H2 Mathematics 2017 Preliminary Exam Paper 1 Question
Answer all questions [100 marks].

1 The sum of the first n terms of a sequence is denoted by S_n. The first term of the sequence is 3 and it is known that $S_3 = 21$ and $S_{10} = 210$. Given that S_n is a quadratic polynomial in n, find S_n in terms of n. [3]

2 Using the substitution $v = \sqrt{x} + 1$, find $\int \frac{1}{x + \sqrt{x}}\,\mathrm{d}x$, where $x > 0$. [3]

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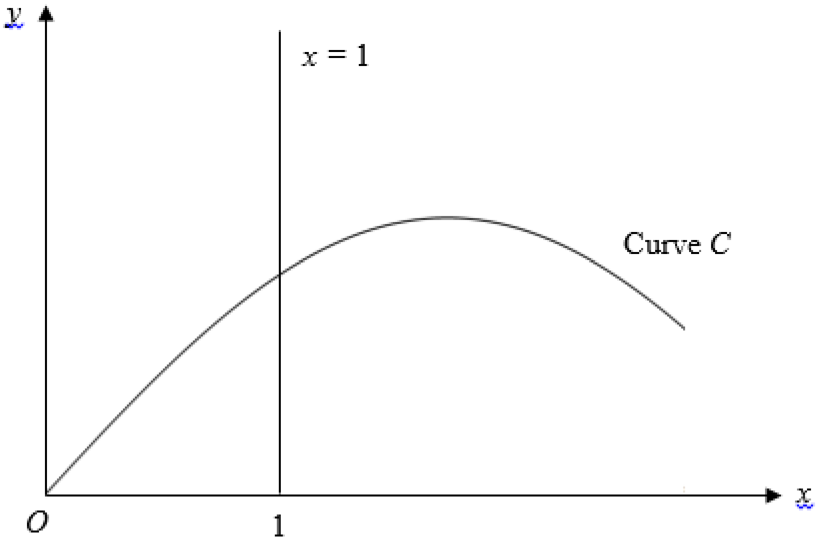


The diagram shows the curve C with equation $y = \sin x$ and the line $x = 1$. With reference to the diagram, a student wrote down the following series

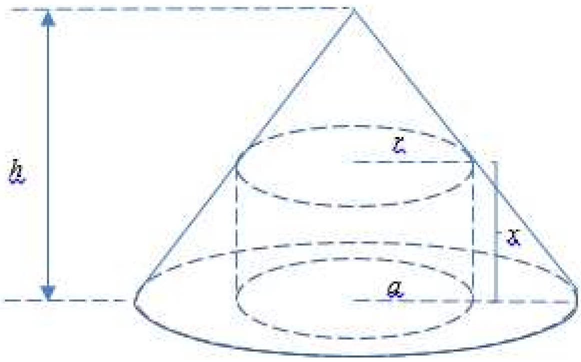
$$S = \frac{1}{n}\left[\sin\left(\frac{1}{n}\right) + \sin\left(\frac{2}{n}\right) + \sin\left(\frac{3}{n}\right) + ... + \sin\left(\frac{n}{n}\right)\right].$$

(i) State what the series represents. [2]

(ii) When $n \to \infty$, $S \to L$. State the geometrical meaning of *L*. Determine the exact value of *L*, leaving your answer in the form $a - \cos b$, where *a* and *b* are constants to be determined. [3]

(iii) What can be said about the value of *S* in relation to the value of *L*? [1]

4 [It is given that the volume of a circular cone with base radius *r* and height *h* is $\frac{1}{3}\pi r^2 h$.]



The diagram above shows a right circular cone with fixed radius *a* and fixed height *h*. A cylinder of radius *r* and height *x* is removed from the cone.

(i) Show that the volume of the remaining shape, *V*, is $\frac{\pi h}{3}\left(a^2-3r^2+\frac{3r^3}{a}\right)$. [2]

(ii) As *r* varies, use differentiation to find the value of *r* that gives the minimum value of *V*, leaving your answer in terms of *a*. [4]

5 A line *L* passes through the points $A(3,-1,\ 0)$ and $B(11,\ 11,\ 4)$.

(i) Find the angle between *L* and the *y*-axis. [2]

(ii) State the geometrical meaning of $\left|\overrightarrow{OB}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right|$. [1]

The point $F(2a+1,\ a,\ a-1)$ is a point on *L*, where *a* is a positive constant.

The point *P* is such that $\overrightarrow{PF}=\begin{pmatrix}3\\0\\2\end{pmatrix}$ and the area of the triangle *AFP* is $\sqrt{\frac{59}{2}}$ units^2.

(iii) Determine the value of *a*. [3]

(iv) The point *C* on *L* is such that the ratio of the area of triangle *AFP* to the area of triangle *FCP* is $2:1$. State the ratio $AF:CF$, justifying your answer. [2]

6 **(i)** Show that $\int e^{2x}\cos x\,dx=\frac{2}{5}e^{2x}\cos x+\frac{1}{5}e^{2x}\sin x+C$. [3]

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	(ii) Find the volume of the solid generated when the region bounded by $y = e^x \sqrt{(\cos x)}$ and $y = -\frac{2}{\pi}x + 1$ between $x = 0$ and $x = \frac{\pi}{2}$ is rotated through 2π radians about the x-axis, leaving your answer in exact form. [4]
7	**(i)** Prove by the method of mathematical induction that $$\sum_{r=1}^{n} \frac{2}{r(r+2)} = \frac{3}{2} - \frac{2n+3}{(n+1)(n+2)}$$ for all positive integers of n. [5] **(ii)** Explain why $\sum_{r=1}^{n} \frac{2}{r(r+2)}$ is a convergent series, and state the value of the sum to infinity. [2] **(iii)** Using the result in part **(i)**, find $\sum_{r=5}^{N} \frac{2}{(r-2)(r-4)}$. [2]
8	Using the substitution $y = ux$, show that the differential equation $$x\frac{dy}{dx} = 3x + y - 2$$ can be reduced to the form $$x^2 \frac{du}{dx} = 3x - 2 .$$ Hence, find the general solution to the differential equation $x\frac{dy}{dx} = 3x + y - 2$. [5] **(i)** State the equation of the locus where the stationary points of the solution curves lie. [1] **(ii)** Sketch, on a single diagram, the graph of the locus found in part **(i)** and two members of the family of solution curves, where the arbitrary constant in the general solution is equal to 1 and -1. [3]
9	It is given that

$$f(x) = \begin{cases} (x-1)^2 + 4 & , \quad k \le x < 3, \\ 3x - 1 & , \quad 3 \le x \le 4, \end{cases}$$

where $k \in \mathbb{R}$, $k < 3$.

(i) Sketch, for $k = 0$, the graph of $y = f(x)$, stating the coordinates of the turning point. Write down the range of f. [3]

(ii) Explain why f^{-1} does not exist. State the smallest value of *k* for f^{-1} to exist. [2]

(iii) Using the value of *k* in part **(ii)**, find f^{-1} in similar form. [4]

(iv) State the geometrical relationship between f and f^{-1}. The point $P(a, b)$, where *a* and *b* are constants, lies on the graph $y = f(x)$. The point *Q* on the graph $y = f^{-1}(x)$ is the point corresponding to *P*. State the coordinates of *Q*. [2]

10 **(a)** It is given that $-1 + i$ is a root of the equation $2z^3 + az^2 + bz + (3 + i) = 0$.

(i) Find the values of the real numbers *a* and *b*. [4]

(ii) Using these values of *a* and *b*, find the other roots of this equation. [3]

(b) It is given that $w = -1 + (\sqrt{3})i$.

(i) Without using a calculator, find an exact expression for w^5. Give your answer in the form $re^{i\theta}$, where $r > 0$ and $0 \le \theta \le 2\pi$. [3]

(ii) Without using a calculator, find the three smallest positive whole number values of *n* for which $\frac{w^*}{w^n}$ is a real number. [4]

11 A curve C_1 is defined parametrically by the equations $x = t - \frac{1}{t}$, $y = t + \frac{1}{t}$, $t \ne 0$.

(i) Sketch C_1, stating the equation of the asymptotes and coordinates of any points of intersection with the *y*-axis. [2]

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(ii) Show that the equation of the normal to C_1 at the point with parameter p is given by

$$y = -\frac{p^2+1}{p^2-1}x + \frac{2(p^2+1)}{p}.$$ [4]

(iii) The normal in part **(ii)** intersects the x-axis at the point A and the y-axis at the point B. Find, in terms of p, an expression for the area of the triangle OAB. [4]

The line l is the normal to C_1 when $p = 2$.

(iv) Find the equation of l. [1]

A curve C_2 is defined parametrically by the equations $x = 3at$, $y = -t^2 + a$, $t \in \mathbb{R}$ where a is a non-zero constant.

(v) Given that l intersects C_2, show that the parameter q of the point(s) of intersection satisfies the equation

$$q^2 - 5aq + 5 - a = 0.$$

Hence, determine the range of values of a such that l intersects C_2 at two distinct points. [3]

12 As part of a project, a group of engineering students design two robots for a game. One robot is called 'Prey' and the other robot is called 'Predator'. The two robots are designed with the following specifications.

'Prey': It is designed to leap 1 m forward for the first leap. Subsequently, it leaps 2.5 cm less than the previous leap distance. 'Prey' shuts down when the leap distance is 0 or when it is caught by 'Predator'.

'Predator': It is designed to leap 2 m forward for the first leap. Subsequently, it leaps 90% of the previous leap distance. 'Predator' shuts down when 'Prey' shuts down or when it catches 'Prey'.

Both robots take each leap at the same time and the number of leaps taken is given by n. 'Predator' starts the game from the starting line while 'Prey' starts the game 7 m in front of 'Predator'.

(i) Find the distance of 'Prey' and of 'Predator' from the starting line after n leaps, leaving your answers in terms of n. [2]

	(ii) Explain why 'Predator' has to catch 'Prey' before 'Predator's distance from the starting line reaches 20 m. [2] **(iii)** Using a graphical method, explain why 'Predator' will not catch 'Prey'. [3] **(iv)** 'Prey' now starts the game 4 m in front of 'Predator'. 'Predator' catches 'Prey' on the k-th leap. Find the value of k. Calculate the distance of 'Predator' from the starting line after completing the k-th leap. [3]

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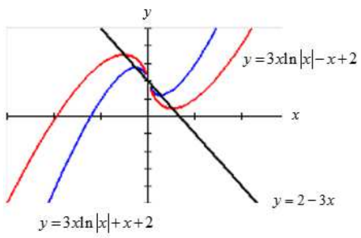
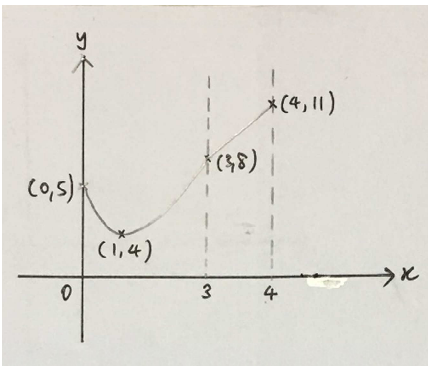
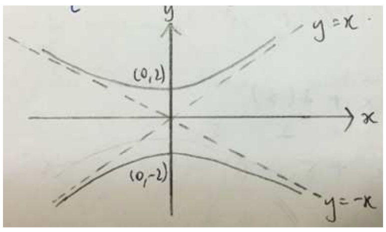
ANNEX B

MI H2 Math PU3 Preliminary Examination Paper 1

QN	Topic Set	Answers
1	Equations and Inequalities	$S_n = 2n^2 + n$
2	Integration techniques	$2\ln\left(\sqrt{x}+1\right)+c$
3	Application of Integration	(i) The sum of the areas of n rectangles with equal width from $x=0$ to $x=1$, where the top right vertex of each rectangle lies on the curve. (ii) L is the actual area under C from $x=0$ to $x=1$, $1-\cos 1$ (iii) Since the sum of the areas of the rectangles in part (i) is larger than the actual area under the curve C, $S > L$
4	Differentiation & Applications	(ii) $r = \frac{2}{3}a$
5	Vectors	(i) 36.7° (ii) The length of projection of $\overrightarrow{OB}$ onto the $z-$axis . (iii) 2 (iv) 2:1
6	Application of Integration	(ii) $\frac{1}{5}\pi e^{\pi} - \frac{2}{5}\pi - \frac{\pi^2}{6}$
7	Sigma Notation and Method of Difference	(ii) $\frac{3}{2}$ (iii) $\frac{3}{2} - \frac{2N-5}{(N-3)(N-2)}$
8	Differential Equations	$y = 3x\ln\lvert x\rvert + Cx + 2$ (i) $y = 2-3x$ (ii) 

9	Functions	(i)  Range of f, $R_f = [4,11]$ (ii) 1 (iii) $f^{-1}(x) = \begin{cases} 1+\sqrt{x-4} & , \quad 4 \le x < 8, \\ \frac{1}{3}x+\frac{1}{3} & , \quad 8 \le x \le 11, \end{cases}$ (iv) The graph $y = f^{-1}(x)$ is the reflection of the graph $y = f(x)$ in the line $y = x$. The coordinates of Q is (b, a).
10	Complex numbers	(a)(i) $a = 6, b = 7$ (a)(ii) $z = -\frac{1}{2} - \frac{1}{2}i$ or $z = -\frac{3}{2} - \frac{1}{2}i$ (b)(i) $32e^{i\left(\frac{4\pi}{3}\right)}$ (b)(ii) 2, 5, 8.
11	Differentiation & Applications	(i) $x = t - \frac{1}{t}, \quad y = t + \frac{1}{t}, \quad t \neq 0.$  (iii) $\frac{2\left(p^2+1\right)}{p^2}\left\|p^2-1\right\|$ or $\frac{2}{p^2}\left\|p^4-1\right\|$ units^2 (iv) $y = -\frac{5}{3}x + 5$ (v) $a < -0.978$ or $a > 0.818$ (3 s.f)
12	AP and GP	(i) $-0.0125n^2 + 1.0125n + 7$; $20\left(1-0.9^n\right)$ (iv) 8; 11.4 m

13	Q13 Topic	

2017 PU3 H2 Prelim II Paper 1 Suggested Solutions

Qn. No.	Question
1	Let $S_n = an^2 + bn + c$ where $a,\ b$ and c are constants $S_1 = 3 \quad \Rightarrow a+b+c=3$ $S_3 = 21 \quad \Rightarrow 9a+3b+c=21$ $S_{10} = 210 \quad \Rightarrow 100a+10b+c=210$ Using GC, $a=2,\ b=1,\ c=0 \quad \Rightarrow \quad S_n = 2n^2+n$
2	$\frac{dv}{dx} = \frac{1}{2x^{\frac{1}{2}}} \Rightarrow \frac{dx}{dv} = 2x^{\frac{1}{2}}$ $\int \frac{1}{x+\sqrt{x}}\ dx = \int \frac{1}{x+x^{\frac{1}{2}}} \frac{dx}{dv} dv$ where $\frac{1}{x+x^{\frac{1}{2}}} \frac{dx}{dv} = \frac{1}{x+x^{\frac{1}{2}}}\left(2x^{\frac{1}{2}}\right) = \frac{2}{\left(x+x^{\frac{1}{2}}\right)\left(x^{-\frac{1}{2}}\right)} = \frac{2}{\left(x^{\frac{1}{2}}+1\right)} = \frac{2}{v}$ $\int \frac{1}{x+\sqrt{x}}\ dx = \int \frac{2}{v} dv$ $= 2\int \frac{1}{v}\ dv$ $= 2\ln v + c$ $= 2\ln\left(\sqrt{x}+1\right)+c$
3	(i) The sum of the areas of n rectangles with equal width from $x=0$ to $x=1$, where the top right vertex of each rectangle lies on the curve. (ii) L is the actual area under C from $x=0$ to $x=1$. $L = \int_0^1 \sin x\ dx$ $= [-\cos x]_0^1$ $= -\cos 1 + \cos 0$ $= 1 - \cos 1 \qquad$ i.e. $a=1,\ b=1$ (iii) Since the sum of the areas of the rectangles in part (i) is larger than the actual area under the curve C, $S > L$

4

(i)

$$\frac{r}{a}=\frac{h-x}{h}$$

$$\therefore x=h-\frac{hr}{a}$$

$$V=\frac{1}{3}\pi a^2 h-\pi r^2\left(h-\frac{hr}{a}\right)$$

$$=\frac{\pi h}{3}\left(a^2-3r^2+\frac{3r^3}{a}\right)\text{ (shown)}$$

(ii)

$$\frac{\mathrm{d}V}{\mathrm{d}r}=\frac{\pi h}{3}\left(-6r+\frac{9r^2}{a}\right)$$

For max/min volume: $\frac{\mathrm{d}V}{\mathrm{d}r}=0$

$$\frac{\pi h}{3}\left(-6r+\frac{9r^2}{a}\right)=0$$

$$-6r+\frac{9r^2}{a}=0$$

$$r\left(-6+\frac{9r}{a}\right)=0$$

$r=0$ (reject as $r>0$) or $r=\frac{2}{3}a$

Method 1 (2nd derivative test)

$$\frac{\mathrm{d}^2V}{\mathrm{d}r^2}=\frac{\pi h}{3}\left(-6+\frac{18r}{a}\right)$$

At $r=\frac{2}{3}a$:

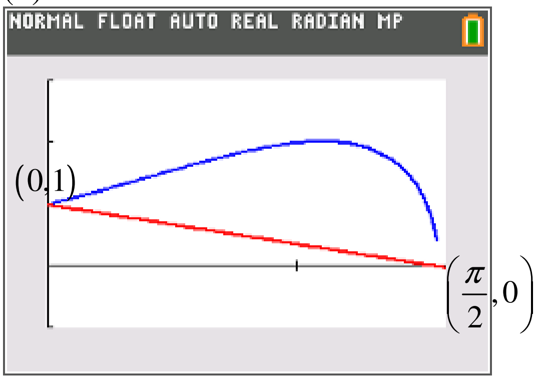
$$\frac{\mathrm{d}^2V}{\mathrm{d}r^2}=\frac{\pi h}{3}\left(-6+\frac{18}{a}\left(\frac{2}{3}a\right)\right)=2\pi h>0$$

Therefore the volume is a minimum when $r=\frac{2}{3}a$.

Method 2 (1st derivative test)

r	$\frac{2}{3}a^-$	$\frac{2}{3}a$	$\frac{2}{3}a^+$
$\frac{\mathrm{d}V}{\mathrm{d}r}$	Negative	Zero	Positive

	Therefore the volume is a minimum when $r=\frac{2}{3}a$.
5	(i) $\overrightarrow{AB}=\begin{pmatrix}8\\12\\4\end{pmatrix}=4\begin{pmatrix}2\\3\\1\end{pmatrix}$ The required angle, $\theta=\cos^{-1}\frac{\begin{pmatrix}2\\3\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}}{\sqrt{14}\sqrt{1}}=36.7^{\circ}$ (1 d.p) (ii) The length of projection of $\overrightarrow{OB}$ onto the $z-$axis . (iii) $\frac{1}{2}\left\|\overrightarrow{AF}\times\overrightarrow{PF}\right\|=\sqrt{\frac{59}{2}}$ $\frac{1}{2}\left\|\begin{pmatrix}2a-2\\a+1\\a-1\end{pmatrix}\times\begin{pmatrix}3\\0\\2\end{pmatrix}\right\|=\sqrt{\frac{59}{2}}$ $\left\|\begin{pmatrix}2(a+1)\\1-a\\-3(a+1)\end{pmatrix}\right\|=2\sqrt{\frac{59}{2}}$ $\sqrt{4(a+1)^2+(1-a)^2+9(a+1)^2}=2\sqrt{\frac{59}{2}}$ $13(a+1)^2+(1-a)^2=118$ $14a^2+24a-104=0$ $7a^2+12a-52=0$ $(7a+26)(a-2)=0$ $a=-\frac{26}{7}$ (rejected as $a>0$) or $a=2$ Accept: Using GC, $a=2$ or $a=-3.7143$ (rejected as $a>0$) (iv)

	Both triangles have the same height (h). $AF:CF$ $=$ Area of triangle AFP : Area of triangle FCP $=2:1$
6	(i) $\int e^{2x}\cos x\,dx=\frac{1}{2}e^{2x}\cos x+\int\frac{1}{2}e^{2x}\sin x\,dx$ $=\frac{1}{2}e^{2x}\cos x+\frac{1}{2}\left[\frac{1}{2}e^{2x}\sin x-\frac{1}{2}\int e^{2x}\cos x\,dx\right]$ $=\frac{1}{2}e^{2x}\cos x+\frac{1}{4}e^{2x}\sin x-\frac{1}{4}\int e^{2x}\cos x\,dx$ $\frac{5}{4}\int e^{2x}\cos x\,dx=\frac{1}{2}e^{2x}\cos x+\frac{1}{4}e^{2x}\sin x+C_1$ $\int e^{2x}\cos x\,dx=\frac{2}{5}e^{2x}\cos x+\frac{1}{5}e^{2x}\sin x+C$ (ii)  Volume = $\pi\int y^2\,dx$ $=\pi\int_0^{\frac{\pi}{2}}e^{2x}\cos x\,dx-\frac{1}{3}\pi(1)^2\left(\frac{\pi}{2}\right)$ $=\pi\left[\frac{2}{5}e^{2x}\cos x+\frac{1}{5}e^{2x}\sin x\right]_0^{\frac{\pi}{2}}-\frac{\pi^2}{6}$ $=\pi\left[\frac{1}{5}e^{\pi}\sin\frac{\pi}{2}-\frac{2}{5}e^{0}\cos 0\right]-\frac{\pi^2}{6}$ $=\frac{1}{5}\pi e^{\pi}-\frac{2}{5}\pi-\frac{\pi^2}{6}$
7	(i) Let P_n be the statement $\sum_{r=1}^{n}\frac{2}{r(r+2)}=\frac{3}{2}-\frac{2n+3}{(n+1)(n+2)}$ for $n\in Z^{+}$. Prove P_1 is true.

LHS = $\dfrac{2}{(1)(1+2)}=\dfrac{2}{3}$

RHS = $\dfrac{3}{2}-\dfrac{2(1)+3}{(1+1)(1+2)}=\dfrac{4}{6}=\dfrac{2}{3}=\text{LHS}$

P_1 is true.

Assume that P_k is true for some $k\in Z^+$ i.e. $\sum_{r=1}^{k}\dfrac{2}{r(r+2)}=\dfrac{3}{2}-\dfrac{2k+3}{(k+1)(k+2)}$.

Prove P_{k+1} is true i.e. $\sum_{r=1}^{k+1}\dfrac{2}{r(r+2)}=\dfrac{3}{2}-\dfrac{2k+5}{(k+2)(k+3)}$.

LHS

$=\sum_{r=1}^{k}\dfrac{2}{r(r+2)}+T_{k+1}$

$=\dfrac{3}{2}-\dfrac{2k+3}{(k+1)(k+2)}+\dfrac{2}{(k+1)(k+3)}$

$=\dfrac{3}{2}-\dfrac{(2k+3)(k+3)-2(k+2)}{(k+1)(k+2)(k+3)}$

$=\dfrac{3}{2}-\dfrac{2k^2+7k+5}{(k+1)(k+2)(k+3)}$

$=\dfrac{3}{2}-\dfrac{(2k+5)(k+1)}{(k+1)(k+2)(k+3)}$

$=\dfrac{2k+5}{(k+2)(k+3)}=\text{RHS}$

P_{k+1} is true

Since P_1 is true, and P_k is true implies P_{k+1} is true,
by Mathematical Induction, $\sum_{r=1}^{n}\dfrac{2}{r(r+2)}=\dfrac{3}{2}-\dfrac{2n+3}{(n+1)(n+2)}$ for $n\in Z^+$.

(ii)

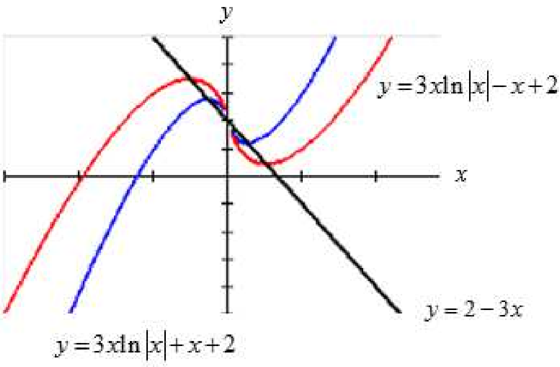
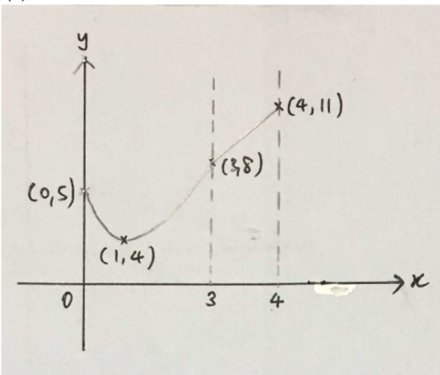
$n\to\infty, \dfrac{2n+3}{(n+1)(n+2)}\to 0, \sum_{r=1}^{n}\dfrac{2}{r(r+2)}\to\dfrac{3}{2}$

The series converges to a value. $\therefore$ the series is a convergent series.

$\sum_{r=1}^{\infty}\dfrac{2}{r(r+2)}=\dfrac{3}{2}$

(iii)

	$\sum_{r=5}^{N}\frac{2}{(r-2)(r-4)}$ $\xrightarrow[r=l+4]{\text{letting}}\sum_{l+4=5}^{l+4=N}\frac{2}{(l+4-2)(l+4-4)}$ $=\sum_{l=1}^{N-4}\frac{2}{l(l+2)}$ $=\frac{3}{2}-\frac{2(N-4)+3}{(N-4+1)(N-4+2)}$ $=\frac{3}{2}-\frac{2N-5}{(N-3)(N-2)}$
8	$y=ux$ $\frac{\mathrm{d}y}{\mathrm{d}x}=u+x\frac{\mathrm{d}u}{\mathrm{d}x}$ $x\left(u+x\frac{\mathrm{d}u}{\mathrm{d}x}\right)=3x+ux-2$ $ux+x^2\frac{\mathrm{d}u}{\mathrm{d}x}=3x+ux-2$ $x^2\frac{\mathrm{d}u}{\mathrm{d}x}=3x-2$ (shown) $\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{3x-2}{x^2}$ $\int\frac{\mathrm{d}u}{\mathrm{d}x}\ \mathrm{d}x=\int\frac{3x-2}{x^2}\ \mathrm{d}x$ $\int\ \mathrm{d}u=\int\frac{3}{x}-\frac{2}{x^2}\ \mathrm{d}x$ $u=3\ln\lvert x\rvert+\frac{2}{x}+C$ $\frac{y}{x}=3\ln\lvert x\rvert+\frac{2}{x}+C$ $y=3x\ln\lvert x\rvert+Cx+2$ (i) For stationary points, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $\Rightarrow x(0)=3x+y-2$ $\Rightarrow y=2-3x$ The equation of the locus is $y=2-3x$. (ii)

	
9	(i)  Range of f, $R_f=[4,11]$ (ii) A horizontal line, $y=k,\ 4<k\le 5$ intersects the graph of $y=f(x)$ at 2 points. f is not a one-one function. Hence, f^{-1} does not exist. For f^{-1} to exist, the minimum value of k is 1 . (iii) Let $y=f(x)$ For $1\le x<3$: Let $y=(x-1)^2+4$ $x=1\pm\sqrt{y-4}$ $x=1+\sqrt{y-4}$ since $1\le x<3$ $f^{-1}(x)=1+\sqrt{x-4},\ 4\le x<8$ For $3\le x\le 4$: Let $y=3x-1$ $x=\frac{1}{3}y+\frac{1}{3}$ $f^{-1}(x)=\frac{1}{3}x+\frac{1}{3},\ 8\le x\le 11$

	$\mathrm{f}^{-1}(x)=\begin{cases}1+\sqrt{x-4} & , \quad 4\le x<8,\\ \frac{1}{3}x+\frac{1}{3} & , \quad 8\le x\le 11,\end{cases}$ (iv) The graph $y=\mathrm{f}^{-1}(x)$ is the reflection of the graph $y=\mathrm{f}(x)$ in the line $y=x$. The coordinates of Q is (b,a).
10	(a)(i) Since $-1+\mathrm{i}$ is a root of $2z^3+az^2+bz+(3+\mathrm{i})=0$, $2(-1+\mathrm{i})^3+a(-1+\mathrm{i})^2+b(-1+\mathrm{i})+(3+\mathrm{i})=0$ $4+4\mathrm{i}+a(-2\mathrm{i})-b+b\mathrm{i}+3+\mathrm{i}=0$ Comparing real parts: $4-b+3=0\Rightarrow b=7$ Comparing imaginary parts: $4-2a+b+1=0\Rightarrow a=6$ (a)(ii) $2z^3+6z^2+7z+(3+\mathrm{i})=0$ $\left[z-(-1+\mathrm{i})\right]\left[2z^2+(4+2\mathrm{i})z+(1+2\mathrm{i})\right]=0$ $z=-1+\mathrm{i}$ (given) or $z=\dfrac{-(4+2\mathrm{i})\pm\sqrt{\left[(4+2\mathrm{i})^2-4(2)(1+2\mathrm{i})\right]}}{2(2)}$ $=\dfrac{(-4-2\mathrm{i})\pm 2}{4}$ $z=-\frac{1}{2}-\frac{1}{2}\mathrm{i}$ or $z=-\frac{3}{2}-\frac{1}{2}\mathrm{i}$ (b)(i) $\left\|-1+\mathrm{i}\sqrt{3}\right\|=2$ $\arg\left(-1+\mathrm{i}\sqrt{3}\right)=\pi-\tan^{-1}\left(\sqrt{3}\right)=\frac{2\pi}{3}$ $w^5=\left(2\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}\right)}\right)^5=32\mathrm{e}^{\mathrm{i}\left(\frac{10\pi}{3}\right)}=32\mathrm{e}^{\mathrm{i}\left(\frac{4\pi}{3}\right)}$ (b)(ii) $\dfrac{w^*}{w^n}=\dfrac{2\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{3}\right)}}{\left[2\mathrm{e}^{\mathrm{i}\left(\frac{2\pi}{3}\right)}\right]^n}=2^{1-n}\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{3}-\frac{2n\pi}{3}\right)}$

Method 1

$$2^{1-n}\mathrm{e}^{\mathrm{i}\left(-\frac{2\pi}{3}-\frac{2n\pi}{3}\right)} = 2^{1-n}\left[\cos\left(-\frac{2\pi}{3}-\frac{2n\pi}{3}\right)+\mathrm{i}\sin\left(-\frac{2\pi}{3}-\frac{2n\pi}{3}\right)\right]$$

$$= 2^{1-n}\left[\cos\left(\frac{2\pi}{3}+\frac{2n\pi}{3}\right)-\mathrm{i}\sin\left(\frac{2\pi}{3}+\frac{2n\pi}{3}\right)\right]$$

Since $\frac{w^*}{w^n}$ is a real number,

$$\sin\left(\frac{2\pi}{3}+\frac{2n\pi}{3}\right)=0$$

$$\frac{2\pi}{3}+\frac{2n\pi}{3}=\pi,2\pi,3\pi,4\pi,5\pi,6\pi...$$

$$2\pi+2n\pi=3\pi,6\pi,9\pi,12\pi,15\pi,18\pi...$$

$$2n\pi=\pi,4\pi,7\pi,10\pi,13\pi,16\pi...$$

$$n=\frac{1}{2},2,\frac{7}{2},5,\frac{13}{2},8,...$$

The 3 smallest positive whole number values of n are 2, 5 and 8.

Method 2

Since $\frac{w^*}{w^n}$ is a real number, $\arg\left(\frac{w^*}{w^n}\right)=k\pi,\ k\in\mathbb{Z}$

$$-\frac{2n\pi}{3}-\frac{2\pi}{3}=k\pi$$

$$n=-1-\frac{3k}{2}$$

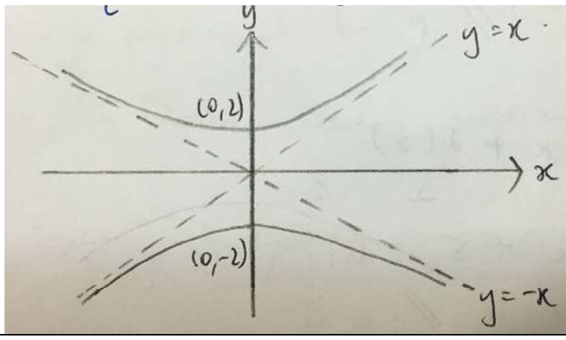
At $k=-2$: $n=2$
At $k=-4$: $n=5$
At $k=-6$: $n=8$

The 3 smallest positive whole number values of n are 2, 5 and 8.

11

(i)

$x=t-\frac{1}{t}, \quad y=t+\frac{1}{t}, \ t\neq 0.$



(ii)

$x = t - \frac{1}{t}, \quad y = t + \frac{1}{t}$

$\frac{dx}{dt} = 1 + \frac{1}{t^2} = \frac{t^2+1}{t^2}, \quad \frac{dy}{dt} = 1 - \frac{1}{t^2} = \frac{t^2-1}{t^2}$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{t^2-1}{t^2+1}$

When $t = p$, the gradient of normal $= -\frac{p^2+1}{p^2-1}$

The required equation of normal:

$y - \left(p + \frac{1}{p}\right) = -\frac{p^2+1}{p^2-1}\left[x - \left(p - \frac{1}{p}\right)\right]$

$y = -\frac{p^2+1}{p^2-1}x + \frac{p^2+1}{p^2-1}\left(p - \frac{1}{p}\right) + p + \frac{1}{p}$

$y = -\frac{p^2+1}{p^2-1}x + \frac{p^2+1}{p^2-1}\left(\frac{p^2-1}{p}\right) + \frac{p^2+1}{p}$

$y = -\frac{p^2+1}{p^2-1}x + \frac{2(p^2+1)}{p}$ (shown)

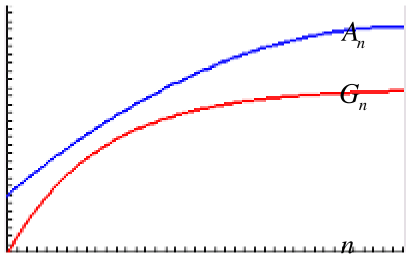
(iii)

When $x = 0$, $y = \frac{2(p^2+1)}{p} \Rightarrow B\left(0, \ \frac{2(p^2+1)}{p}\right)$

When $y = 0$, $-\frac{p^2+1}{p^2-1}x + \frac{2(p^2+1)}{p} = 0$

$x = \frac{2(p^2-1)}{p} \Rightarrow A\left(\frac{2(p^2-1)}{p}, \ 0\right)$

	Area of triangle *OAB* $=\left\|\frac{1}{2}\left[\frac{2\left(p^2+1\right)}{p}\right]\left[\frac{2\left(p^2-1\right)}{p}\right]\right\|$ $=2\left\|\frac{\left(p^2+1\right)\left(p^2-1\right)}{p^2}\right\|$ $=\frac{2}{p^2}\left\|\left(p^2+1\right)\left(p^2-1\right)\right\|$ $=\frac{2\left(p^2+1\right)}{p^2}\left\|p^2-1\right\|$ or $\frac{2}{p^2}\left\|p^4-1\right\|$ units^2 (iv) When $p=2$, the equation of the normal is $y=-\frac{2^2+1}{2^2-1}x+\frac{2\left(2^2+1\right)}{2}$ $y=-\frac{5}{3}x+5$ The equation of l is $y=-\frac{5}{3}x+5.$ (v) By substitution, $-q^2+a=-\frac{5}{3}(3aq)+5$ $q^2-5aq+5-a=0$ (shown) For l to intersect C_2 at 2 distinct points, $b^2-4ac>0$ $(-5a)^2-4(1)(5-a)>0$ $25a^2+4a-20>0$ $a<-0.978$ or $a>0.818$ (3 s.f)
12	(i) Distance of 'Prey' from starting line, A_n $=\frac{n}{2}\left[2(1)+(n-1)(-0.025)\right]+7=-0.0125n^2+1.0125n+7$ Distance of 'Predator' from starting line, G_n $=\frac{2\left(1-0.9^n\right)}{1-0.9}=20\left(1-0.9^n\right)$ (ii)

	The sum to infinity $= \dfrac{2}{1-0.9} = 20$ Hence, 'Predator' has to catch 'Prey' before its distance from the starting line reaches 20 m. (iii) To determine when the leap distance of 'Prey' becomes 0 (if it is not caught): $1+(n-1)(-0.025)>0$ $n<41$ If not caught, 'Prey' will leap 40 times before the leap distance becomes 0. Plot, for $0 \le n \le 40$, the graphs of $A_n = -0.0125n^2 + 1.0125n + 7$ and $G_n = 20\left(1-0.9^n\right)$ as follows:  For $0 \le n \le 40$, since the two curves do not intersect, 'Predator' will not catch 'Prey'. (iv) When 'Predator' catches 'Prey', $-0.0125n^2 + 1.0125n + 4 = 20\left(1-0.9^n\right)$ Using GC, $n = 7.2557$ or 12.012 (rejected) 'Predator' catches 'Prey' on the 8th leap $\Rightarrow$ $k = 8$. The required distance $= 20\left(1-0.9^8\right) = 11.4$ m (3 s.f)

H2 Mathematics 2017 Preliminary Exam Paper 2 Question
Answer all questions [100 marks].

1 The curve C has the equation $4(x-1)^2+9y^2=36$.

(i) Sketch, for $y\geq 0$, the curve C, stating the coordinates of the end points and the turning point. [3]

(ii) By adding a suitable graph to your sketch in part **(i)**, solve the inequality

$$2\sqrt{\left[1-\frac{(x-1)^2}{9}\right]}+2-(x-1)^2\geq 0.$$ [2]

(iii) Hence, solve the inequality $2\sqrt{\left[1-\frac{(\mathrm{e}^x-1)^2}{9}\right]}\geq\left(\mathrm{e}^x-1\right)^2-2$. [2]

2 Two loci in the Argand diagram are given by the equations

$$|z-2+2\mathrm{i}|=1 \quad \text{and} \quad \arg z=-\frac{\pi}{6}.$$

The complex numbers z_1 and z_2, where $|z_1|<|z_2|$, correspond to the points of intersection of these loci.

(i) Draw an Argand diagram to show both loci, and mark the points represented by z_1 and z_2. [3]

(ii) Find the two values of z which represent points on $|z-2+2\mathrm{i}|=1$ such that $|z-z_1|=|z-z_2|$. [4]

(iii) Given that the complex number w satisfies $|w-2+2\mathrm{i}|\leq 1$ and $\arg w\leq -\frac{\pi}{6}$, find the range of values of $\arg(w+3\mathrm{i})$. [3]

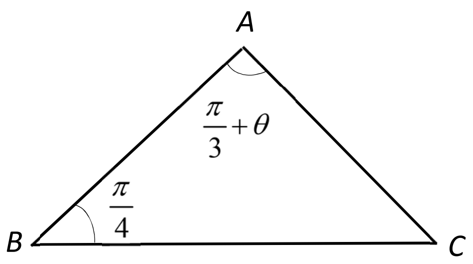
3 **(a)** It is given that $\tan^{-1}y=\ln(1+x)$.

(i) Show that $(1+x)\frac{\mathrm{d}y}{\mathrm{d}x}=1+y^2$. [1]

(ii) By successively differentiating this result, find the Maclaurin series for $\tan\left[\ln\left(1+x\right)\right]$, up to and including the term in x^3. [3]

(iii) It is given that $\mathrm{f}(x)=\mathrm{e}^x\tan\left[\ln\left(1+x\right)\right]$. Using your answer to part **(a)(ii)**, estimate the value of $\mathrm{f}'\left(\frac{1}{2}\right)$. [3]

(b) The diagram shows triangle ABC, where $AC=k$ cm, $BC=h$ cm, $\angle BAC=\frac{\pi}{3}+\theta$ and $\angle ABC=\frac{\pi}{4}$.



Given that θ is a sufficiently small angle, show that

$$\frac{h}{k}\approx\frac{\sqrt{2}}{4}\left[2\sqrt{3}+2\theta-\left(\sqrt{3}\right)\theta^2\right].$$ [3]

4 The plane π_1 contains the line $l_1:\mathbf{r}=\begin{pmatrix}1\\2\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\1\end{pmatrix}$, where $\lambda\in\mathbb{R}$, and is parallel to the line

$l_2:\mathbf{r}=\begin{pmatrix}1\\0\\1\end{pmatrix}+\mu\begin{pmatrix}2\\0\\1\end{pmatrix}$, where $\mu\in\mathbb{R}$.

(i) Find the vector equation of π_1 in scalar product form. [2]

(ii) Find the position vector of the foot of the perpendicular from the point $A(1,0,1)$ to the plane π_1. [3]

(iii) Find the position vector of the point A', which is the reflection of A about π_1. [2]

(iv) Given that the angle between $l_3:\mathbf{r}=\begin{pmatrix}1\\1\\1\end{pmatrix}+\alpha\begin{pmatrix}1\\-1\\0\end{pmatrix}$, where $\alpha\in\mathbb{R}$, and the plane

$\pi_2 : ax + 2y - z = 3$, where $a \in \mathbb{R}$, is $\dfrac{\pi}{4}$, find the value of a. [2]

(v) Find the line of intersection between the planes π_1 and π_2. [1]

(vi) π_3 has equation $bx + y + z = c$, where $b, c \in \mathbb{R}$. Given that π_1, π_2 and π_3 have no points in common, describe the geometrical relationship between the three planes. What can be said about the values of b and c? [3]

5 Resilience Primary School has 500 students who are either Chinese, Indian or Malay, as seen in the table below.

	Chinese	Indian	Malay
Boys	114	8	93
Girls	122	77	86

The National Eye Centre wishes to conduct a survey at Resilience Primary School to find out the number of hours students spend on electronic devices each week, using a sample of 50 students.

(i) Explain how stratified sampling can be carried out in this context. [2]

(ii) Give two reasons why systematic sampling may not be appropriate. [2]

6 In another survey conducted by the National Eye Centre, it was found that p% are boys and the remaining are girls. The probability that a randomly chosen boy wears spectacles is 0.3 and the probability that a randomly chosen girl wears spectacles is 0.24.

(i) Find the value of p, given that the probability that a randomly chosen child wears spectacles is 0.267. [2]

(ii) For a general value of p, the probability that a randomly chosen child that wears spectacles is a girl is denoted by $\mathrm{f}(p)$. Show that $\mathrm{f}(p) = \dfrac{4(100-p)}{(400+p)}$. Prove by differentiation that f is a decreasing function for $0 \le p \le 100$, and explain what this statement means in the context of the question. [5]

7 In this question you should state clearly all distributions that you use, together with the values of the appropriate parameters.

The mass, in grams, of broccoli and carrots are normally distributed with means and standard deviations as shown in the table below.

	Mean (g)	Standard deviation (g)
Broccoli	μ	σ
Carrot	180	15

(i) Given that the probability that the mass of a randomly chosen broccoli does not exceed 250g is 0.788 and the probability that the mass of a randomly chosen broccoli exceeds 236g is 0.625, find the values of μ and σ. [3]

(ii) Find the probability that the mass of a randomly chosen broccoli lies within 5 grams of a randomly chosen carrot. [2]

(iii) 120 broccoli are randomly chosen. Using a suitable approximation, find the probability that there are fewer than 90 broccoli with a mass not exceeding 250g.[3]

(iv) Determine, with explanation, whether the mass of a vegetable chosen randomly from a basket containing an equal number of broccoli and carrots follows a normal distribution. [1]

8 The table gives the values of eight observations of bivariate data, x and y.

x	1	2	3	4	5	6	7	8
y	5	1	18	23	28	31	33	34

(i) Draw a scatter diagram for these values, labelling the axes clearly. Determine the outlier by labelling it as P in your scatter diagram. [2]

(ii) By omitting P, explain if $y = ax^2 + b$ or $y = a\ln x + b$ is the better model for the data. [2]

(iii) Using the more appropriate model found in part **(ii)**, calculate the equation of the least-squares regression line. [1]

(iv) Interpret, in the context of the question, the least squares estimates of a and b. [2]

(v) Use the regression line found in part **(iii)** to predict the value of y when x = 4.5. Comment on the reliability of your answer. [2]

9 Based on past records, the mean number of rainy days per year in Singapore was reported as 178. The authorities suspect that due to global warming, the number of rainy days has changed. A random sample of 12 years is taken and the number of rainy days per year, X, is summarised by

$$\sum(x-8)=2017.7, \qquad \sum x^2=372\,500.$$

(i) Calculate the unbiased estimates of the mean and variance of X. [2]

(ii) Test, at the 5% level of significance, whether the mean number of rainy days per year has changed. State any assumptions used in your calculations. [4]

(iii) Explain, in the context of the question, the meaning of the p-value. [1]

(iv) The population variance is found to be 9 and the assumption used in part **(ii)** holds true. A test at the 5% level of significance whether the mean number of rainy days per year has changed was conducted. Find the range of values of $\bar{x}$ such that the null hypothesis is not rejected. [3]

10 **(a)** Find the number of ways in which the letters of the word MILLENNIUM can be arranged if

(i) there are no restrictions, [1]

(ii) the first and last letters are the same, and the letters E and U must be separated. [2]

Four letters are randomly selected from the letters of the word MILLENNIUM to form a code word. Find the number of possible code words that can be formed. [2]

(b) Mr See (together with his wife and daughter) and Mrs Saw (together with her husband and two sons) came to visit their former teacher Mdm Rain during Millennium Institute's Homecoming Day. Find the number of ways Mr See and his family, Mrs Saw and her family, and Mdm Rain can be arranged if

(i) they are around a table with ten indistinguishable chairs, such that the children are seated together. [2]

(ii) the two empty chairs are removed and Mr See's daughter is seated between her parents and the See family are to be seated directly opposite Mdm Rain. [3]

11 In this question you should state clearly all distributions that you use, together with the values of the appropriate parameters.

The number of people queuing to buy coffee at CoffeeVille in a period of 1 minute during lunch hour (12pm to 2pm) is a random variable with an average number of 2.9.

State, in context, a condition under which a Poisson distribution would be a suitable probability model. [1]

Assume that the number of people queuing to buy coffee at CoffeeVille in a period of 1 minute during the lunch hour follows the distribution $\text{Po}(2.9)$.

(i) State the most probable number of people queuing in 1 minute. [1]

(ii) Find the probability that in a period of 3 minutes, there are at most 5 people queuing to buy coffee. [2]

(iii) *N* periods of 3 minutes are taken. Given that the probability that at least 7 periods of 3 minutes have at most 5 people queuing to buy coffee is more than 0.99, find the least value of *N*. [3]

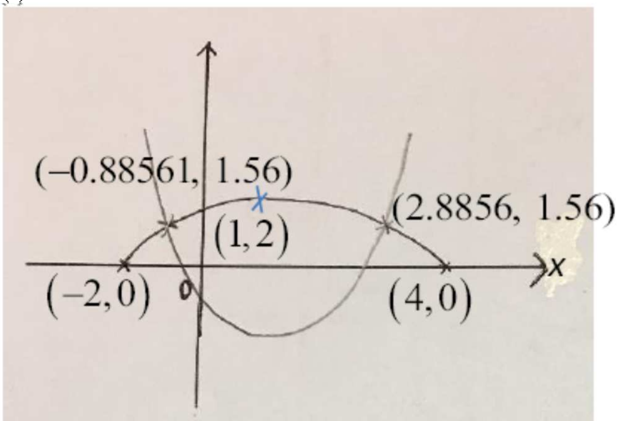
(iv) A random sample of 120 periods of 3 minutes is taken. Using a suitable approximation, find the probability that more than 12 periods of 3 minutes have exactly 4 people queuing. [3]

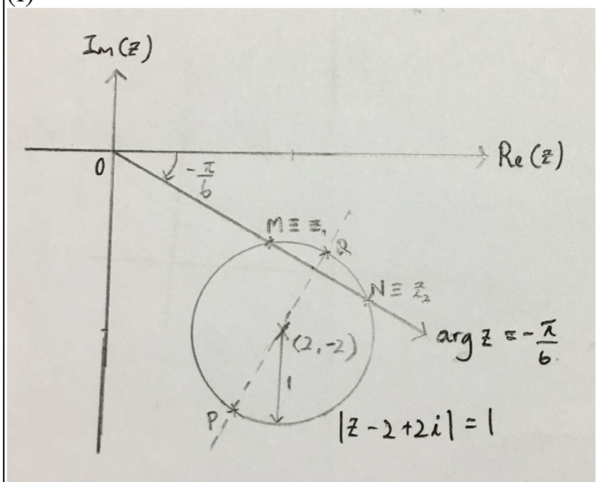
(v) Explain why the Poisson model would probably not be valid if applied to the operating hours of CoffeeVille from 11am to 10pm. [1]

− End Of Paper −

ANNEX B

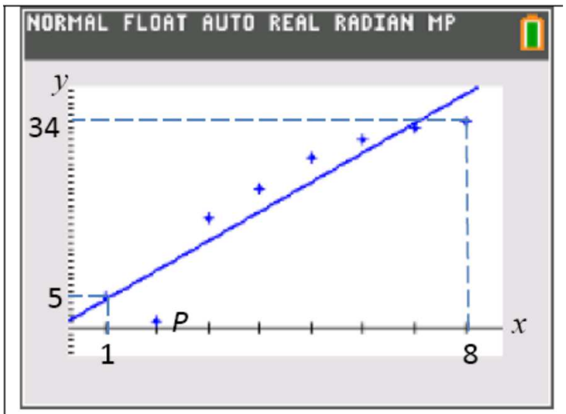
MI H2 Math PU3 Preliminary Examination Paper 2

<table>
<tr><th>QN</th><th>Topic Set</th><th>Answers</th></tr>
<tr><td>1</td><td>Equations and Inequalities</td><td>(i)


(ii) $-0.886 \le x \le 2.89$ (3 s.f)
(iii) $x \le 1.06$ (3 s.f)</td></tr>
<tr><td>2</td><td>Complex numbers</td><td>(i)


(ii) $z=\frac{5}{2}-\left(2-\frac{\sqrt{3}}{2}\right)\mathrm{i}$, $z=\frac{3}{2}-\left(2+\frac{\sqrt{3}}{2}\right)\mathrm{i}$,
(iii) $0 \le \arg[z+3\mathrm{i}] \le 0.927$ (3 s.f)</td></tr>
<tr><td>3</td><td>Maclaurin series</td><td>(a)(ii) $y = x-\frac{1}{2}x^2+\frac{2}{3}x^3+\ldots$
(a)(iii) 2</td></tr>
<tr><td>4</td><td>Vectors</td><td>(i) $\underset{\sim}{r}\cdot\begin{pmatrix}1\\1\\-2\end{pmatrix}=-3$</td></tr>
</table>

		(ii) $\overrightarrow{ON} = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{5}{3} \end{pmatrix}$ (iii) $\overrightarrow{OA'} = \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{7}{3} \end{pmatrix}$ (iv) $a = -\frac{1}{4}$ (v) $\underset{\sim}{r} = \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}, \beta \in \mathbb{R}$ (vi) Either: the three planes are the sides of a triangular prism. OR: π_3 is parallel to the line of intersection of π_1 and π_2, but does not contain it, $b = -\frac{5}{4}$, $c \neq 6$
5	Sampling	(ii) $k = \frac{500}{50} = 10$ Since $k = 10 > 8 =$ number of Indian boys available, there is a possibility the Indian boys may not be represented. Systematic sampling does not ensure equal proportions of students being taken from each strata.
6	P&C, Probability	(i) $p = 45$ (ii) As the percentage of boys in the survey increases, the percentage that a girl wears spectacles decreases.
7	Normal Distribution	(i) $\mu \approx 240$, $\sigma \approx 12.5$ (ii) 0.00200 (iii) 0.129 (iv) It will not follow normal distribution as the mass of a randomly chosen vegetable from a basket containing an equal number of broccoli and carrot follows a bimodal distribution.

8	Q8 Topic	(i)  (ii) $y = a\ln x + b$ (iii) $y \approx 4.01 + 14.5\ln x\ (3 \text{ s.f.})$ (iv) The expected value of y when $\ln x$ is 0 is 4.01. For every increase in $\ln x$ by 1 unit, expected value of y increases by 14.5 units. (v) $y = 25.9$, Reliable because $x = 4.5$ lies within the data range and $\lvert r \rvert$ is close to 1
9	Hypothesis Testing	(i) $\bar{x} \approx 176$, $s^2 \approx 17.9$ (accept 17.2) (ii) p-value = 0.156 (accept 0.149) (iii) p-value is the smallest level of significance for which the null hypothesis of the mean number of rainy days per year is 178 will be rejected. (iv) $176 < \bar{x} < 180$
10	P&C, Probability	(a)(i) 226 800 (a)(ii) 15 120 (a)(last part) 876 (b)(i) 15120 (b)(ii) 48
11	DRV	Average number of people queuing to buy coffee is a constant (i) 2 (ii) 0.135 (iii) 104 (iv) 0.00135 (v) Mean number of people queuing varies throughout the day.

H2 Further Mathematics 2017 Midyear Exam Paper 1 Solution

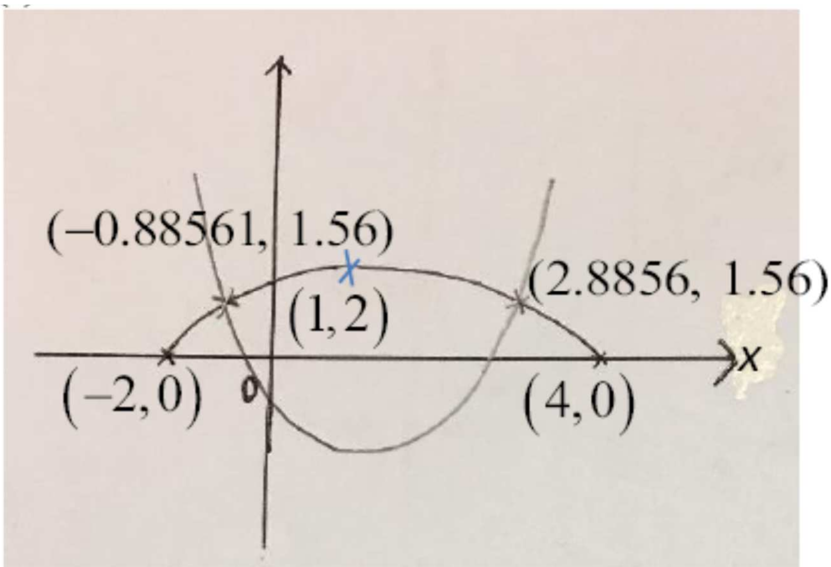
<table>
<tr><td>1</td><td>The curve C has the equation $4(x-1)^2+9y^2=36$.

(i) Sketch, for $y \geq 0$, the curve C, stating the coordinates of the end points and the turning point. [3]

(ii) By adding a suitable graph to your sketch in part (i), solve the inequality

$$2\sqrt{\left[1-\frac{(x-1)^2}{9}\right]}+2-(x-1)^2 \geq 0. \quad [2]$$

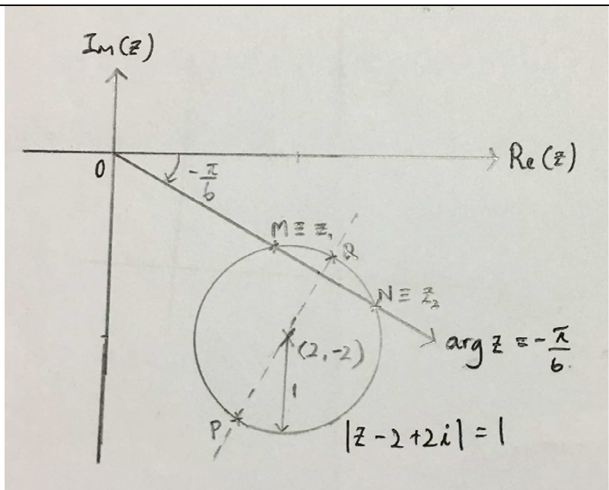
(iii) Hence, solve the inequality $2\sqrt{\left[1-\frac{(e^x-1)^2}{9}\right]} \geq \left(e^x-1\right)^2-2$. [2]</td></tr>
<tr><td></td><td>Solution:
(i)



$4(x-1)^2+9y^2=36$
$y^2=\frac{36-4(x-1)^2}{9}$
$y^2=4\left[1-\frac{(x-1)^2}{9}\right]$
$y=2\sqrt{1-\frac{(x-1)^2}{9}}$ (for $y \geq 0$)

(ii)</td></tr>
</table>

	$2\sqrt{1-\frac{(x-1)^2}{9}}+2-(x-1)^2 \geq 0$ $2\sqrt{1-\frac{(x-1)^2}{9}} \geq (x-1)^2-2$ The suitable graph to be added is $y=(x-1)^2-2$. From the graph, $-0.88561 \leq x \leq 2.8856$ $-0.886 \leq x \leq 2.89$ (3 s.f) (iii) By comparison, $x \to e^x$ $0 \leq e^x \leq 2.8856$ $\ln e^x \leq \ln 2.8856$ $x \leq 1.06$ (3 s.f)
2	Two loci in the Argand diagram are given by the equations $$\lvert z-2+2i\rvert = 1 \quad \text{and} \quad \arg z = -\frac{\pi}{6}.$$ The complex numbers z_1 and z_2, where $\lvert z_1\rvert < \lvert z_2\rvert$, correspond to the points of intersection of these loci. **(i)** Draw an Argand diagram to show both loci, and mark the points represented by z_1 and z_2. [3] **(ii)** Find the two values of z which represent points on $\lvert z-2+2i\rvert=1$ such that $\lvert z-z_1\rvert = \lvert z-z_2\rvert$. [4] **(iii)** Given that the complex number w satisfies $\lvert w-2+2i\rvert \leq 1$ and $\arg w \leq -\frac{\pi}{6}$, find the range of values of $\arg(w+3i)$. [3]
	Solution: (i) $\lvert z-2+2i\rvert = 1 \Rightarrow \lvert z-(2-2i)\rvert = 1$ $\arg z = -\frac{\pi}{6}$



(ii)

The 2 values of z are as indicated as P and Q on the diagram.

$b = (1)\cos\frac{\pi}{6}$; $a = (1)\sin\frac{\pi}{6}$

$b = \frac{\sqrt{3}}{2}$; $b = \frac{1}{2}$

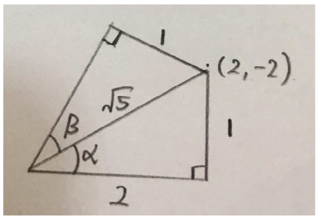
At Q: $z = \left(2 + \frac{1}{2}\right) - \left(2 - \frac{\sqrt{3}}{2}\right)\mathrm{i}$

At P: $z = \left(2 - \frac{1}{2}\right) - \left(2 + \frac{\sqrt{3}}{2}\right)\mathrm{i}$

The 2 values of z are

$\frac{5}{2} - \left(2 - \frac{\sqrt{3}}{2}\right)\mathrm{i}$ and $z = \frac{3}{2} - \left(2 + \frac{\sqrt{3}}{2}\right)\mathrm{i}$.

(iii)



Smallest value of $\arg\left[z - (-3\mathrm{i})\right] = 0$

Since $\alpha = \beta$,

Largest value of $\arg\left[z - (-3\mathrm{i})\right] = 2\tan^{-1}\frac{1}{2} = 0.927$ (3 s.f)

$\therefore 0 \le \arg\left[z + 3\mathrm{i}\right] \le 0.927$ (3 s.f)

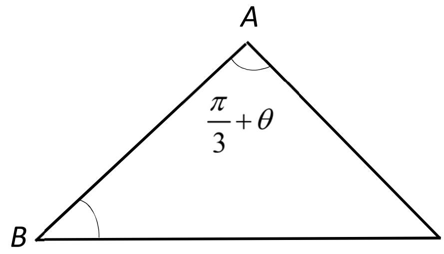
3 **(a)** It is given that $\tan^{-1} y = \ln(1 + x)$.

(i) Show that $(1 + x)\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + y^2$. [1]

(ii) By successively differentiating this result, find the Maclaurin series for $\tan\left[\ln\left(1+x\right)\right]$, up to and including the term in x^3. [3]

(iii) It is given that $\mathrm{f}(x)=\mathrm{e}^x\tan\left[\ln\left(1+x\right)\right]$. Using your answer to part **(a)(ii)**, estimate the value of $\mathrm{f}'\left(\frac{1}{2}\right)$. [3]

(b) The diagram shows triangle *ABC*, where $AC=k$ cm, $BC=h$ cm, $\angle BAC=\frac{\pi}{3}+\theta$ and $\angle ABC=\frac{\pi}{4}$.



Given that θ is a sufficiently small angle, show that

$$\frac{h}{k}\approx\frac{\sqrt{2}}{4}\left[2\sqrt{3}+2\theta-\left(\sqrt{3}\right)\theta^2\right].$$ [3]

Solution:

(i)

$\tan^{-1}y=\ln\left(1+x\right)$

Differentiating both sides with respect to x:

$$\frac{1}{1+y^2}\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{1+x}$$

$$\left(1+x\right)\frac{\mathrm{d}y}{\mathrm{d}x}=1+y^2 \text{ (shown)}$$

(ii)

$$(1+x)\frac{dy}{dx}=1+y^2$$

Differentiating both sides with respect to x:

$$(1+x)\frac{d^2y}{dx^2}+\frac{dy}{dx}=2y\frac{dy}{dx}\Rightarrow(1+x)\frac{d^2y}{dx^2}+(1-2y)\frac{dy}{dx}=0$$

Differentiating both sides with respect to x:

$$(1+x)\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}+(1-2y)\frac{d^2y}{dx^2}+(-2)\left(\frac{dy}{dx}\right)^2=0$$

$$(1+x)\frac{d^3y}{dx^3}+2(1-y)\frac{d^2y}{dx^2}-2\left(\frac{dy}{dx}\right)^2=0$$

When $x=0,\ y=0,\frac{dy}{dx}=1,\frac{d^2y}{dx^2}=-1,\frac{d^3y}{dx^3}=4$

$$y=0+(1)x+(-1)\frac{x^2}{2!}+(4)\frac{x^3}{3!}+...$$

$$y=x-\frac{1}{2}x^2+\frac{2}{3}x^3+...$$

(iii)

$$\begin{aligned}f(x)&=e^x\tan\left[\ln(1+x)\right]\\&=e^x\left(x-\frac{1}{2}x^2+\frac{2}{3}x^3+...\right)\\&=\left(1+x+\frac{1}{2}x^2+...\right)\left(x-\frac{1}{2}x^2+\frac{2}{3}x^3+...\right)\\&=x-\frac{1}{2}x^2+\frac{2}{3}x^3+x^2-\frac{1}{2}x^3+-\frac{1}{2}x^3+...\\&=x+\frac{1}{2}x^2+\frac{2}{3}x^3+...\end{aligned}$$

$$f(x)=x+\frac{1}{2}x^2+\frac{2}{3}x^3+...$$

$$f'(x)=1+x+2x^2+...$$

$$f'\left(\frac{1}{2}\right)=1+\frac{1}{2}+2\left(\frac{1}{2}\right)^2+...\approx 2$$

(b)

	$\dfrac{\sin\left(\dfrac{\pi}{3}+\theta\right)}{h}=\dfrac{\sin\left(\dfrac{\pi}{4}\right)}{k}$ $\sin\left(\dfrac{\pi}{3}+\theta\right)=\dfrac{h}{k}\left(\dfrac{1}{\sqrt{2}}\right)$ $\dfrac{h}{k}\left(\dfrac{1}{\sqrt{2}}\right)=\sin\left(\dfrac{\pi}{3}\right)\cos(\theta)+\cos\left(\dfrac{\pi}{3}\right)\sin(\theta)$ from MF15 $\dfrac{h}{k}\left(\dfrac{1}{\sqrt{2}}\right)\approx\left(\dfrac{\sqrt{3}}{2}\right)\left(1-\dfrac{\theta^2}{2}\right)+\dfrac{\theta}{2}$ $\dfrac{h}{k}\left(\dfrac{1}{\sqrt{2}}\right)\approx\dfrac{\sqrt{3}}{2}-\dfrac{\sqrt{3}\theta^2}{4}+\dfrac{\theta}{2}$ $\dfrac{h}{k}\left(\dfrac{1}{\sqrt{2}}\right)\approx\dfrac{1}{4}\left(2\sqrt{3}+2\theta-\sqrt{3}\theta^2\right)$ $\dfrac{h}{k}\approx\dfrac{\sqrt{2}}{2}\left(\sqrt{3}+\theta-\dfrac{\sqrt{3}}{2}\theta^2\right)$ (shown)

4

The plane π_1 contains the line $l_1 : \mathbf{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\1\end{pmatrix}$, where $\lambda \in \mathbb{R}$, and is parallel to the line $l_2 : \mathbf{r} = \begin{pmatrix}1\\0\\1\end{pmatrix} + \mu\begin{pmatrix}2\\0\\1\end{pmatrix}$, where $\mu \in \mathbb{R}$.

(i) Find the vector equation of π_1 in scalar product form. [2]

(ii) Find the position vector of the foot of the perpendicular from the point $A(1, 0, 1)$ to the plane π_1. [3]

(iii) Find the position vector of the point A', which is the reflection of A about π_1.[2]

(iv) Given that the angle between $l_3 : \mathbf{r} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \alpha\begin{pmatrix}1\\-1\\0\end{pmatrix}$, where $\alpha \in \mathbb{R}$, and the plane $\pi_2 : ax + 2y - z = 3$, where $a \in \mathbb{R}$, is $\frac{\pi}{4}$, find the value of *a*. [2]

(v) Find the line of intersection between the planes π_1 and π_2. [1]

(vi) π_3 has equation $bx + y + z = c$, where $b, c \in \mathbb{R}$. Given that π_1, π_2 and π_3 have no points in common, describe the geometrical relationship between the three planes. What can be said about the values of *b* and *c*? [3]

Solution:

(i)

$$\underset{\sim}{n} = \begin{pmatrix}1\\1\\1\end{pmatrix} \times \begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}1\\1\\-2\end{pmatrix}$$

$$\underset{\sim}{r} \cdot \begin{pmatrix}1\\1\\-2\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} \cdot \begin{pmatrix}1\\1\\-2\end{pmatrix} \Rightarrow \underset{\sim}{r} \cdot \begin{pmatrix}1\\1\\-2\end{pmatrix} = -3$$

(ii)

<u>Method 1:</u>

$$l_{AN} : \underset{\sim}{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \alpha \in \mathbb{R}$$

$$\overrightarrow{ON} = \begin{pmatrix} 1+\alpha \\ \alpha \\ 1-2\alpha \end{pmatrix}, \text{ for some } \alpha \in \mathbb{R}$$

Since N is the intersection point of line AN and plane,

$$\begin{pmatrix} 1+\alpha \\ \alpha \\ 1-2\alpha \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = -3$$

$$1+\alpha+\alpha-2+4\alpha = -3$$

$$\alpha = -\frac{1}{3}$$

$$\overrightarrow{ON} = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{5}{3} \end{pmatrix}$$

Method 2:

$$\overrightarrow{AN} = \left(\overrightarrow{AB} \cdot \frac{\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}}{\sqrt{6}} \right) \frac{\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}}{\sqrt{6}}, \text{ where } \overrightarrow{OB} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\overrightarrow{ON} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \left(\begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \cdot \frac{\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}}{\sqrt{6}} \right) \frac{\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}}{\sqrt{6}}$$

$$\overrightarrow{ON} = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{5}{3} \end{pmatrix}$$

(iii)

$$\overrightarrow{ON} = \frac{\overrightarrow{OA} + \overrightarrow{OA'}}{2}$$

<table>
<tr><td></td><td>
$\overrightarrow{OA'} = 2\overrightarrow{ON} - \overrightarrow{OA} = \begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ \frac{7}{3} \end{pmatrix}$
(iv)
$\pi_2 : \underset{\sim}{r} \cdot \begin{pmatrix} a \\ 2 \\ -1 \end{pmatrix} = 3$
$\sin\theta = \dfrac{\left| \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ 2 \\ -1 \end{pmatrix} \right|}{\left(\sqrt{2}\right)\left(\sqrt{\left(a^2+4+1\right)}\right)}$

Since $\theta = \dfrac{\pi}{4}$,
$\dfrac{\sqrt{2}}{2} = \dfrac{|a-2|}{\left(\sqrt{2}\right)\sqrt{\left(a^2+5\right)}}$
$\sqrt{\left(a^2+5\right)} = |a-2|$
$\left(a^2+5\right) = a^2 - 4a + 4$
$a = -\dfrac{1}{4}$
(v)
Using GC:

Equation of line of intersection:
$\underset{\sim}{r} = \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}, \beta \in \mathbb{R}$
(vi)
Geometrical interpretation:
Either: the three planes are the sides of a triangular prism

OR: π_3 is parallel to the line of intersection of π_1 and π_2, but does not contain it.

$\pi_3 : \underset{\sim}{r} \cdot \begin{pmatrix} b \\ 1 \\ 1 \end{pmatrix} = c$, $\begin{pmatrix} b \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} = 0 \Rightarrow b = -\dfrac{5}{4}$
$\begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} b \\ 1 \\ 1 \end{pmatrix} \neq c \Rightarrow c \neq 6$
</td></tr>
<tr><td>5</td><td>Resilience Primary School has 500 students who are either Chinese, Indian or Malay, as</td></tr>
</table>

	seen in the table below.

	Chinese	Indian	Malay
Boys	114	8	93
Girls	122	77	86

The National Eye Centre wishes to conduct a survey at Resilience Primary School to find out the number of hours students spend on electronic devices each week, using a sample of 50 students.

(i) Explain how stratified sampling can be carried out in this context. [2]

(ii) Give two reasons why systematic sampling may not be appropriate. [2]

Solution:

(i)

	Chinese	Indian	Malay
Boys	$\frac{114}{500}\times 50 \approx 11$	$\frac{8}{500}\times 50 \approx 1$	$\frac{93}{500}\times 50 \approx 9$
Girls	$\frac{122}{500}\times 50 \approx 12$	$\frac{77}{500}\times 50 \approx 8$	$\frac{86}{500}\times 50 \approx 9$

Split the students into the stratas for Chinese, Indian, Malay boys or girls as shown in the table above. Arrange the students within each strata in alphabetical order (for example). Using simple random sampling, obtain the required number in each strata.

(ii)

$$k = \frac{500}{50} = 10$$

Since $k = 10 > 8 =$ number of Indian boys available, there is a possibility the Indian boys may not be represented.

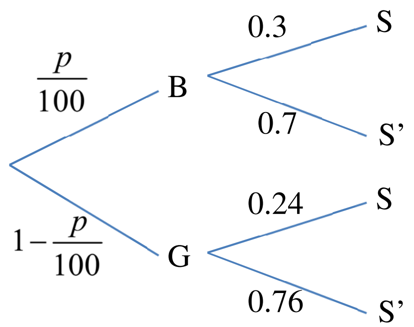
Systematic sampling does not ensure equal proportions of students being taken from each strata.

6 In another survey conducted by the National Eye Centre, it was found that p% are boys and the remaining are girls. The probability that a randomly chosen boy wears spectacles is 0.3 and the probability that a randomly chosen girl wears spectacles is 0.24.

(i) Find the value of p, given that the probability that a randomly chosen child wears spectacles is 0.267. [2]

(ii) For a general value of p, the probability that a randomly chosen child that wears spectacles is a girl is denoted by $\mathrm{f}(p)$. Show that $\mathrm{f}(p)=\dfrac{4(100-p)}{(400+p)}$. Prove by differentiation that f is a decreasing function for $0 \le p \le 100$, and explain what this statement means in the context of the question. [5]

Solution:



B = Boys

G = Girls

S = wear spectacles

S' = don't wear spectacles

(i)

$$\frac{p}{100}(0.3)+\left(1-\frac{p}{100}\right)(0.24)=0.267$$

$$0.0006p=0.027$$

$$p=45$$

(ii)

$$\mathrm{P}(\text{Girl}\,|\,\text{spectacles})=\frac{0.24\left(1-\frac{p}{100}\right)}{0.3\left(\frac{p}{100}\right)+0.24\left(1-\frac{p}{100}\right)}$$

$$=\frac{0.24-0.0024p}{0.003p+0.24-0.0024p}$$

$$=\frac{0.0024(100-p)}{0.0006(400+p)}$$

$$\mathrm{f}(p)=\frac{4(100-p)}{(400+p)} \text{ (shown)}$$

$$\mathrm{f}'(p)=\frac{(400+p)(-4)-(400-4p)}{(400+p)^2}$$

$$=\frac{-2000}{(400+p)^2}$$

Since $(400+p)^2>0$,

$$\mathrm{f}'(p)=\frac{-2000}{(400+p)^2}<0,\ \forall p\in\mathbb{R}$$

$\therefore$ f is a decreasing function.

Context: As the percentage of boys in the survey increases, the percentage that a girl wears spectacles decreases.

7 In this question you should state clearly all distributions that you use, together with the values of the appropriate parameters.

The mass, in grams, of broccoli and carrots are normally distributed with means and standard deviations as shown in the table below.

	Mean (g)	**Standard deviation (g)**
Broccoli	μ	σ
Carrot	180	15

(i) Given that the probability that the mass of a randomly chosen broccoli does not exceed 250g is 0.788 and the probability that the mass of a randomly chosen broccoli exceeds 236g is 0.625, find the values of μ and σ. [3]

(ii) Find the probability that the mass of a randomly chosen broccoli lies within 5 grams of a randomly chosen carrot. [2]

(iii) 120 broccoli are randomly chosen. Using a suitable approximation, find the probability that there are fewer than 90 broccoli with a mass not exceeding 250g. [3]

(iv) Determine, with explanation, whether the mass of a vegetable chosen randomly from a basket containing an equal number of broccoli and carrots follows a normal distribution. [1]

Solution:
Let X and Y be the random variable, the mass of a broccoli and the mass of a carrot respectively

$X \sim \mathrm{N}\left(\mu, \sigma^2\right)$, $Y \sim \mathrm{N}\left(180, 15^2\right)$

(i) $\mathrm{P}(X \le 250) = 0.788$

$$\mathrm{P}\left(Z \le \frac{250-\mu}{\sigma}\right) = 0.788$$

$$\frac{250-\mu}{\sigma} = 0.79950$$

$\mu + 0.79950\sigma = 250 \quad ---(1)$

$\mathrm{P}(X > 236) = 0.625$

$$\mathrm{P}\left(Z \le \frac{236-\mu}{\sigma}\right) = 0.375$$

$$\frac{236-\mu}{\sigma} = -0.31864$$

$\mu - 0.31864\sigma = 236 \quad ---(2)$

	Using GC: $\mu \approx 239.99 \approx 240$ (3 s.f.) and $\sigma \approx 12.521 \approx 12.5$ (3 s.f.) (ii) $X - Y \sim \mathrm{N}(59.99, 381.78)$ $\mathrm{P}(\lvert X-Y \rvert \le 5) = \mathrm{P}(-5 \le X - Y \le 5)$ $= 0.00200$ (3 s.f.) (iii) Let W be the random variable, the number of broccoli with mass not exceeding 250g $W \sim \mathrm{B}(120, 0.788)$ Since $n = 120 > 50$, $np = 94.56 > 5$, $nq = 25.44 > 5$ $W \sim \mathrm{N}(94.56, 20.047)$ approx. $\mathrm{P}(W < 90) = \mathrm{P}(W \le 89)$ $= \mathrm{P}(W < 89.5)$ (using Continuity Correction) $= 0.129$ (3 s.f.) (iv) It will not follow normal distribution as the mass of a randomly chosen vegetable from a basket containing an equal number of broccoli and carrot follows a bimodal distribution.
8	The table gives the values of eight observations of bivariate data, x and y.

x	1	2	3	4	5	6	7	8
y	5	1	18	23	28	31	33	34

(i) Draw a scatter diagram for these values, labelling the axes clearly. Determine the outlier by labelling it as P in your scatter diagram. [2]

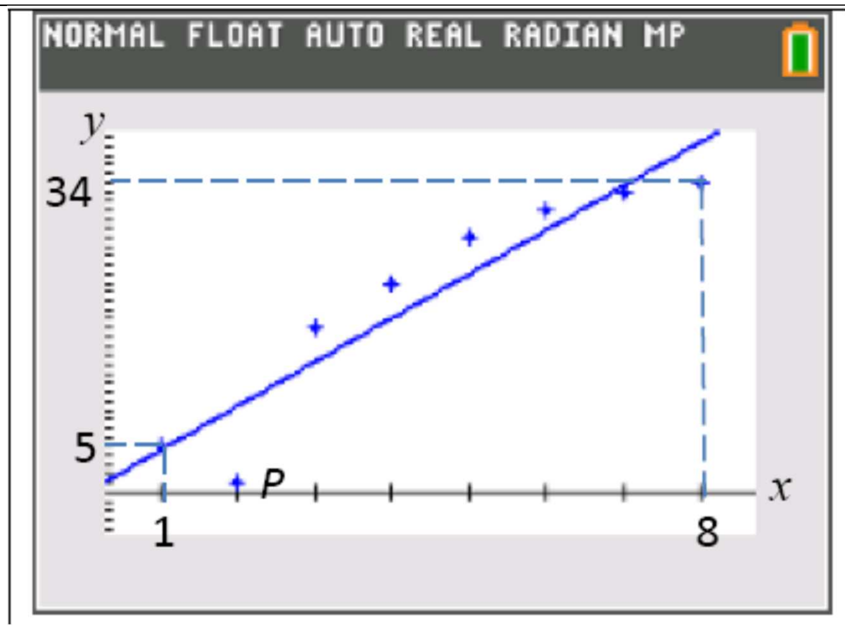
(ii) By omitting P, explain if $y = ax^2 + b$ or $y = a\ln x + b$ is the better model for the data. [2]

(iii) Using the more appropriate model found in part **(ii)**, calculate the equation of the least-squares regression line. [1]

(iv) Interpret, in the context of the question, the least squares estimates of a and b. [2]

(v) Use the regression line found in part **(iii)** to predict the value of y when x = 4.5. Comment on the reliability of your answer. [2]

Solution:
(i)



(ii) $y = ax^2 + b$: $r = 0.880$ (3 s.f.)

$y = a\ln x + b$: $r = 0.994$ (3 s.f.)

Since $y = a\ln x + b$ has $|r|$ closer to 1, $y = a\ln x + b$ is the better model.

(iii)

$y = 4.0144 + 14.518\ln x$

$\approx 4.01 + 14.5\ln x$ (3 s.f.)

(iv)

The expected value of y when $\ln x$ is 0 is 4.01.

For every increase in $\ln x$ by 1 unit, expected value of y increases by 14.5 units.

(v)

At $x = 4.5$, $y = 4.0144 + 14.518\ln(4.5) = 25.9$ (3 s.f.)

Reliable because $x = 4.5$ lies within the data range and $|r|$ is close to 1

9 Based on past records, the mean number of rainy days per year in Singapore was reported as 178. The authorities suspect that due to global warming, the number of rainy days has changed. A random sample of 12 years is taken and the number of rainy days per year, X, is summarised by

$$\sum(x-8) = 2017.7, \qquad \sum x^2 = 372\,500.$$

(i) Calculate the unbiased estimates of the mean and variance of X. [2]

(ii) Test, at the 5% level of significance, whether the mean number of rainy days per year has changed. State any assumptions used in your calculations. [4]

(iii) Explain, in the context of the question, the meaning of the p-value. [1]

	(iv) The population variance is found to be 9 and the assumption used in part **(ii)** holds true. A test at the 5% level of significance whether the mean number of rainy days per year has changed was conducted. Find the range of values of $\overline{x}$ such that the null hypothesis is not rejected. [3]
	Solution: (i) $\overline{x} = \frac{2017.7}{12} + 8 \approx 176.14 \approx 176$ (3 s.f.) <u>Method 1</u> $s^2 = \frac{1}{n-1}\left(\sum x^2 - n\left(\overline{x}\right)^2\right)$ $= \frac{1}{11}\left(372500 - 12(176.14)^2\right)$ $\approx 17.855 \approx 17.9$ (3 s.f.) <u>Method 2</u> $\sum x = 2017.7 + 8(12) = 2113.7$ $s^2 = \frac{1}{n-1}\left(\sum x^2 - \frac{\left(\sum x\right)^2}{n}\right)$ $= \frac{1}{11}\left(372\,500 - \frac{(2113.7)^2}{12}\right)$ $= 17.214 \approx 17.2$ (3 sf) (ii) Let *X* be the random variable, the number of rainy days per year in Singapore $H_0 : \mu = 178$ $H_1 : \mu \neq 178$ Assume H_0 is true. $\alpha = 0.05$. Assume *X* follows normal distribution. Since $n = 12 < 50$, population variance unknown, $T \sim t(11)$ approx. 2 tail t-test used. <u>Method 1:</u> Using GC, *p*-value = 0.156 (3 s.f.) > 0.05 if $s^2 = 17.855$ used [Alt: *p*-value = 0.149 (3 s.f.) > 0.05 if $s^2 = 17.214$ used]

	Do not reject H_0 <u>Method 2:</u> Test-statistic value: $t = \frac{176.14 - 178}{\sqrt{\frac{17.855}{12}}} \approx -1.52$ (3 s.f.) if $s^2 = 17.855$ used [Alt: $t = \frac{176.14 - 178}{\sqrt{\frac{17.214}{12}}} \approx -1.55$ (3 s.f.) if $s^2 = 17.214$ used] Critical region: $t \le -2.20$ (3 s.f.) or $t \ge 2.20$ (3 s.f.) Since test-statistic does not lie in the critical region, H_0 is not rejected. There is insufficient evidence at 5% level of significance to conclude that the mean number of rainy days per year has changed. (iii) <u>Either</u> *p*-value is the smallest level of significance for which the null hypothesis of the mean number of rainy days per year is 178 will be rejected. <u>Or</u> *p*-value is twice the probability of obtaining a test statistic less than or equal to -1.52, assuming the null hypothesis of the mean number of rainy days per year is 178 is true. (iv) $H_0 : \mu = 178$ $H_1 : \mu \neq 178$ Assume H_0 is true. Since *X* is normal, $\overline{X} \sim N\left(178, \frac{9}{12}\right)$ 2 tail *z*-test used. Since H_0 is not rejected at the 5% level of significance, $-1.9600 < \frac{\overline{x} - 178}{\left(\sqrt{\left(\frac{3}{4}\right)}\right)} < 1.9600$ $-1.9600\sqrt{\left(\frac{3}{4}\right)} < \overline{x} - 178 < 1.9600\sqrt{\left(\frac{3}{4}\right)}$ $176 < \overline{x} < 180$ (3 s.f.)
10	**(a)** Find the number of ways in which the letters of the word MILLENNIUM can be arranged if **(i)** there are no restrictions, [1]

(ii) the first and last letters are the same, and the letters E and U must be separated. [2]

Four letters are randomly selected from the letters of the word MILLENNIUM to form a code word. Find the number of possible code words that can be formed. [2]

(b) Mr See (together with his wife and daughter) and Mrs Saw (together with her husband and two sons) came to visit their former teacher Mdm Rain during Millennium Institute's Homecoming Day. Find the number of ways Mr See and his family, Mrs Saw and her family, and Mdm Rain can be arranged if

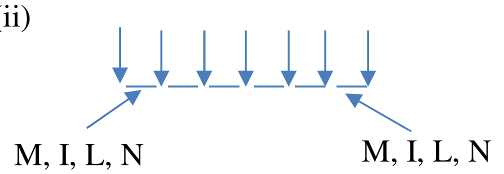
(i) they are around a table with ten indistinguishable chairs, such that the children are seated together. [2]

(ii) the two empty chairs are removed and Mr See's daughter is seated between her parents and the See family are to be seated directly opposite Mdm Rain. [3]

Solution:

(a)(i) No. of ways = $\dfrac{10!}{2!2!2!2!} = 226\,800$

(ii)



$$\text{No. of ways } = {}^{4}C_1 \times \frac{6!}{2!2!2!} \times {}^{7}C_2 \times 2!$$
$$= 15\,120$$

(a)(last part)

<u>Case 1: 2 Repeats</u>

$$\text{No. of ways } = {}^{4}C_2 \times \frac{4!}{2!2!} = 36$$

<u>Case 2: 1 Repeat</u>

$$\text{No. of ways } = {}^{4}C_1 \times {}^{5}C_2 \times \frac{4!}{2!} = 480$$

<u>Case 3: No Repeat</u>

$$\text{No. of ways } = {}^{6}C_4 \times 4! = 360$$

Total ways = 876

(b)(i)

$$\text{No. of ways } = \frac{8!}{8(2!)} \times 3!$$

	$= 15\ 120$ (b)(ii) No. of ways $= \frac{2!}{2} \times 2! \times 4!$ $= 48$
11	In this question you should state clearly all distributions that you use, together with the values of the appropriate parameters. The number of people queuing to buy coffee at CoffeeVille in a period of 1 minute during lunch hour (12pm to 2pm) is a random variable with an average number of 2.9. State, in context, a condition under which a Poisson distribution would be a suitable probability model. [1] Assume that the number of people queuing to buy coffee at CoffeeVille in a period of 1 minute during the lunch hour follows the distribution $\text{Po}(2.9)$. **(i)** State the most probable number of people queuing in 1 minute. [1] **(ii)** Find the probability that in a period of 3 minutes, there are at most 5 people queuing to buy coffee. [2] **(iii)** *N* periods of 3 minutes are taken. Given that the probability that at least 7 periods of 3 minutes have at most 5 people queuing to buy coffee is more than 0.99, find the least value of *N*. [3] **(iv)** A random sample of 120 periods of 3 minutes is taken. Using a suitable approximation, find the probability that more than 12 periods of 3 minutes have exactly 4 people queuing. [3] **(v)** Explain why the Poisson model would probably not be valid if applied to the operating hours of CoffeeVille from 11am to 10pm. [1]
	Solution: Average number of people queuing to buy coffee is a constant (i) Let *X* be the random variable, for the number of people queuing to buy coffee in 1 min. $X \sim \text{Po}(2.9)$ Using GC: Mode = 2 (ii) Let *Y* be the random variable, for the number of people queuing to buy coffee in 3 min.

$Y \sim \text{Po}(8.7)$

$\text{P}(Y \leq 5) = 0.13516 \approx 0.135$ (3 s.f.)

(iii)

Let W be the random variable, for the number of periods of 3 min with $Y \leq 5$

$W \sim \text{B}(n, 0.13516)$

$\text{P}(W \geq 7) > 0.99$

$1 - \text{P}(W \leq 6) > 0.99$

$\text{P}(W \leq 6) < 0.01$

Using GC:

N	$\text{P}(W \leq 6)$
103	0.0104 > 0.01
104	0.00947 < 0.01
105	0.00864 < 0.01

Least value of N is 104

(iv)

Let V be the random variable, for the number of periods of 3 min with $Y = 4$

$V \sim \text{B}(120, 0.039765)$

Since $n = 120 > 50$, $np = 4.7718 < 5$

$V \sim \text{Po}(4.7718)$ approx.

$\text{P}(V > 12) = 1 - \text{P}(V \leq 12) \approx 0.00135$ (3 s.f.)

(v)

Mean number of people queuing varies throughout the day.