NANYANG JUNIOR COLLEGE
JC2 PRELIMINARY EXAMINATION

Higher 1

MATHEMATICS **8865/01**

Paper 1 **13 September 2017**

3 hours

Additional Materials: Cover Page
Answer Paper
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagram or graph.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of 6 printed pages.

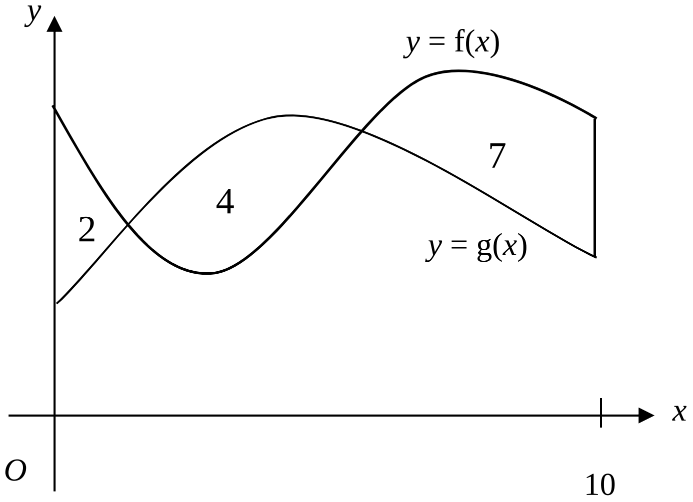
Pure Mathematics [40 marks]

1 Find the range of values of a such that $ax^2+4ax+7a>2x+2$ for all real values of x. [4]

2 The graphs of two functions are shown with the areas of the regions between the curves indicated.

(i) What is the total area between the curves for $0\le x\le 10$? [1]

(ii) What is the value of $\int_0^{10} f(x)-g(x)\,dx$? [3]



3 Given that u_n is a quadratic polynomial in n and the first three terms are as follows:

$u_1=3$, $u_2=1$ and $u_3=-5$.

(i) Find u_n in terms of n. [4]

(ii) Using an algebraic method, find the value of n for which u_n is greater than 2. [2]

4 The gradient of a curve $y = \mathrm{f}\left(x\right)$ at any point $\left(x, y\right)$ is given by $\mathrm{f}'\left(x\right) = 4\mathrm{e}^{1-x} - 2\mathrm{e}^{-3-2x}$.

(i) Find the exact x-coordinate of the stationary point on the curve, leaving your answer in the form of $a\ln 2 + b$ where a and b are constants to be determined. [2]

(ii) Given that the curve intersects the x-axis at the point where $x = -4$ find the equation of the curve, giving your answer in exact form. [3]

(iii) Sketch the graph of $y = \mathrm{f}\left(x\right)$, giving the equations of any asymptotes and the coordinates of the points of intersection with the x- and y-axes. [3]

(iv) Find the equation of the tangent at the point $\left(-4, 0\right)$, leaving answer in terms of e. Hence find the exact area of the region bounded by the curve, the tangent at the point $\left(-4, 0\right)$ and the y-axis. [5]

5 NY Press Holdings is an English-language online news media based in Singapore. It is known that the online news will be taken down after 24 hours. Suppose that the number of viewers (in thousands) at t^{th} hour after the release of the news is given by

$$S(t) = 50\mathrm{e}^{-0.04t} \quad , \quad 0 \le t \le 24$$

(i) Show that the number of viewers (in thousands) in the hour just before the news was taken down is 19.1. [1]

(ii) What is the total number of viewers for the period of 24 hours? [3]

(iii) State an assumption needed for the model to be valid. [1]

Assume that on the average, the number of viewers per hour is 32,000 and the company receives 50 cents for every viewing. In order to attract more viewings, the company uses an advertising site. It is known that for each cent spent on advertising, the number of viewings per hour will increase by 1000.

(iv) Given that the company targets to achieve 50,000 viewings per hour, how much must the company spent on advertising in order to achieve its target? Hence state the amount of revenue (in thousand dollars) per hour. [2]

It is given that the hourly profit achieved by the company after investing x cents on advertising is $\$(16000 + 180x - 10x^2)$.

(v) Use differentiation to find the maximum hourly profit (in thousand dollars), proving that it is maximum. [6]

Statistics [60 marks]

6 A group of 10 students from various sports CCA are gathered for a photoshoot. The group consists of 3 basketball players, 4 tennis players and 3 players from other sport. The 10 students are arranged randomly in a line.

(i) In how many different ways can the 10 people be arranged in a line? [1]

(ii) Find the probability that no 2 basketball players are next to each other? [2]

(iii) The basketball players are all separated. Find the probability that the 4 tennis players are next to each other. [3]

7 A fixed number, n, of students is observed and the number of those students wearing spectacles is denoted by S.

(i) State, in context, two assumptions needed for S to be well modelled by a binomial distribution. [2]

Assume now that S has the distribution $B(n, p)$.

(ii) Given that $n = 15$ and $P(S = 0 \text{ or } S = 1) = 0.3$, write down an equation for the value of p, and find this value numerically. Hence find $P(2 \le S < 8)$. [3]

It is now given that $n = 30$ and $p = 0.7$.

(iii) Suppose that there are 50 groups of 30 students being observed, find the probability that the mean number of students wearing spectacles is at most 20. [3]

8 Two badminton players, Derek and Benjamin, met in a match, and the winner of the game is the first player to win two sets. The probability that Derek wins a match is p. From the second set onwards, the probability that he wins a set is

- p times of the previous probability if he wins in the preceding set,
- 0.4 if he did not win in the preceding set.

(i) Construct a tree diagram to represent all the possible outcomes. [2]

(ii) Find p if the probability of Derek winning the game is 0.6528. [3]

(iii) Given that $p = 0.64$, find the probability of Derek winning the second set if he wins the game. [3]

9 The rate of growth, y, of a particular organism in a laboratory is believed to depend in some way on the controlled temperature, x°C. The table shows the results of 8 experiments.

Experiment	1	2	3	4	5	6	7	8
Temperature, x°C.	6	9	11	14	18	22	25	28
Rate of growth, y	5	13	15	a	20	24	26	30

The regression line of y on x is given by $y = 0.993x + 2.11$.

(i) Find the value of a, giving your answer to the nearest integer. [2]

(ii) Give a sketch of your scatter diagram for the 8 sets of data, as shown on your calculator. [2]

(iii) Calculate the product moment correlation coefficient, giving your answer correct to 4 decimal place and comment on its value in relation to your scatter diagram. [2]

(iv) Estimate the temperature in the laboratory when the growth rate of the organism is 18. Comment on the reliability of this prediction. [2]

(v) Explain why it might be unsuitable to use the equation y on x to estimate how the rate of growth of the organism when the temperature is 30°C. [1]

10 The owner of AAA Tuition Company which has many centres in Singapore claims that the mean H1 Mathematics score of their students is 80. A random sample of 50 students from one of the centres is taken. Their mathematics scores, x, are summarised by

$$\sum(x-80) = -40, \qquad \sum(x-80)^2 = 450.$$

Find unbiased estimates of the population mean and variance. [3]

(i) Test, at a 5% level of significance, whether the owner has overstated the mean H1 Mathematics score of their students. [4]

(ii) Suppose a teacher in one of the tuition centres now claims that the mean H1 Mathematics score of a randomly chosen student is not 80. Without carrying out another test, state, with a reason, whether the conclusion in part **(ii)** would remain the same. [2]

Another large random sample of n students gives a mean H1 Mathematics score of 78.8. Given that the population standard deviation of the H1 Mathematics score is now known to be 5, find the largest value of n to conclude that the owner's claim is valid at the same level of significance. [5]

[Turn over

11 The masses, in kilograms, of the honeydews and durians, sold by a supermarket have independent normal distributions with means and standard deviations as shown in the following table.

	Mean mass	Standard deviation
Honeydew	1.8	b
Durian	1.5	0.3

(i) The mass of the honeydews is a random variable denoted by H. It is known that $\text{P}(H < 2k) = 0.95$ and $\text{P}(H < k) = 0.25$ where k is a constant. Find the value of b. [3]

For the rest of the question, use $b = 0.4$.

(ii) One honeydew and one durian are chosen at random. Find the probability that the mass of the honeydew is more than 1.7 kg and the mass of the durian is less than 1.7 kg. [2]

(iii) Three honeydews are chosen at random. Find the probability that one of the honeydews has mass less than 1.6 kg and one honeydew has mass more than 1.7 kg each. [4]

(iv) Find the probability that three times the mass of a randomly chosen durian is within ± 0.8kg of the mass of two randomly chosen honeydews. [3]

(v) Honeydews cost \$3 per kilogram and durians cost \$10 per kilogram. Find the mean and variance of the cost of two honeydews and hence find the probability that the total cost of two honeydews and one durian is greater than \$28.50.

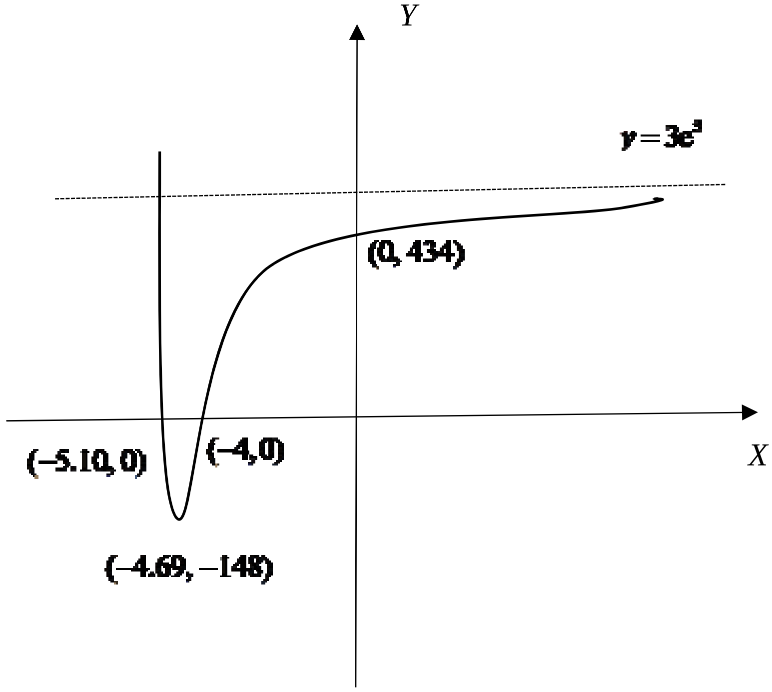
[3]

--- End of Paper ---

no	Solution
1.	$ax^2+4ax+7a>2x+2 \quad \Rightarrow \quad ax^2+(4a-2)x+\left(7a-2\right)>0$ For $Ax^2+Bx+C>0$ for all real values of x, we must have: $A>0$ and $B^2-4AC<0$ That is, $a>0$ and $(4a-2)^2-4a(7a-2)<0$ $(4a-2)^2-4a(7a-2)<0 \quad \Rightarrow \quad 16a^2-16a+4-28a^2+8a<0$ $\Rightarrow \quad -12a^2-8a+4<0 \quad \Rightarrow \quad 3a^2+2a+1>0$ $\Rightarrow \quad 3a^2+2a-1>0$ $\Rightarrow \quad -12a^2-8a+4<0 \quad \Rightarrow \quad 3a^2+2a-1>0$ $\Rightarrow \quad (3a-1)(a+1)>0$ Thus, $a>\frac{1}{3}$ or $a<-1$ Since $a>0$, we have $a>\frac{1}{3}$
2	(a) Total area = 2 + 4 + 7 = 13 (b) $\int_0^{10} f(x)-g(x)\,\mathrm{d}x$ $=\int_0^{a} f(x)-g(x)\,\mathrm{d}x+\int_a^{b} f(x)-g(x)\,\mathrm{d}x+\int_b^{10} f(x)-g(x)\,\mathrm{d}x$ $=\int_0^{a} f(x)-g(x)\,\mathrm{d}x-\int_a^{b} g(x)-f(x)\,\mathrm{d}x+\int_b^{10} f(x)-g(x)\,\mathrm{d}x$ $=2-4+7$ $=5$
3	$u_n=an^2+bn+c$ When n = 1, $3=a+b+c$...........(1) n = 2, $1=4a+2b+c$...........(2) n =3, $-5=9a+3b+c$...........(3) Using GC, a = -2, b = 4, c = 1. $u_n=-2n^2+4n+1$

	$-2n^2+4n+1>2$ $2n^2-4n+1<0$ $2(n^2-2n)+1<0$ $2(n-1)^2-2+1<0$ $2(n-1)^2-1<0$ $1-\frac{1}{\sqrt{2}}<n<1+\frac{1}{\sqrt{2}}$ $\text{Sin}\,ce\ n\in\mathbf{Z}^+, n=1$
4	(i) $\text{f}'(x)=4\text{e}^{1-x}-2\text{e}^{-3-2x}$ At stationary point, $\frac{\text{d}y}{\text{d}x}=0$ $4\text{e}^{1-x+1}-2\text{e}^{-3-2x}=0$ $4\text{e}^{1-x}=2\text{e}^{-3-2x}$ $\frac{\text{e}^{-3-2x}}{\text{e}^{1-x}}=\frac{4}{2}$ $\text{e}^{-3-2x-1+x}=2$ $\text{e}^{-x-4}=2$ $-x-4=\ln 2$ $\therefore x=-\ln 2-4$ (ii) $y=\int 4\text{e}^{1-x}-2\text{e}^{-3-2x}\ \text{d}x$ $=4\int \text{e}^{1-x}\ \text{d}x-2\int \text{e}^{-3-2x}\ \text{d}x$ $=4\left(\frac{\text{e}^{1-x}}{-1}\right)-2\left(\frac{\text{e}^{-3-2x}}{-2}\right)+C$ $y=-4\text{e}^{1-x}+\text{e}^{-3-2x}+C$ Given that the curve cut the x-axis at $x=-4$, so $0=-4\text{e}^{1+4}+\text{e}^{-3+8}+C$ $0=-4\text{e}^5+\text{e}^5+C$ $C=3\text{e}^5$ Equation of curve: $y=-4\text{e}^{1-x}+\text{e}^{-3-2x}+3\text{e}^5$

(iii)



(iv)

$$f'(x) = 4e^{-x+1} - 2e^{-2x-3}$$

Gradient of the tangent at the point (-4, 0)

$$= f'(-4) = 4e^{4+1} - 2e^{8-3} = 2e^5$$

Equation of the tangent at the point (-4, 0):

$$y - 0 = 2e^5(x+4)$$
$$y = 2e^5 x + 8e^5$$

(v)

Area required

$$= \int_{-4}^{0} \left(2e^5 x + 8e^5\right) - \left(-4e^{1-x} + e^{-3-2x} + 3e^5\right) \mathrm{d}x$$
$$= \int_{-4}^{0} 2e^5 x + 5e^5 + 4e^{1-x} - e^{-3-2x} \ \mathrm{d}x$$
$$= \left[\frac{2e^5}{2} x^2 + 5e^5 x + \frac{4}{-1} e^{1-x} - \frac{1}{-2} e^{-3-2x} \right]_{-4}^{0}$$

$$=\left[e^5x^2+5e^5x-4e^{1-x}+\frac{1}{2}e^{-3-2x}\right]_{-4}^{0}$$

$$=\left(0+0-4e^1+\frac{1}{2}e^{-3}\right)-\left(16e^5-20e^5-4e^5+\frac{1}{2}e^{-3+8}\right)$$

$$=\frac{1}{2}e^{-3}+\frac{15}{2}e^5-4e$$

5

(i) $S(24)=50e^{-0.04\times24}=19.1$ (thousand papers)

(ii) The total circulation in the 24 hrs

$$\int_0^{24}50e^{-0.04t}\,dt=\left[\frac{50}{-0.04}e^{-0.041t}\right]_0^{24}=\left[-1250e^{-0.04t}\right]_0^{24}$$

$$=\left(-1250e^{-0.96}\right)-(-1250)=771.38$$

(iii) Assume the number of viewers at t th hour follows a continuous random variable.

(iv) To increase the circulation from 32 thousand papers to 50 thousand papers, the amoubt spent on advertising will be 50-32 = 18 cents.

Hence total revenue = 50 x (0.5) = 25 (thousand dollars)

(iv) Let R be the hourly profit,

$$R=16000+180x-10x^2$$

$$\frac{dR}{dx}=180-20x$$

For max R, $\frac{dR}{dx}=0$

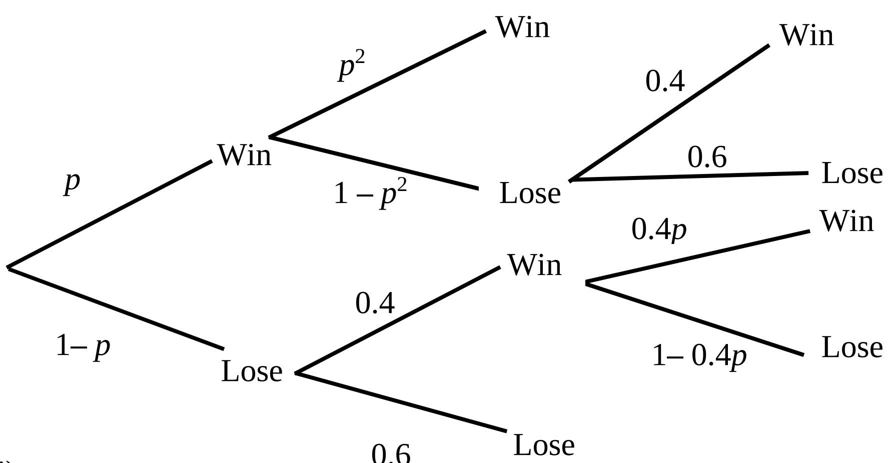
$$x=\frac{180}{20}=9$$

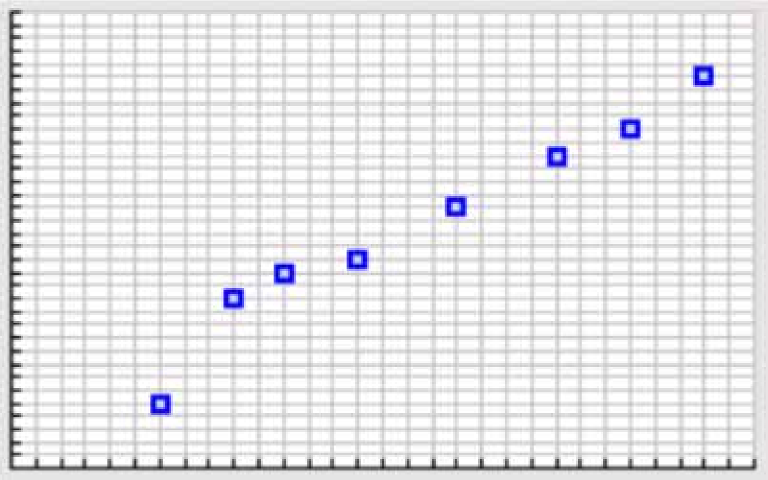
x	9^-	9	9^+
$\frac{dR}{dx}$	+ve	0	-ve
slope	/	-	\

When x = 9, R is maximum

Maximum $R=16000+180(9)-10(9)^2=16810$

= 1.68 (thousand dollars)

6	(i) No of way = 10! = 3628800 P(no 2 basketball players are next to one another) (ii) $= \frac{7!\binom{8}{3}3!}{3628800} = \frac{7}{15} = 0.467$ P(tennis players are together|basketball players are separated) (iii) $= \frac{\frac{4!4!\binom{5}{3}3!}{3628800}}{\frac{7}{15}} = \frac{1}{49} = 0.0204$
7	2 assumptions 1. Whether the student wear spectacles is independent of one another. 2. The probability of selecting a student wearing spectacle is constant. (ii)S~B(15,p) $P(S=0 \text{ or } S=1)=0.3$ $P(S=0)+P(S=1)=0.3$ $(1-p)^{15}+15p(1-p)^{14}=0.3$ $(1-p)^{15}+15p(1-p)^{14}-0.3=0$ Using GC, $p=0.155$ $P(2 \le S < 8)$=0.699 (iii) Let X be no. of the students wearing spectacles out of 30 students. $X \sim B(30, 0.7)$ Since n = 50 is large, by Central Limit theorem, $\overline{X} \sim N(21, \frac{6.3}{50})$ approximately $P(\overline{X} \le 20) = 0.00242$
8(i)	 (ii)

	$P(WW)+P(WLW)+P(LWW)=0.6528$ $p^3+p(1-p^2)(0.4)+(1-p)(0.4)(0.4p)=0.6528$ $0.6p^3-0.16p^2+0.56p-0.6528=0$ Using GC, $p=0.8$ (iii) P(wins second set | wins the game) $=\dfrac{0.64^3+0.36(0.4)(0.4\times0.64)}{0.64^3+0.64(1-0.64^2)(0.4)+0.36(0.4)(0.4\times0.64)}=\dfrac{0.299008}{0.4501504}$ = 0.664
9i	$\bar{x}=16.625$ Since the point $(\bar{x},\bar{y})$ will pass through the regression line *y* on *x*, $\bar{y}=18.619$. $\bar{y}=\dfrac{5+13+15+a+20+24+26+30}{8}$ $a=16$ (ii)  (iii) r = 0.9799 (4 dp) The *r* value is close to 1, shows that most of the points lie close to the best fit line through the curve. This shows that there is strong positive linear correlation between the growth rate of the organisms and the temperature.

(iv) When $y = 18$, sub into $y = 0.993x + 2.11$

$$18 = 0.993x + 2.11$$

$$x = 16.002 = 16.0\,^{0}\text{C}.$$

The estimate is reliable because the growth rate of the organism is within the data range and r is close to 1.

(v) The estimate is unreliable as $x = 30^{o}c$ is out of the data range of x.

10

Unbiased estimate of population mean, $\bar{x} = \frac{-40}{50} + 80 = 79.2$

Unbiased estimate of population variance,

$$s^2 = \frac{1}{49}\left[450 - \frac{(-40)^2}{50}\right] = \frac{418}{49} = 8.53061 \approx 8.53$$

(i) Let X denote the H1 Mathematics score of a student and μ the population mean H1 Mathematics score.

To test $H_0 : \mu = 80$

$H_1 : \mu < 80$

Level of significance: 5%

Test statistic: Under H_0, $\bar{X} \sim N(80, \frac{8.53061}{50})$ approximately, by Central Limit Theorem since n is large

Reject H_0 if p-value ≤ 0.05

$\bar{x} = 79.2$, $n = 50$, $s = \sqrt{\frac{418}{49}}$

From GC, p-value = 0.026385

Since p-value < 0.05, there is sufficient evidence to reject H_0 and conclude at 5% significant level that there is sufficient evidence that the owner has overstated their mean H1 Mathematics score.

(ii) Not the same conclusion

Note $H_1 : \mu \neq 80$ (teacher's claim) and p-value =

$0.026385 \times 2 = 0.05277 > 0.05$

Do not reject H_0, i.e insufficient evidence to support teacher's claim that the mean H1 Mathematics score is not 80

To test $H_0 : \mu = 80$

$H_1 : \mu \neq 80$

at 5% level of significance

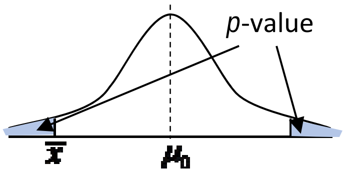
Under H_0, $\bar{X} \sim N(80, \frac{5^2}{n})$ approximately, by Central Limit Theorem since n is large

Reject H_0 if p-value ≤ 0.05

owner's claim is valid => do not reject H_0

$\therefore p$-value > 0.05

$$P\left(\bar{X} \le \bar{x}\right) > \frac{0.05}{2}, \text{ i.e. } P\left(\bar{X} \le \bar{x}\right) > 0.025 \qquad \text{or}$$



$$P\left(Z \le \frac{78.8-80}{5/\sqrt{n}}\right) > 0.025$$

$$\frac{78.8-80}{5/\sqrt{n}} > -1.95996$$

$$-0.24\sqrt{n} > -1.95996$$

$$\sqrt{n} < 8.1665$$

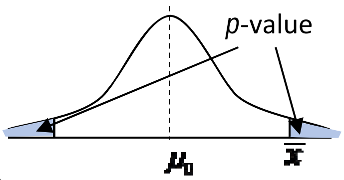
$$n < 66.692$$

$$n \le 66$$

$$P\left(\bar{X} \ge \bar{x}\right) > \frac{0.05}{2}, \text{ i.e. } P\left(\bar{X} \ge \bar{x}\right) > 0.025$$

$$1 - P\left(\bar{X} < \bar{x}\right) > 0.025$$

$$P\left(\bar{X} < \bar{x}\right) < 0.975$$



$$P\left(Z < \frac{78.8-80}{5/\sqrt{n}}\right) < 0.975$$

$$\frac{78.8-80}{5/\sqrt{n}} < 1.95996$$

$$-0.24\sqrt{n} < 1.95996$$

$$\sqrt{n} > -8.1665$$

$$\sqrt{n} > 0$$

The largest n is 66

11i	$P(H < 2k) = 0.95$

	$P(Z < \frac{2k-1.8}{b}) = 0.95$ $\frac{2k-1.8}{b} = 1.6448$.........(1) $P(H < k) = 0.25$ $P(Z < \frac{k-1.8}{b}) = 0.25$ $\frac{k-1.8}{b} = -0.67449$.........(2) Solving equation (1) and (2), $k = 1.39446$ $b = 0.601$ (ii) Let *H* be the mass of a randomly chosen honeydew and *D* be the mass of a randomly chosen durian. $H \sim N(1.8, 0.4^2)$ $D \sim N(1.5, 0.3^2)$ P(mass of the honeydew is more than 1.7 kg and the mass of the durian is less than 1.7 kg) = $P(H > 1.7) \times P(D < 1.7) = 0.448$ (iii) Probability = $P(H < 1.6) \times P(1.6 \le H \le 1.7) \times P(H > 1.7) \times 3! = 0.103$ (iv) $H_1 + H_2 \sim N(3.6, 0.32)$ $3D \sim N(4.5, 0.81)$ $3D - (H_1 + H_2) \sim N(0.9, 1.13)$ $P(-0.8 \le 3D - (H_1 + H_2) \le 0.8) = 0.408$ (v) $C_H = 3(H_1 + H_2) \sim N(10.8, 2.88)$ $C_H + C_D \sim N(25.8, 11.88)$ $P(C_H + C_D > 28.5) = 0.217$