CANBERRA SECONDARY SCHOOL

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2024 Preliminary Examination

Secondary Four Express

ADDITIONAL MATHEMATICS
4049/01

22 August 2024
2 hours 15 minutes
1130h – 1345h

Name: ______________________________ ( ) **Class:** ________

READ THESE INSTRUCTIONS FIRST

Write your full name, class and index number on all work you hand in.
Write in dark blue or black pen.
You may use a pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is 90.

FOR MARKER'S USE		
	Marks Awarded	Max Marks
		90

This question paper consists of 19 printed pages including the cover page.

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2+bx+c=0, a\neq 0,$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Binomial expansion

$$(a+b)^n=a^n+\binom{n}{1}a^{n-1}b+\binom{n}{2}a^{n-2}b^2+\ldots+\binom{n}{r}a^{n-r}b^r+\ldots+b^n,$$

where n is a positive integer and $\binom{n}{r}=\frac{n!}{(n-r)!r!}=\frac{n(n-1)...(n-r+1)}{r!}$.

2. TRIGONOMETRY

Identities

$$\sin^2 A+\cos^2 A=1$$
$$\sec^2 A=1+\tan^2 A$$
$$\operatorname{cosec}^2 A=1+\cot^2 A$$
$$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$$
$$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$$
$$\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}$$
$$\sin 2A=2\sin A\cos A$$
$$\cos 2A=\cos^2 A-\sin^2 A=2\cos^2 A-1=1-2\sin^2 A$$
$$\tan 2A=\frac{2\tan A}{1-\tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$
$$a^2=b^2+c^2-2bc\cos A$$
$$\Delta=\frac{1}{2}bc\sin A$$

Answer **all** the questions

1 Express $\dfrac{-3x^2+12x-1}{(3x+1)(x-1)^2}$ in partial fractions. [5]

2 **(a)** Find the amplitude and period of

(i) $3\cos x$, [1]

(ii) $1-4\sin 2x$ [2]

(b) Sketch, on the same diagram, the curves $y=3\cos x$ and $y=1-4\sin 2x$ for $0^\circ \le x \le 360^\circ$. [3]

(c) State the number of solution(s) for which the equation $4\sin 2x = 1-3\cos x$ has. [1]

3 **(a)** **(i)** Find the first three terms in the expansion, in ascending powers of x, of $\left(2-\frac{x}{3}\right)^7$. [2]

(ii) Hence find the value of p, where p is an integer, such that the coefficient of x^2 in the expansion of $(p+x)^2\left(2-\frac{x}{3}\right)^7$ is $-\frac{32}{3}p^2$. [4]

3 **(b)** By considering the general term in the binomial expansion of $\left(x^2 - \frac{1}{2x}\right)^{17}$, explain why there is no independent term in this expansion. [4]

4 The equation of a curve is $y = ax^2 + 4x + 3a - 7$, where a is a constant.
The line $y = 2ax - 5$ is a tangent to the curve as point Q.

(i) Find the possible values of a. [4]

(ii) Given that the gradient of the line $y = 2ax - 5$ is positive, find the coordinates of Q. [2]

5 Mr Lu invested a sum of money in a bank in 1980. The value of the investment, V in thousands dollars can be modelled by an equation of the form $V = V_0 k^t$, where V_0 and k are constants and t is the number of years since Mr Lu invested the sum of money.
The table below gives some values of V and t.

t (years)	5	10	15	20	25
V (in thousands $)	192	370	714	1374	2642

(i) On the grid provided in the next page, plot lg$V$ against t and draw a straight line graph to illustrate the information.

The vertical lg$V$-axis should start at 1.8. [3]

Use your graph to estimate

(ii) the initial value of the investment, [2]

(iii) the value of k, [3]

(iv) the time for the initial value of the investment to increase by 50%. [2]

6 **(a)** Find the range of values of x for which $2x(x-3) \geq 3-5x$. [3]

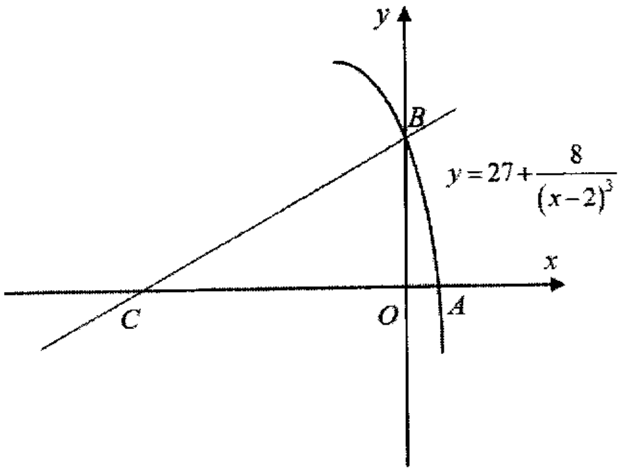
(b) Find the greatest value of k such that the line $2y+k=x$ does not intersect the curve $y=2x+\frac{6}{x}$, where k is an integer. [4]

7 The equation of a curve is $y = 2x^2 - 6x + 9$.

(a) By expressing $2x^2 - 6x + 9$ in the form $a(x+b)^2 + c$, where a, b and c are constants, explain if it is possible for the curve to have a value smaller than $\frac{9}{2}$. [2]

(b) The line $y = 3x + 5$ intersects the curve $y = 2x^2 - 6x + 9$ at points P and Q. Find the length of PQ and express your answer in exact form. [4]

8



The diagram shows part of the curve $y=27+\frac{8}{(x-2)^3}$ which crosses the axes at points A and B.

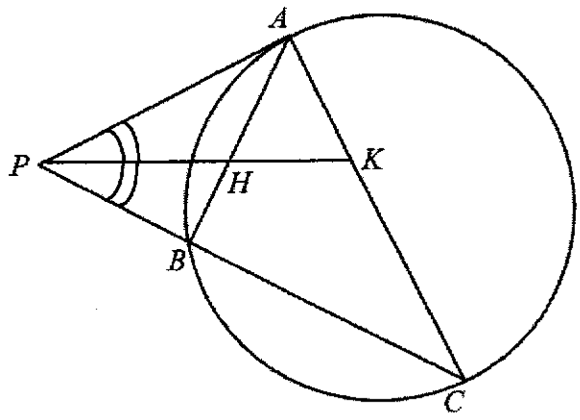
(a) Explain why the curve does not have a stationary point. [3]

(b) Find the coordinates of A and of B. [2]

8 **(c)** Find the equation of the normal to the curve at B. [3]

(d) The normal to the curve at B cuts the x-axis at C.
Find the area of triangle BOC. [2]

9 In the diagram, AP is a tangent to the circle passing through A, B and C.
AC is the diameter of the circle and PBC is a straight line.
The bisector of angle APC meets AB at H and AC at K.



(i) Show that triangle BAP is similar to triangle BCA. [2]

(ii) Prove that $AH = AK$. [2]

9 **(iii)** Given that triangle BHP is similar to triangle AKP, prove that $AH \times HP = KP \times BH$. [2]

10 The equation of a curve is $y = \frac{e^{2x}}{1-x}$, where $x > \frac{3}{2}$.

Find $\frac{dy}{dx}$ and explain whether y is always increasing or decreasing. [5]

11 A curve is such that $\frac{d^2y}{dx^2} = 36e^{3x} - e^{-x}$.
The curve intersects the y-axis at $Q(0,5)$ and the tangent to the curve at Q has a gradient of 14.
Find the equation of the curve. [7]

12 A particle P travels in a straight line and its velocity, v m/s, is given by $v = 4\sin^2\left(\frac{t}{2}\right) - 1$, where t is the time in seconds after passing a fixed point O.

Find

(i) the initial velocity of P, [1]

(ii) the velocity when the acceleration first reaches 2 m/s^2, [4]

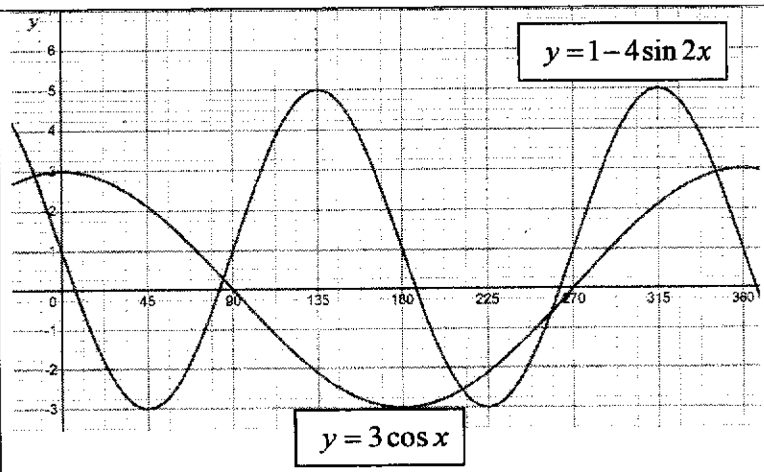
12 **(iii)** the value of t when the particle is first instantaneously at rest after passing the fixed point O, [3]

(iv) the total distance travelled during the 3rd second of the motion. [3]

End of Paper

4E Additional Mathematics Preliminary Examination 2024 Marking Scheme

Qns	Suggested Solutions	Remarks
1	$\frac{-3x^2+12x-1}{(3x+1)(x-1)^2}=\frac{A}{3x+1}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$ $-3x^2+12x-1=A(x-1)^2+B(3x+1)(x-1)+C(3x+1)$	[M1]
	When $x=1$, $-3+12-1=C(4)$ $C=2$	[M1]
	When $x=-\frac{1}{3}$, $-3\left(-\frac{1}{3}\right)^2+12\left(-\frac{1}{3}\right)-1=A\left(-\frac{4}{3}\right)^2$ $A=-3$	[M1]
	When $x=1$, $A=-3$, $C=2$ $-1=(-3)(-1)^2+B(1)(-1)+(2)(0+1)$ $B=0$	[M1]
	$\frac{-3x^2+12x-1}{(3x+1)(x-1)^2}=-\frac{3}{3x+1}+\frac{2}{(x-1)^2}$	[A1]
2(a)(i)	$3\cos x$ Amplitude = 3 Period = 360°	[B1]
2(a)(ii)	$1-4\sin 2x$ Amplitude = 4	[B1]
	Period $=\frac{360^\circ}{2}$ $=180^\circ$	[B1]

2(b)		[M1] Correct shape [M1] Smoothness [M1] Intersection
2(c)	4 solutions	[B1]
3(a)(i)	$\left(2-\frac{x}{3}\right)^7=2^7+\binom{7}{1}(2)^6\left(-\frac{x}{3}\right)+\binom{7}{2}(2)^5\left(-\frac{x}{3}\right)^2+\ldots$ $=128-\frac{448}{3}x+\frac{224}{3}x^2+\ldots$	[M1] [A1]
3(a)(ii)	$(p+x)^2\left(2-\frac{x}{3}\right)^7=\left(p^2+2px+x^2\right)\left(128-\frac{448}{3}x+\frac{224}{3}x^2+\ldots\right)$ $=\frac{224}{3}p^2x^2-\frac{896}{3}px^2+128x^2+\ldots$ $=\left(\frac{224}{3}p^2-\frac{896}{3}p+128\right)x^2+\ldots$ $\frac{224}{3}p^2-\frac{896}{3}p+128=-\frac{32}{3}p^2$ $256p^2-896p+384=0$ $2p^2-7p+3=0$ $(p-3)(2p-1)=0$ $p=3$ $p=\frac{1}{2}$ (rejected)	[M1] [M1] [M1] [A1]
3(b)	$\left(x^2-\frac{1}{2x}\right)^{17}$	

	$T_{r+1}=\binom{17}{r}\left(x^2\right)^{17-r}\left(-\frac{1}{2x}\right)^r$	[M1]
	$=\binom{17}{r}\left(x^{34-2r}\right)(-1)^r\left(\frac{1}{2}\right)^r\left(\frac{1}{x}\right)^r$	
	$=\binom{17}{r}(-1)^r\left(\frac{1}{2}\right)^r x^{34-3r}$	[M1]
	Let $x^{34-3r}=x^0$	[M1]
	$34-3r=0$ $r=\frac{34}{3}$ $=11\frac{1}{3}$	
	Since r is <u>not an integer</u>, there is no independent term.	[A1]
4(i)	$y=ax^2+4x+3a-7\cdots(1)$ $y=2ax-5\cdots(2)$ Sub (2) into (1)	
	$ax^2+4x+3a-7=2ax-5$	
	$ax^2+(4-2a)x+3a-2=0$	[M1]
	$b^2-4ac=0$	
	$(4-2a)^2-4a(3a-2)=0$	[M1]
	$16-16a+4a^2-12a^2+8a=0$ $-8a^2-8a+16=0$ $a^2+a-2=0$	
	$(a+2)(a-1)=0$	[M1]
	$a=-2, a=1$	[A1]
4(ii)	Since gradient is positive, $a=1$. $y=x^2+4x-4\cdots(1)$ $y=2x-5\cdots(2)$ Sub (2) into (1)	

$x^2 + 4x - 4 = 2x - 5$

$x^2 + 2x + 1 = 0$

$(x+1)^2 = 0$

$x = -1$ [M1]

Sub $x = -1$,

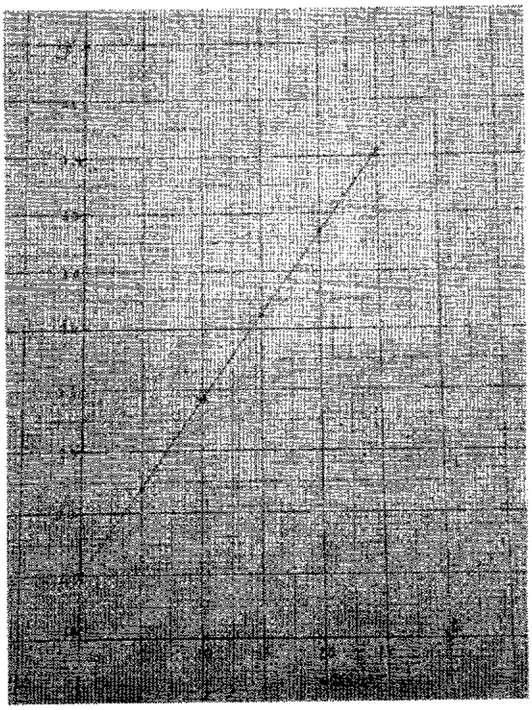
$y = -2 - 5$

$y = -7$

$Q(-1, -7)$ [A1]

5(i)

t (years)	5	10	15	20	25
V (in thousands \$)	192	370	714	1374	2642
lgV	2.28	2.57	2.85	3.14	3.42



[G1] Axes

[G1] Points plotted correctly

[G1] Straight line passing through all plotted points.

5(ii)	From graph $\lg V = 2$ $V = 10^2$ $V = 100$ The initial value of investment is \$100 000.	[M1] [A1]
5(iii)	$V = V_o k^t$ $\lg V = t \lg k + \lg V_o$ From graph, $\text{Gradient} = \frac{3.08 - 2.4}{19 - 7} = \frac{17}{300}$ $\lg k = \frac{17}{300}$ $k = 1.13937$ $k \approx 1.14$	 [M1] [M1] [A1]
5(iv)	Increase by 50% → investment is at \$150 000 $V = 150$ $\lg V \approx 2.18$ From graph, when $\lg V = 2.18$, $t = 3.25$ years	 [M1] [A1]
6(a)	$2x(x-3) \geq 3 - 5x$ $2x^2 - 6x \geq 3 - 5x$ $2x^2 - x - 3 \geq 0$ $(x+1)(2x-3) \geq 0$ $x \leq -1$ or $x \geq 1\frac{1}{2}$	[M1] Expansion [M1] Factorisation [A1]
6(b)	$2y + k = x \cdots (1)$ $y = 2x + \frac{6}{x} \cdots (2)$ Sub (2) into (1)	

	$2\left(2x+\frac{6}{x}\right)+k=x$ $4x+\frac{12}{x}+k=x$ $4x^2+12+kx=x^2$ $3x^2+kx+12=0$ $b^2-4ac<0$ $k^2-4(3)(12)<0$ $k^2-144<0$ $(k+12)(k-12)<0$ $-12<k<12$ Therefore, the greatest value of k is 11.	[M1] [M1] [M1] [A1]
7(a)	$y=2x^2-6x+9$ $=2\left(x^2-3x\right)+9$ $=2\left[\left(x-\frac{3}{2}\right)^2-\frac{9}{4}\right]+9$ $=2\left(x-\frac{3}{2}\right)^2+\frac{9}{2}$ Since $\left(\frac{3}{2},\frac{9}{2}\right)$ is a minimum point, it is not possible for the curve to have a value smaller than $\frac{9}{2}$.	[M1] [A1]
7(b)	$y=3x+5\cdots(1)$ $y=2x^2-6x+9\cdots(2)$ Sub (1) into (2) $2x^2-6x+9=3x+5$ $2x^2-9x+4=0$ $(x-4)(2x-1)=0$ $x=4, x=\frac{1}{2}$	[M1]

	Sub $x=4$ $y=3(4)+5$ $=17$ $(4,17)$ Sub $x=\frac{1}{2}$ $y=3\left(\frac{1}{2}\right)+5$ $=\frac{13}{2}$ $\left(\frac{1}{2},\frac{13}{2}\right)$	[M1]
	Length PQ $=\sqrt{\left(4-\frac{1}{2}\right)^2+\left(17-\frac{13}{2}\right)^2}$ $=\sqrt{\frac{490}{4}}$ $=\frac{\sqrt{490}}{2}$ or $\frac{7\sqrt{10}}{3}$ or $\frac{7}{2}\sqrt{10}$	[M1] [A1]
8(a)	$y=27+\frac{8}{(x-2)^3}$ $\frac{dy}{dx}=-24(x-2)^{-4}$ $=-\frac{24}{(x-2)^4}$ Since $\frac{dy}{dx}\neq 0$ for all $x\neq 2$, the curve does not have a stationary point.	[M1] [M1] [A1]
8(b)	Sub $x=0$	

	$y = 27 + \frac{8}{(-2)^3}$ $= 26$ $B(0,26)$ Sub $y = 0$ $0 = 27 + \frac{8}{(x-2)^3}$ $-27 = \frac{8}{(x-2)^3}$ $-27(x-2)^3 = 8$ $(x-2)^3 = -\frac{8}{27}$ $x-2 = -\frac{2}{3}$ $x = 1\frac{1}{3}$ $A\left(1\frac{1}{3},0\right)$	[B1] [B1]
8(c)	Sub $x = 0$ $\frac{dy}{dx} = -\frac{24}{(-2)^4}$ $= -\frac{3}{2}$ Gradient of normal $= \frac{2}{3}$ $y - 26 = \frac{2}{3}(x-0)$ $y = \frac{2}{3}x + 26$	 [M1] [M1] [A1]
8(d)	Sub $y = 0$	

	$\frac{2}{3}x=-26$ $x=-39$ Area of triangle BOC $=\frac{1}{2}\times 39\times 26$ $=507$ units^2	[M1] [A1]
9(i)	$\angle PAB=\angle BCA$ (alternate segment Thm) $\angle ABC=90^\circ$ (angle in semi-circle) $\angle ABP=90^\circ$ (angle on a straight line) $\therefore \angle ABP=\angle ABC=90^\circ$ $\therefore \Delta BAP$ is similar to ΔBCA (AA)	[M1] [A1]
9(ii)	Let $\angle AKH=x$, $\angle KAP=90^\circ$ (tangent perpendicular to radius) $\angle APH=90^\circ-x$ (sum of angles in triangle) $\angle HPB=90^\circ-x$ (angle bisector) $\angle ABC=90^\circ$ (angle in semi-circle) $\angle ABP=90^\circ$ (angle on a straight line) $\angle BHP=180^\circ-90^\circ-(90^\circ-x)$ (sum of angles in triangle) $=x$ $\angle AHK=x$ (vertically opposite angles) $\therefore$ Since $\angle AHK=\angle AKH, AH=AK$ (isosceles triangle)	[M1] [A1]
9(iii)	Since ΔBHP is similar to ΔAKP $\frac{AK}{BH}=\frac{KP}{HP}$ $\Rightarrow \frac{AH}{BH}=\frac{KP}{HP}$ ($AK=AH$ from (ii)) $\Rightarrow AH\times HP=KP\times BH$ (Proved)	[M1] [A1]

10	$y=\dfrac{e^{2x}}{1-x}$ $\dfrac{dy}{dx}=\dfrac{2e^{2x}(1-x)+e^{2x}}{(1-x)^2}$ $=\dfrac{3e^{2x}-2xe^{2x}}{(1-x)^2}$ $=\dfrac{e^{2x}(3-2x)}{(1-x)^2}$ Since $x>\dfrac{3}{2}$, $(1-x)^2>0$ $\dfrac{1}{(1-x)^2}>0$ $\dfrac{(3-2x)}{(1-x)^2}<0$ $\dfrac{e^{2x}(3-2x)}{(1-x)^2}<0$ $\dfrac{dy}{dx}<0$ Since $\dfrac{dy}{dx}<0$, y is always decreasing.	[M1] [M1] [M1] [M1] [A1]
11	$\dfrac{d^2y}{dx^2}=36e^{3x}-e^{-x}$ $\dfrac{dy}{dx}=\int 36e^{3x}-e^{-x}dx$ $=\dfrac{36}{3}e^{3x}+e^{-x}+c$ $=12e^{3x}+e^{-x}+c$ Sub $\dfrac{dy}{dx}=14, x=0$ $14=12+1+c$ $c=1$	[M1] [M1] [M1]

	$\frac{dy}{dx}=12e^{3x}+e^{-x}+1$ $y=\int 12e^{3x}+e^{-x}+1dx$ $y=4e^{3x}-e^{-x}+x+d$ Sub $x=0, y=5$ $5=4-1+0+d$ $d=2$ $y=4e^{3x}-e^{-x}+x+2$	[M1] [M1] [M1] [A1]
12(i)	When $t=0,\ v=4\sin^2\left(\frac{0}{2}\right)-1\ =\ -1$ m/s $\therefore$ initial velocity = –1 m/s	[B1]
12(ii)	When acceleration, $a = 2$ m/s^2 $\frac{\mathrm{d}v}{\mathrm{d}t}=8\sin\left(\frac{t}{2}\right)\times\cos\left(\frac{t}{2}\right)\times\frac{1}{2}$ $4\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)=2$ $2\sin t=2$ $\sin t=1$ $t=\frac{\pi}{2}$ $\therefore$ when $a = 2$ m/s^2, $t=\frac{\pi}{2}$ and $v=4\sin^2\left(\frac{\pi}{4}\right)-1$ $v=4\times\left(\frac{1}{\sqrt{2}}\right)^2-1\ =1$ m/s	[M1] [M1] [M1] [A1]
12(iii)	Instantaneous rest $\Rightarrow\ v=0$ $\Rightarrow\ 4\sin^2\left(\frac{t}{2}\right)-1=0$ $\Rightarrow\ \sin\left(\frac{t}{2}\right)=\pm\frac{1}{2}$ $\Rightarrow\ \frac{t}{2}=\frac{\pi}{6},\frac{5\pi}{6},\frac{13\pi}{6},.....,$ For first instantaneous rest, $\frac{t}{2}=\frac{\pi}{6}$ $\Rightarrow\ t=\frac{\pi}{3}$ *or* 1.05 second	[M1] [M1] [A1]

12(iv)	$\cos t = 1 - 2\sin^2\left(\frac{t}{2}\right)$ $2\sin^2\left(\frac{t}{2}\right) = 1 - \cos t$ $4\sin^2\left(\frac{t}{2}\right) = 2 - 2\cos t$ $S = \int 4\sin^2\left(\frac{t}{2}\right) - 1 dt$ $= \int 2 - 2\cos t - 1 dt$ $= \int 1 - 2\cos t dt$ $= t - 2\sin t + c$	[M1] Double angle formula
	When $t = 0$, $S = 0$, $c = 0$ When $t = 2$ $S = 2 - 2\sin(2)$ ≈ 0.18140 When $t = 3$ $S = 3 - 2\sin(3)$ ≈ 2.7177	[M1] Finding S either $t = 2$ or $t = 3$
	Distance travelled during the 3rd second $2.7177 - 0.18140 \approx 2.54$ m	[A1]