
ii	$\sum_{r=2}^{n}\left[7.9-4.9\left(0.98^{r-2}\right)\right]$ $=\sum_{r=2}^{n}7.9-4.9\sum_{r=2}^{n}\left(0.98^{r-2}\right)$ $=7.9(n-1)-4.9\left[\dfrac{1\left(1-0.98^{n-1}\right)}{1-0.98}\right]$ $=7.9(n-1)-245\left(1-0.98^{n-1}\right)$ $=7.9n+245\left(0.98^{n-1}\right)-252.9$
iii	Let $\mathrm{f}(n)=7.9n+245\left(0.98^{n-1}\right)-252.9$. Note that $\mathrm{f}(n)$ is increasing in n Consider $7.9n+245\left(0.98^{n-1}\right)-252.9>200$ <table><tr><td>44</td><td>197.47</td></tr><tr><td>45</td><td>203.32</td></tr><tr><td>46</td><td>209.21</td></tr></table> Using GC, $n\geq 45$ He will not have sufficient money at the 45th toll station.
11i	$\int e^{-2x}\sin x\ \mathrm{d}x$ $=(-\cos x)\left(e^{-2x}\right)-\int(-\cos x)\left(-2e^{-2x}\right)\ \mathrm{d}x$ $=-e^{-2x}\cos x-2\left[(\sin x)\left(e^{-2x}\right)-\int\sin x\left(-2e^{-2x}\right)\ \mathrm{d}x\right]$ $=-e^{-2x}\cos x-2e^{-2x}\sin x-4\int e^{-2x}\sin x\ \mathrm{d}x$ $5\int e^{-2x}\sin x\ \mathrm{d}x=-e^{-2x}\cos x-2e^{-2x}\sin x+C$ $\int e^{-2x}\sin x\ \mathrm{d}x=-\dfrac{2}{5}e^{-2x}\sin x-\dfrac{1}{5}e^{-2x}\cos x+A$
11ii	$\dfrac{\mathrm{d}x}{\mathrm{d}t}=-e^{-t}$ $\qquad$ $\dfrac{\mathrm{d}y}{\mathrm{d}t}=-e^{-t}\sin t+e^{-t}\cos t$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{-e^{-t}\sin t+e^{-t}\cos t}{-e^{-t}}=\sin t-\cos t$

At $t=\frac{\pi}{2}$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\sin\left(\frac{\pi}{2}\right)-\cos\left(\frac{\pi}{2}\right)=1-0=1$, so gradient of normal = -1

$x=e^{-\pi/2}$, $\quad y=e^{-\pi/2}\sin\frac{\pi}{2}=e^{-\pi/2}$

Equation of normal: $\quad y-e^{-\pi/2}=-1(x-e^{-\pi/2}) \quad \Rightarrow \quad y=-x+2e^{-\pi/2}$

11iii



Area

$$=\int_{e^{-\pi/2}}^{1} e^{-t}\sin t-\left(-x+2e^{-\pi/2}\right)\mathrm{d}x$$

$$=\int_{\pi/2}^{0} e^{-t}\sin t(-e^{-t})\mathrm{d}t+\int_{e^{-\pi/2}}^{1} x-2e^{-\pi/2}\ \mathrm{d}x$$

$$=-\int_{\pi/2}^{0} e^{-2t}\sin t\ \mathrm{d}t+\left[\frac{x^2}{2}\right]_{e^{-\pi/2}}^{1}-\left[2e^{-\pi/2}x\right]_{e^{-\pi/2}}^{1}$$

$$=-\left[-\frac{2}{5}e^{-2x}\sin x-\frac{1}{5}e^{-2x}\cos x\right]_{\pi/2}^{0}+\left[\frac{1}{2}-\frac{e^{-\pi}}{2}\right]-\left[2e^{-\pi/2}\left(1-e^{-\pi/2}\right)\right]$$

$$=\frac{2}{5}e^{0}\sin 0+\frac{1}{5}e^{0}\cos 0-\frac{2}{5}e^{-\pi}\sin\left(\frac{\pi}{2}\right)-\frac{1}{5}e^{-\pi}\cos\frac{\pi}{2}+\frac{1}{2}-\frac{e^{-\pi}}{2}-2e^{-\pi/2}+2e^{-\pi}$$

$$=\frac{1}{5}-\frac{2}{5}e^{-\pi}+\frac{1}{2}-\frac{e^{-\pi}}{2}-2e^{-\pi/2}+2e^{-\pi}$$

$$=\frac{11}{10}e^{-\pi}-2e^{-\pi/2}+\frac{7}{10}$$

Alternative:

Area = $\int_{e^{-\pi/2}}^{1} e^{-t}\sin t-\frac{1}{2}(e^{-\pi/2})(2e^{-\pi/2}-e^{-\pi/2})+\frac{1}{2}(1-2e^{-\pi/2})^2$

[When x =1, y = 1+ $2e^{-\pi/2}$]

11iv

For normal to meet curve again,

Substitute parametric eqns into $y=-x+2e^{-\pi/2}$

$e^{-t}\sin t=-e^{-t}+2e^{-\pi/2}$

$e^{-t}(\sin t+1)-2e^{-\pi/2}=0$

Using GC, $t=-1.92148,\ \ -1.0145,\ \ 1.5707$ (rej, this is $\frac{\pi}{2}$)

So $q=-1.92$ and $r=-1.01$ (to 3 sf)

ANDERSON JUNIOR COLLEGE
2017 Preliminary Examination
H2 Mathematics Paper 2 (9758/02)

Duration: 3 hours

	Section A: Pure Mathematics [40 marks]
1	At the intensive care unit of a hospital, patients of a particular condition receive a certain treatment drug through an intravenous drip at a constant rate of 30mg per hour. Due to the limited capacity for absorption by the body, the drug is lost from a patient's body at a rate proportional to x , where x is the amount of drug (in mg) present in the body at time t (in hours). It is assumed that there is no presence of the drug in any patient prior to admission to the hospital. (i) Form a differential equation involving x and t and show that $x=\frac{30}{k}\left(1-\mathrm{e}^{-kt}\right)$ where k is a positive constant. [4] (ii) If there is more than 1000mg of drug present in a patient's body, it is considered an overdose. Suppose the drug continues to be administered, determine the range of values of k such that a patient will have an overdose. [2] For a particular patient, $k=\frac{1}{50}$. (iii) Find the time required for the amount of the drug present in the patient's body to be 200mg. [3]
2	The polynomial P(z) has real coefficients. The equation P(z) = 0 has a root $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $0<\theta<\pi$. Write down a second root in terms of r and θ, and hence show that a quadratic factor of P(z) is $z^2-2rz\cos\theta+r^2$. [2] Let P(z) = $z^3+az^2+15z+18$ where a is a real number. One of the roots of the equation P(z) = 0 is $3\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}\right)}$. By expressing P(z) as a product of two factors with real coefficients, find a and the other roots of P(z) = 0. [4] Deduce the roots of the equation $18z^3+15z^2+az+1=0$. [2]

3	Planes Π_1 and Π_2 are defined by $$\Pi_1 : x - 2y + 2z = 7, \quad \Pi_2 : \mathbf{r} \cdot \begin{pmatrix} a \\ 3 \\ -1 \end{pmatrix} = 8.$$ where a is a constant. **(i)** The point P has position vector $-2\mathbf{i} + \mathbf{j} + \mathbf{k}$. Find the position vector of F, the foot of the perpendicular from P to plane Π_1. Hence, or otherwise, find the shortest distance from P to plane Π_1. [5] **(ii)** Line m passes through the point F and is parallel to both planes Π_1 and Π_2. Find the vector equation of line m. [2] **(iii)** It is given that the point $Q(1, -4, -1)$ lies on line m. Find the value of a. [3] **(iv)** Find the length of projection of $\overrightarrow{PQ}$ on the x-y plane. [3]
4	The function f is defined by $$\mathrm{f} : x \mapsto \frac{\mathrm{e}^x - 1}{\mathrm{e} - 1} \quad \text{for} \quad x \in \mathbb{R}.$$ Sketch the graph of $y = \mathrm{f}(x)$ and state the range of f. [3] Another function h is defined by $$\mathrm{h} : x \mapsto \begin{cases} (x-1)^2 + 1 & \text{for} \quad x \le 1 \\ 1 - \dfrac{\lvert 1 - x \rvert}{2} & \text{for} \quad 1 < x \le 4 \end{cases}$$ Sketch the graph of $y = \mathrm{h}(x)$ for $x \le 4$ and explain why the composite function $\mathrm{f}^{-1}\mathrm{h}$ exists. Hence find the exact value of $\left(\mathrm{f}^{-1}\mathrm{h}\right)^{-1}(3)$. [7]

Section B: Probability and Statistics [60 marks]

5 A vehicle insurance company classifies the drivers it insures as class A, B and C according to whether they are of low risk, medium risk or high risk with regard to having an accident. The company estimates that 30% of the drivers who are insured are class A and 50% are class B. The probability that a class A driver will have at least one accident in any 12 month period is 0.01, the corresponding probabilities for class B and C are 0.03 and 0.06 respectively.

(i) Find the probability that a randomly chosen driver will have at least one accident in a 12-month period. [2]

(ii) The company sold a policy to a driver and within 12 months, the driver had at least one accident. Find the probability that the driver is of class C. [2]

(iii) Three drivers insured by the company are chosen randomly. Find the probability that all three drivers are of class C and exactly one of them had at least one accident in a 12-month period. [3]

6 In an experiment to investigate the decay of organic material over time, a bag of leaf litter was allowed to sit for a 20-week period in a moderately forested area.

The table below shows the weight of the remaining leaf litter (y kg) when x number of weeks have passed.

x	1	2	4	6	8	9	11	15	20
y	60.9	51.8	34.7	26.2	14.0	12.3	8.2	3.1	1.4

(i) Draw a scatter diagram of these data. [1]

(ii) Find the equation of the regression line of y on x and calculate the corresponding estimated value of y when $x = 17$.
Comment on the suitability of the linear model for these data. [3]

The variable W is defined as $W = \ln y$.

(iii) Find the product moment correlation coefficient between W and x. [1]

(iv) It is given that the weight of the leaf litter in the bag was 75.0 kg initially. Using an appropriate regression line, estimate how long it takes for the weight of the leaf litter to drop to half its initial value, giving your answer to one decimal place. [3]
Give two reasons why you would expect this estimate to be reliable. [2]

7	**(a)** The Health Promotion Board of a certain country claims that the average number of hours of sleep of working adults is at most 6 hours per day. To investigate this claim, the editor of a magazine plans to conduct a survey on a sample of adults travelling to work by train. **(i)** Explain why this method of sampling will not give a random sample for the purpose of the investigation. [1] The editor of another magazine interviewed a random sample of 50 working adults and their number of hours of sleep per day, *x*, are summarised as follows: $\sum x = 320$, $\sum x^2 = 2308.5$ **(ii)** Test at the 5% level of significance whether there is any evidence to doubt the Health Promotion Board's claim. State with a reason, whether it is necessary to assume that the number of hours of sleep per day follows a normal distribution. [5] **(b)** The Health Promotion Board carried out their own survey on another random sample of 50 working adults. The sample yielded an average of 6.14 hours of sleep per day and a standard deviation of 2.1 hours. If the sample does not provide significant evidence at the 5% level of significance that the mean number of hours of sleep per day of working adults differs from μ_o hours, find the range of values of μ_o [4]
8	A biscuit manufacturer produces both cream and chocolate biscuits. Biscuits are chosen randomly and packed into boxes of 10. The number of cream biscuits in a box is denoted by *X*. **(a)** On average, the proportion of cream biscuits is *p*. Given that P(*X* = 1 or 2) = 0.15, write down an equation for the value of *p*. Hence find the value(s) of *p* numerically. [3] **(b)** It is given instead that the biscuit manufacturer produces 3 times as many cream biscuits as chocolate biscuits. **(i)** Find the most likely value of *X*. [2] **(ii)** A random sample of 18 boxes is taken. Find the probability that at least 3 but fewer than 7 boxes have equal numbers of cream and chocolate biscuits. [3] A box of biscuits is sold at \$10. The manufacturer gives a discount of \$2 per box to its premium customers. The mean and variance of the number of boxes sold per day to each type of customers (assuming independence) are as follows:

	Mean	Variance
Number of boxes sold at usual price	180	64
Number of boxes sold at discounted price	840	169

Find the approximate probability that the total amount collected per month from the sales of biscuits is not less than \$255,000, assuming that there are 30 days in a month. [4]

9 Four families arrive at Science Centre together. Mr and Mrs *A* brought their 2 children while Mr *B* brought his 2 children. Mr and Mrs *C* brought their 3 children while Mrs *D* brought her only child. All these 14 people have to go through a gate one at a time to enter the centre.

(i) In how many different ways can they go through the gate if each family goes in one after another? [2]

There are two experiments at the ***Science Magic Experience*** station.

(ii) In one experiment, participants are to be in groups of twos or threes. In how many different ways can the 8 children from the four families be grouped among themselves? [3]

(iii) In another experiment, the four families have to hold hands to form two separate circles of equal size to experience a science phenomenon. Each circle must have exactly four children and members of the same family must be in the same circle. Find the number of ways of arranging these 14 people in the two circles such that there is no more than one adult between any two children. [3]

10 Males and females visiting an amusement park have heights, in centimetres, which are normally distributed with means and standard deviations as shown in the following table:

	Mean (cm)	Standard deviation (cm)
Male	165	12
Female	155	σ

It is found that 38.29% of the females have heights between 150 cm and 160 cm.

(i) Show that $\sigma = 10.0$ cm, correct to 3 significant figures. [2]

(ii) Find the probability that the height of a randomly chosen female is within 20 cm of three-quarter the height of a randomly chosen male. State an assumption that is necessary for the calculation to be valid. [4]

The amount, \$$X$, a visitor has to pay for a popular ride in the park is \$10 if the visitor's height is at least 120 cm but less than 150 cm, and \$$m$ if the visitor's height is 150 cm and above. If the visitor's height is less than 120 cm, he/she does not need to pay for the ride.

(iii) Assuming that a visitor purchasing a ticket for the ride is equally likely to be a male or female, find in terms of m, the probability distribution of X. [3]

Given that the expected amount a visitor will pay for a ride is \$17.93, show that $m = 20.00$, correct to 2 decimal places. [1]

(iv) Three visitors were randomly chosen. Find the probability that the total amount they paid for a ride together is more than \$40. [3]

End of paper

ANNEX B

AJC H2 Math JC2 Preliminary Examination Paper 2

<table>
<tr><th>QN</th><th>Topic Set</th><th>Answers</th></tr>
<tr><td>1</td><td>Differential Equations</td><td>(ii) $0 < k < 0.03$
(iii) $t = 7.16\text{h}$ or 7h 9min</td></tr>
<tr><td>2</td><td>Complex numbers</td><td>$a = 5$
$3e^{i\left(\frac{2\pi}{3}\right)}$, $3e^{i\left(-\frac{2\pi}{3}\right)}$ and $-2 = 2e^{i(\pi)}$
$z = \frac{1}{3}e^{i\left(-\frac{2\pi}{3}\right)}, \frac{1}{3}e^{i\left(\frac{2\pi}{3}\right)}, -\frac{1}{2}$</td></tr>
<tr><td>3</td><td>Vectors</td><td>(i) $\overrightarrow{OF} = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$; 3
(ii) $\underset{\sim}{r} = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -4 \\ 1+2a \\ 3+2a \end{pmatrix}$ where $\mu \in \mathbb{R}$
(iii) $5/2$
(iv) $\sqrt{34}$</td></tr>
<tr><td>4</td><td>Functions</td><td>$1 - \sqrt{e^2 + e}$</td></tr>
<tr><td>5</td><td>P&C, Probability</td><td>(i) 0.03
(ii) 0.4
(iii) 0.00127 (to 3sf)</td></tr>
<tr><td>6</td><td>Correlation & Linear Regression</td><td>(ii) $y = 49.7 - 3.09x$; - 2.85
(iii) - 0.998
(iv) 3.6 weeks</td></tr>
<tr><td>7</td><td>Hypothesis Testing</td><td>(a)(ii) do not reject H_0
(b) $5.55 < \mu_o < 6.73$</td></tr>
<tr><td>8</td><td>Binomial Distribution</td><td>(a) $5p(1-p)^8(2+7p) = 0.15$; $p = 0.0162$ or $p = 0.408$
(b)(i) 8 (ii) 0.0843 (iii) 0.798</td></tr>
<tr><td>9</td><td>P&C, Probability</td><td>(i) 829, 440
(ii) 385
(iii) 20736</td></tr>
<tr><td>10</td><td>Normal Distribution</td><td>(ii) 0.201
<table>
<tr><th>x (in $)</th><th>P(X = x)</th></tr>
<tr><td>0</td><td>0.00016056</td></tr>
<tr><td>10</td><td>0.20693</td></tr>
<tr><td>m</td><td>0.79291</td></tr>
</table>
</td></tr>
</table>

		(iii) 0.889
11	Q11 Topic	
12	Q12 Topic	
13	Q13 Topic	

1(i)	$\frac{dx}{dt}=30-kx, \quad k>0$ $\Rightarrow \int \frac{1}{30-kx}\,dx=\int dt$ $\Rightarrow -\frac{1}{k}\ln\lvert 30-kx\rvert = t+C$ $\Rightarrow \ln\lvert 30-kx\rvert = -kt-kC$ $\Rightarrow \lvert 30-kx\rvert = e^{-kt-kC}$ $\Rightarrow 30-kx=Ae^{-kt}, \quad$ where $A=\pm e^{-kC}$ $\Rightarrow x=\frac{1}{k}\left(30-Ae^{-kt}\right)$ At $t=0$, $x=0 \Rightarrow 0=\frac{1}{k}\left(30-Ae^{0}\right) \Rightarrow A=30$ $\Rightarrow x=\frac{1}{k}\left(30-30e^{-kt}\right)=\frac{30}{k}\left(1-e^{-kt}\right)$
1(ii)	For patient to have overdose, $x=\frac{30}{k}\left(1-e^{-kt}\right)>1000$ Since for $t>0$, $0<e^{-kt}<1$, so $0<1-e^{-kt}<1$ $\frac{30}{k}>\frac{30}{k}\left(1-e^{-kt}\right)>1000$ $0<k<\frac{30}{1000}=0.03$
(iii)	At $x=200, \quad 200=30(50)\left(1-e^{-\frac{t}{50}}\right)$ $1-e^{-\frac{t}{50}}=\frac{2}{15}$ $t=50\ln\left(\frac{15}{13}\right)$ Using GC, t = 7.16h or 7h 9min
2	Second root is $re^{-i\theta}$. Quadratic factor of P(z) is $\left(z-re^{i\theta}\right)\left(z-re^{-i\theta}\right)$ $=z^2-\left(re^{i\theta}+re^{-i\theta}\right)z+\left(re^{i\theta}\right)\left(re^{-i\theta}\right)$ $=z^2-r\left(e^{i\theta}+e^{-i\theta}\right)z+r^2$ $=z^2-r\left(\cos\theta+i\sin\theta+\cos\theta-i\sin\theta\right)z+r^2$ $=z^2-\left(2r\cos\theta\right)z+r^2$ root of the equation is $3e^{i\left(\frac{2\pi}{3}\right)}$. So $r=3$ and $\theta=\frac{2\pi}{3}$. Quadratic factor is $z^2-2(3)\left(\cos\frac{2\pi}{3}\right)z+9 = z^2+3z+9$

	hence $z^3+az^2+15z+18=(z^2+3z+9)(z+2)$ By comparing z^2 term, $a=5$ The roots of the equation $z^3+az^2+15z+18=0$ are $3e^{i\left(\frac{2\pi}{3}\right)}$, $3e^{i\left(-\frac{2\pi}{3}\right)}$ and $-2=2e^{i(\pi)}$
	$18z^3+15z^2+az+1=0$ $z^3\left(18+15\left(\frac{1}{z}\right)+a\left(\frac{1}{z^2}\right)+\left(\frac{1}{z^3}\right)\right)=0$ Since $z\neq 0$, and let $w=\frac{1}{z}$ We have $w^3+aw^2-2w+18=0$ Hence $w=3e^{i\left(\frac{2\pi}{3}\right)},3e^{i\left(-\frac{2\pi}{3}\right)},-2$ $\frac{1}{z}=3e^{i\left(\frac{2\pi}{3}\right)},3e^{i\left(-\frac{2\pi}{3}\right)},-2$ Since $\left\|\frac{1}{z}\right\|=\frac{1}{\|z\|}$ and $\arg\left(\frac{1}{z}\right)=-\arg(z)$ So $z=\frac{1}{3}e^{i\left(-\frac{2\pi}{3}\right)},\frac{1}{3}e^{i\left(\frac{2\pi}{3}\right)},-\frac{1}{2}$ are the roots of $18z^3+15z^2+az+1=0$
3(i)	Equation of line through point *P* and perpendicular to π_1 is $\mathbf{r}=\begin{pmatrix}-2\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\2\end{pmatrix},\quad \lambda\in\mathbb{R}$ Since *F* lies on plane π_1 , $(-2+\lambda)-2(1-2\lambda)+2(1+2\lambda)=7 \quad\Rightarrow\quad \lambda=1$ $\overrightarrow{OF}=\begin{pmatrix}-2\\1\\1\end{pmatrix}+1\begin{pmatrix}1\\-2\\2\end{pmatrix}=\begin{pmatrix}-1\\-1\\3\end{pmatrix}$ $\overrightarrow{PF}=\overrightarrow{OF}-\overrightarrow{OP}=\begin{pmatrix}-1\\-1\\3\end{pmatrix}-\begin{pmatrix}-2\\1\\1\end{pmatrix}=\begin{pmatrix}1\\-2\\2\end{pmatrix}$ shortest distance from *P* to plane Π_1 = $\left\|\overrightarrow{PF}\right\|=\sqrt{1^2+(-2)^2+2^2}=3$
3(ii)	Line *m* is parallel to both planes: $\begin{pmatrix}1\\-2\\2\end{pmatrix}\times\begin{pmatrix}a\\3\\-1\end{pmatrix}=\begin{pmatrix}2-6\\-(-1-2a)\\3+2a\end{pmatrix}=\begin{pmatrix}-4\\1+2a\\3+2a\end{pmatrix}$ Equation of this line $m: r=\begin{pmatrix}-1\\-1\\3\end{pmatrix}+\mu\begin{pmatrix}-4\\1+2a\\3+2a\end{pmatrix}$ where $\mu\in\mathbb{R}$
3(iii)	$Q(1,-4,-1)$ lies on line *m*, $-1-4\mu=1$ --- (1) $-1+(1+2a)\mu=-4$ --- (2) $3+(3+2a)\mu=-1$ --- (3)

From (1) : $\mu = -\frac{1}{2}$
From (2) : $a = 5/2$
From (3) : $a = 5/2$. Hence the value of a is 5/2

<u>Alternative method</u>

$$\overrightarrow{FQ} = \begin{pmatrix}1\\-4\\-1\end{pmatrix} - \begin{pmatrix}-1\\-1\\3\end{pmatrix} = \begin{pmatrix}2\\-3\\-4\end{pmatrix}$$

Since line m contains F and is parallel to π_1 , line m lies on π_1.

Since line m is on π_1, Q is on π_1. hence $\overrightarrow{FQ}$ is // π_1 and $\perp n_1$

$$\begin{pmatrix}2\\-3\\-4\end{pmatrix} \bullet \begin{pmatrix}a\\3\\-1\end{pmatrix} = 0$$

$$2a - 9 + 4 = 0$$

$$a = 5/2$$

(iv) <u>Method 1 (dot product)</u>

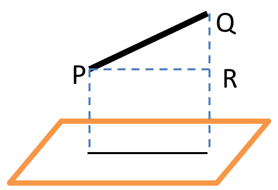
$$\overrightarrow{PQ} = \begin{pmatrix}1\\-4\\-1\end{pmatrix} - \begin{pmatrix}-2\\1\\1\end{pmatrix} = \begin{pmatrix}3\\-5\\-2\end{pmatrix}$$ and normal to the x-y plane = $\begin{pmatrix}0\\0\\1\end{pmatrix}$

$$QR = \left| \overrightarrow{PQ} \bullet \begin{pmatrix}0\\0\\1\end{pmatrix} \right| = \left| \begin{pmatrix}3\\-5\\-2\end{pmatrix} \bullet \begin{pmatrix}0\\0\\1\end{pmatrix} \right| = 2$$

length of projection of $\overrightarrow{PQ}$ on the x-y plane

= PR

$= \sqrt{PQ^2 - 2^2} = \sqrt{(3^2 + 5^2 + 2^2 - 2^2} = \sqrt{34}$



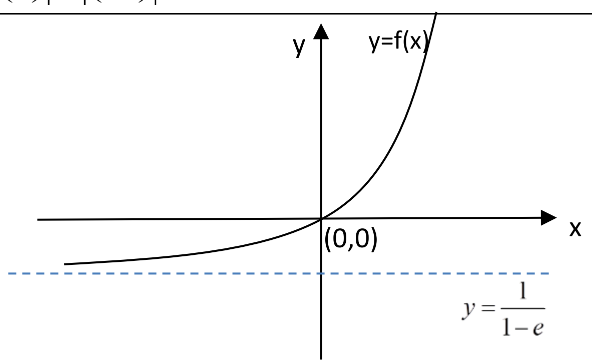
<u>Method 2 (cross product)</u>

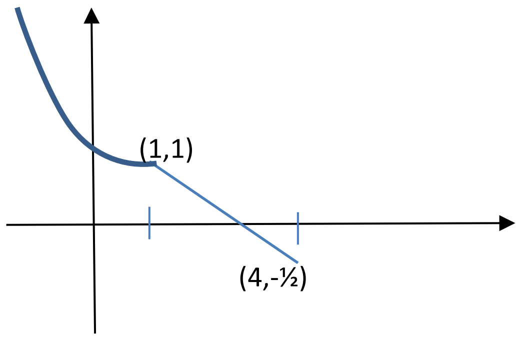
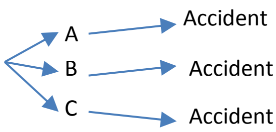
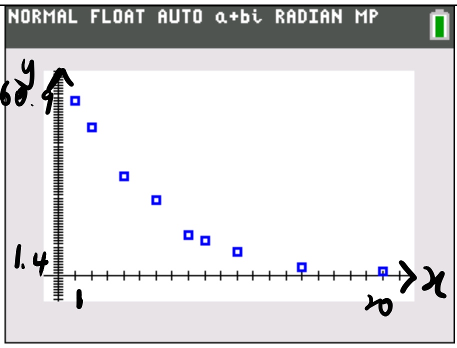
length of projection of $\overrightarrow{PQ}$ on the x-y plane

$$= PR = \left| \overrightarrow{PQ} \times \begin{pmatrix}0\\0\\1\end{pmatrix} \right| = \left| \begin{pmatrix}3\\-5\\-2\end{pmatrix} \times \begin{pmatrix}0\\0\\1\end{pmatrix} \right| = \left| \begin{pmatrix}-5\\-3\\0\end{pmatrix} \right| = \sqrt{5^2 + 3^2} = \sqrt{34}$$

4 Soln:

$$\mathrm{R_f} = \left(\frac{1}{1-e}, \infty \right)$$



4	 $\boxed{R_h = \left[-\frac{1}{2}, \infty\right)}$ $D_{f^{-1}} = R_f = \left(\frac{1}{1-e}, \infty\right) = (-0.582, \infty)$ Hence $R_h \subseteq D_{f^{-1}}$, so $f^{-1}h$ exists . Let $\left(f^{-1}h\right)^{-1}(3) = k$ $\Rightarrow f^{-1}h(k) = 3$ $\Rightarrow h(k) = f(3)$ $\Rightarrow h(k) = \frac{e^3-1}{e-1} = e^2+e+1$ Since $e^2+e+1 > 1$, hence $h(x) = (x-1)^2+1$ $(k-1)^2+1 = e^2+e+1$ $\Rightarrow k = 1 \pm \sqrt{e^2+e}$ Alternative method: use $f^{-1}(x)$ Let $\left(f^{-1}h\right)^{-1}(3) = k$ $\Rightarrow f^{-1}h(k) = 3$ $\Rightarrow \ln\left[1+(e-1)h(k)\right] = 3$ $\Rightarrow 1+(e-1)h(k) = e^3$ $\Rightarrow h(k) = \frac{e^3-1}{e-1} = e^2+e+1$ Since x <1, hence the exact value of $\left(f^{-1}h\right)^{-1}(3) = 1-\sqrt{e^2+e}$.
5(i)	Required probability $= (0.3\times0.01)+(0.5\times0.03)+(0.2\times0.06)$ = 0.03 
5(ii)	P(class C/accident) $= \frac{P(\text{accident} \cap \text{class C})}{P(\text{accident})}$ $= \frac{0.2\times0.06}{0.03} = 0.4$
5(iii)	P(all three drivers are of class C and exactly one have accident) $= (0.2\times0.94)^2 \times 0.2\times0.06\times\frac{3!}{2!}$ = 0.00127 (to 3sf)
6(i)	

6(ii)	Regression line of y on x is $y = 49.7 - 3.09x$ When x = 17, y = -2.8466…= - 2.85 The linear model is not suitable since 1) the negative value of y is impossible or 2) the scatter diagram shows a curved relationship between the two variables.
6(iii)	Product moment correlation coefficient between W and x = -0.997837…= - 0.998
6(iv)	Since x is the controlled variable, we use the regression line of $\ln y$ on x : $\ln y = 4.3549 - 0.20532x$ [from GC] When $y = \frac{1}{2}(75)$, we have $\ln\frac{75}{2} = 4.3549 - 0.20532x$ $\Rightarrow x = 3.5581... = 3.6$ The weight will drop to half its original value in 3.6 weeks. The estimate is reliable since 1) The product moment correlation coefficient between $\ln y$ and x is -0.998 which is very close to -1, showing a strong negative linear correlation between $\ln y$ and x. 2) The estimate is an interpolation, because $y = \frac{1}{2}(75)$ is in the data range $1.4 \le y \le 60.9$.
7a(i)	Only working adults travelling by train will have a chance of being selected. Those who do not travel by train will have no chance of being chosen. Hence not every working adult in the country has an equal chance to be selected – therefore the sample is not a random sample.
7a(ii)	Let X hours be the number of hours of sleep of a randomly chosen adult and μ be the mean of X. To test $H_o : \mu = 6$ vs $H_1 : \mu > 6$ Since sample size is large, by CLT, $\overline{X} \sim N\left(6, \frac{\sigma^2}{50}\right)$ Since population variance σ^2 is unknown, it is replaced by s^2 Under H_o, test statistic $Z = \frac{\overline{X} - 6}{\frac{s}{\sqrt{50}}} \sim N(0,1)$ We use a one-tailed test at 5% level of significance, that is, reject H_o if p-value < 0.05 Sample readings: $\bar{x} = \frac{320}{50} = 6.4$, $s^2 = \frac{1}{49}\left(2308.5 - \frac{(320)^2}{50}\right) = 5.31633$ From GC, p-value = 0.109967 = 0.110 > 0.05 $\Rightarrow$ we **do not reject** H_o. Hence we conclude that there is **insufficient** evidence at the 5% level of significance to doubt the Health Promotion Board's claim. It is not necessary to assume that the number of hours of sleep follow a normal distribution because since the sample size is large, by the Central Limit Theorem, the sample mean follows a normal distribution approximately.

7b

To test $H_o : \mu = \mu_o$ vs $H_1 : \mu \neq \mu_o$

$s^2 = \frac{50}{49}(2.1)^2$, $\bar{x} = 6.14$,

Since H_o is not rejected at the 5% level,

$$-1.95996 < \frac{\bar{x} - \mu_o}{\frac{s}{\sqrt{n}}} < 1.95996$$

$$\Rightarrow \; -1.95996 < \frac{6.14 - \mu_o}{\frac{2.1}{\sqrt{49}}} < 1.95996$$

$$\Rightarrow 6.14 - 1.95996\frac{2.1}{\sqrt{49}} < \mu_o < 6.14 + 1.95996\frac{2.1}{\sqrt{49}}$$

$$\Rightarrow 5.552012 < \mu_o < 6.727988$$

$$\Rightarrow 5.55 < \mu_o < 6.73$$

8 (a)

Let X be the number of cream biscuits per box. $X \sim \mathrm{B}(10, p)$

$\mathrm{P}(X = 1 \text{ or } 2) = 0.15$

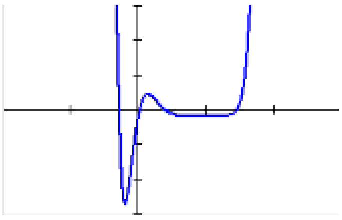
$\mathrm{P}(X = 1) + \mathrm{P}(X = 2) = 0.15$

$^{10}\mathrm{C}_1 p^1(1-p)^9 + {}^{10}\mathrm{C}_2 p^2(1-p)^8 = 0.15$

$10p(1-p)^9 + 45p^2(1-p)^8 = 0.15$

$5p(1-p)^8\left[2(1-p) + 9p\right] = 0.15$

$5p(1-p)^8\left(2 + 7p\right) = 0.15$



Draw $y = 5p(1-p)^8(2+7p) - 0.15$

From G.C.,

$p = 0.0162$ or $p = 0.408$

(other values are 1.45 or -0.288 need to be rejected)

8 (b) (i)

$X \sim \mathrm{B}(10, \frac{3}{4})$. Let $Y_1 = \mathrm{P}(X = x)$.

X	Y1
0	9.5E-7
1	2.9E-5
2	3.9E-4
3	.00309
4	.01622
5	.0584
6	.146
7	.25028
8	.28157
9	.18771
10	.05631

From G.C.,

since $\mathrm{P}(X = 8)$ is the highest,

The most likely no. of cream biscuits = 8

(ii)

Let Y denote the random variable: Number of boxes with $X = 5$.

$Y \sim \mathrm{B}(18, p)$ where $p = \mathrm{P}(X=5) = 0.058399$

$P(3 \leq Y < 7) = P(Y \leq 6) - P(Y \leq 2)$

$= 0.0843$ (3 s.f.)

(iii)

Let U = no. of boxes sold at Usual price

Let D = no. of boxes sold at Discounted price

Let W: Total income per day.

$W = 10U + 8D$

$E(W) = 10E(U) + 8E(D) = 180 \times \$10 + 840 \times \$8 = \$8520$

$\mathrm{Var}(W) = 10^2\mathrm{Var}(U) + 8^2\mathrm{Var}(D) = 64 \times 10^2 + 169 \times 8^2 = 17216$

Let $T = W_1 + W_2 + \ldots + W_{30}$

Since n = 30 is large, by Central Limit Theorem,

$T \sim \mathrm{N}(30 \times 8520,\ 30 \times 17216) = \mathrm{N}(255600,\ \sqrt{516480}^2)$ approximately

$\mathrm{P}(T \geq \$255000) = 0.798$

9(i)

family	A		B		C		D	
	Adult	kids	Adult	kids	Adult	kids	Adult	kids
	2	2	1	2	1	3	1	1

4 family units, No. of ways = $4! \times 4! \times 3! \times 5! \times 2! = 829,440$

9(ii)

Case 1: 3,3,2 No. of ways = $\dfrac{{}^8C_3 \times {}^5C_3 \times {}^2C_2}{2!} = 280$

Case 2: 2,2,2,2 No. of ways = $\dfrac{{}^8C_2 \times {}^6C_2 \times {}^4C_2 \times {}^2C_2}{4!} = 105$

Total no. of ways = 280 + 105 = 385

9(iii)

There is only 1 way to divide the 8 children and the adults into 2 circles to satisfy all conditions. Family A and B (3 adults & 4 kids) must be in 1 circle and Family C & D are in another circle.
Arrange the children in 1 circle : (4-1)!
Slot in adults in between children : ${}^4C_3 \times 3!$

No. of ways = $[(4-1)! \times {}^4C_3 \times 3!] \times [(4-1)! \times {}^4C_3 \times 3!]$
= 20736

10(i)

Let M denote the random variable: Height of a male visitor in cm. $M \sim N(165, 12^2)$
Let F denote the random variable: Height of a female visitor in cm. $F \sim N(155, \sigma^2)$

$$P(150 < F < 160) = 0.3829$$

$$P(-\frac{5}{\sigma} < Z < \frac{5}{\sigma}) = 0.3829$$

$$P(Z < -\frac{5}{\sigma}) = \frac{1-0.3829}{2} = 0.30855$$

From G.C. $-\dfrac{5}{\sigma} = -0.4999646$

$\Rightarrow \sigma = 10.0$ cm (3 sig. fig.)

(ii)

$$\frac{3}{4}M - F \sim N(\frac{3}{4}\times 165 - 155, \frac{9}{16}\times 12^2 + 10^2) = N(-31.25, 181)$$

$$P(\left|\frac{3}{4}M - F\right| \le 20) = P(-20 \le \frac{3}{4}M - F \le 20)$$

$= 0.201$ (3 sig. fig.)

Assumption: The heights of all male and female visitors are independent of one another.

(iii)

Probability Distribution of X:

***x* (in $)**	**P(X = x)**
0	$\frac{1}{2}P(M < 120) + \frac{1}{2}P(F < 120) = 0.00016056$
10	$\frac{1}{2}P(120 \le M < 150) + \frac{1}{2}P(120 \le F < 150) = 0.20693$
m	$\frac{1}{2}P(M \ge 150) + \frac{1}{2}P(F \ge 150) = 0.79291$

Given $E(X) = 17.93 = 0(0.00016056) + 10(0.20693) + m(0.79291)$

$\Rightarrow m = 20.00$ (shown)

(iv)

$P(X_1 + X_2 + X_3 > 40) = P(20, 20, 20) + 3.\ P(20, 20, 10)$
$= 0.889$