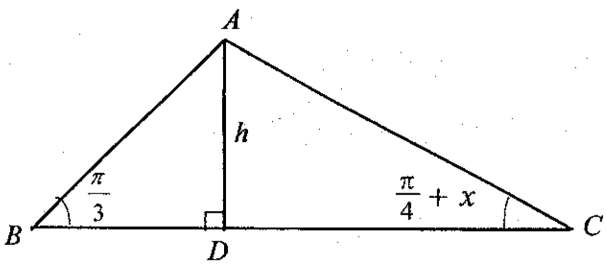
2017 SAJC Prelim Paper 1

Answer all questions [100 marks].

1 The volume of a spherical bubble is increasing at a constant rate of λ cm^3 per second. Assuming that the initial volume of the bubble is negligible, find the exact rate in terms of λ at which the surface area of the bubble is increasing when the volume of the bubble is 20 cm^3. **[5]**

[The volume of a sphere, $V = \dfrac{4}{3}\pi r^3$ and the surface area of a sphere, $A = 4\pi r^2$ where r is the radius of the sphere.]

2 The diagram shows the triangle ABC. It is given that the height AD is h units, $\angle ABD = \dfrac{\pi}{3}$ and $\angle ACD = \dfrac{\pi}{4} + x$.



Show that if x is sufficiently small for x^3 and higher powers of x to be neglected, then

$$BC = \frac{h}{\sqrt{3}} + \frac{h}{\tan\left(\frac{\pi}{4} + x\right)} \approx h\left(p + qx + rx^2\right)$$

for constants p, q, r to be determined in exact form. **[5]**

3 It is given that

$$\mathrm{f}(x) = \begin{cases} b\sqrt{1 - \dfrac{x^2}{a^2}} & \text{for } -a < x \le a \\ -a\sqrt{1 - \dfrac{(x-2a)^2}{a^2}} & \text{for } a < x \le 3a \end{cases}$$

and that $\mathrm{f}(x + 4a) = \mathrm{f}(x)$ for all real values of x, where a and b are real constants and $0 < a < b$.

(i) Sketch the graph of $y = \mathrm{f}(x)$ for $-a \le x \le 8a$. **[3]**

(ii) Use the substitution $x = a\cos\theta$ to find the exact value of $\int_{3a}^{4a} \mathrm{f}(x)\,\mathrm{d}x$ in terms of a and π. **[5]**

(i) State a sequence of transformations that would transform the curve with equation $y = \mathrm{e}^{x^2}$ onto the curve with equation $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \mathrm{e}^{ax^2} - b$, $a > 0$ and $b > 1$. **[2]**

(ii) Sketch the curve $y = \mathrm{f}(x)$ and the curve $y = \dfrac{1}{\mathrm{f}(x)}$.

	You should state clearly the equations of any asymptotes, coordinates of turning points and axial intercepts. **[5]**
	It is given that $\mathbf{u}+\mathbf{v}-\mathbf{w}$ is perpendicular to $\mathbf{u}-\mathbf{v}+\mathbf{w}$, where $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are unit vectors. **(i)** Show that the angle between $\mathbf{v}$ and $\mathbf{w}$ is 60°. **[4]** Referred to the origin O, the points U, V and W have position vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ respectively. **(ii)** Find the exact area of triangle OVW. **[2]** **(iii)** Given that $\mathbf{u}$ and $\mathbf{v}\times\mathbf{w}$ are parallel, find the exact volume of the solid $OUVW$. **[2]** [The volume of a pyramid is $\frac{1}{3}bh$, where b is the base area and h is the height of the pyramid.]
6	**(a)** **(i)** Find $\int e^x \cos nx \, dx$, where n is a positive integer. **[4]** **(ii)** Hence, without the use of a calculator, find $\int_{\pi}^{2\pi} e^x \cos nx \, dx$ when n is odd. **[3]** **(b)** The region bounded by the curve $y=\frac{\sqrt{x}}{16-x^2}$, the y-axis and the line $y=\frac{\sqrt{2}}{12}$ is rotated 2π radians about the x-axis. Find the exact volume of the solid obtained. **[5]**
7	**(i)** Show that for any complex number $z=re^{i\theta}$, where $r>0$, and $-\pi<\theta\le\pi$, $$\frac{z}{z-r}=\frac{1}{2}-\frac{1}{2}\left(\cot\frac{\theta}{2}\right)i.$$ **[3]** **(ii)** Given that $z=2e^{i\left(\frac{\pi}{3}\right)}$ is a root of the equation $z^2-2z+4=0$. State, in similar form, the other root of the equation. **[1]** **(iii)** Using parts **(i)** and **(ii)**, solve the equation $\frac{4w^2}{(w-1)^2}-\frac{4w}{w-1}+4=0$. **[4]**
8	In a training session, an athlete runs from a starting point S towards his coach in a straight line as shown in the diagrams below. When he reaches the coach, he runs back to S along the same straight line. A lap is completed when he returns to S. At the beginning of the training session, the coach stands at A_1 which is 30 m away from S.

After the first lap, the coach moves from A_1 to A_2 and after the second lap, he moves from A_2 to A_3 and so on. The distance between A_i to A_{i+1} is denoted by A_iA_{i+1}, $i \in \mathbb{Z}^+$.

Figure 1

(i) For training regime 1 (shown in Figure 1), the coach ensures that the distance $A_iA_{i+1} = 3$ m for $i \in \mathbb{Z}^+$. Find the least number of laps that the athlete must complete so that he covers a total distance of 3000 m. **[3]**

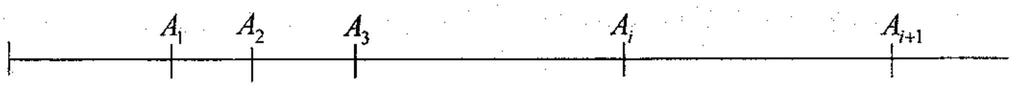


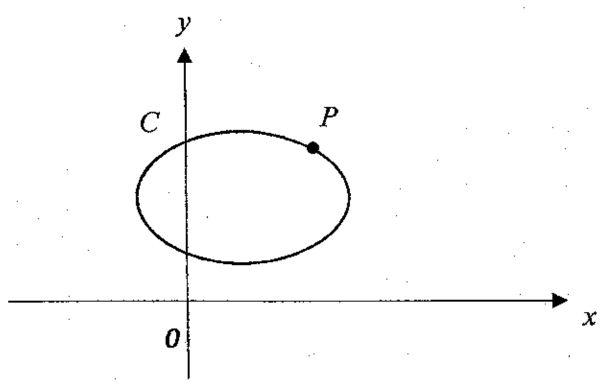
Figure 2

(ii) For training regime 2 (shown in Figure 2), after the first lap, the coach ensures that the distances $A_1A_2 = 2$ m, $A_2A_3 = 6$ m and the distance $A_{i+1}A_{i+2} = 3A_iA_{i+1}$ where $i \in \mathbb{Z}^+$. Show that the distance the coach is away from S **just before** the athlete completed r laps is $\left(3^{r-1} + 29\right)$m.

Hence find the distance run by the athlete after n complete laps. Also find how far the athlete is from the coach after he has run 8 km. **[6]**

9 The diagram below shows the curve C with parametric equations

$$x = 1 + 2\sin\theta, \quad y = 4 + \sqrt{3}\cos\theta, \quad \text{for } -\pi < \theta \le \pi.$$



The point P is where $\theta = \dfrac{\pi}{6}$.

(i) Using a non-calculator method, find the equation of the normal at P. **[4]**

(ii) The normal at the point P cuts C again at point Q, where $\theta=\alpha$. Show that $8\sin\alpha-2\sqrt{3}\cos\alpha=1$ and hence deduce the coordinates of Q. **[3]**

(iii) Find the area of the region bounded by the curve C, the normal at point P and the vertical line passing through the point Q. **[4]**

10 A population of 15 foxes has been introduced into a national park. A zoologist believes that the population of foxes, x, at time t years, can be modelled by the Gompertz equation given by:

$$\frac{\mathrm{d}x}{\mathrm{d}t}=cx\ln\left(\frac{40}{x}\right)$$

where c is a constant.

(i) Using the substitution $u=\ln\left(\frac{40}{x}\right)$, show that the differential equation can be written as $\frac{\mathrm{d}u}{\mathrm{d}t}=-cu$. **[2]**

(ii) Hence find u in terms of t and show that $x=40\mathrm{e}^{-B\mathrm{e}^{-ct}}$, where B is a constant. **[5]**

After 3 years, the population of foxes is estimated to be 20.

(iii) Find the values of B and c. **[3]**

(iv) Find the population of foxes in the long run. **[1]**

(v) Hence, sketch the graph showing the population of foxes over time. **[2]**

11 A computer-controlled machine can be programmed to make plane cuts by keying in the equation of the plane of the cut, and drill holes in a straight line through an object by keying in the equation of the drill line.

A 10cm × 20 cm ×30 cm cuboid is to be cut and drilled. The cuboid is positioned relative to the x-, y-and z-axes as shown in Figure 1.

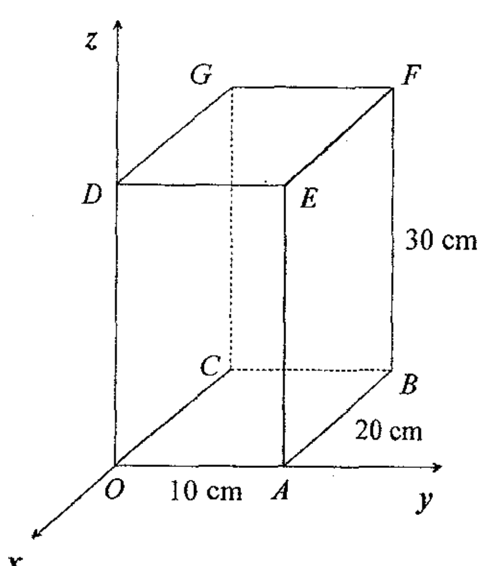


Figure 1

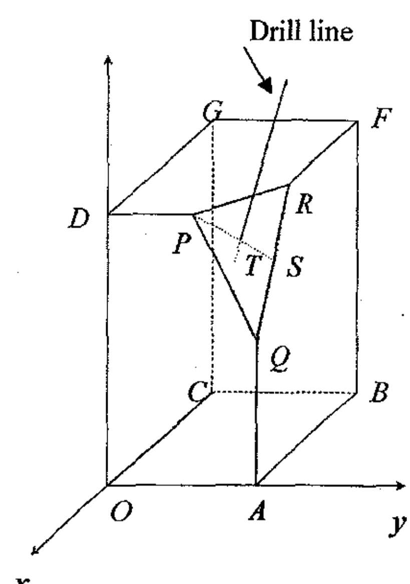


Figure 2

	First, a plane cut is made to remove the corner at E. The cut goes through the points P, Q and R which are the midpoints of the sides ED, EA and EF respectively. **(i)** Show that $\overrightarrow{PQ}=\begin{pmatrix}0\\5\\-15\end{pmatrix}$ and $\overrightarrow{PR}=\begin{pmatrix}-10\\5\\0\end{pmatrix}$. **[2]** **(ii)** Find the cartesian equation of the plane, p that contains P, Q and R. **[2]** **(iii)** Find the acute angle between p and the plane $DEFG$. **[2]** A hole is then drilled perpendicular to triangle PQR, as shown in Figure 2. The hole passes through the triangle at the point T which divides the line PS in the ratio of 4:1, where S is the midpoint of QR. **(iv)** Show that the point T has coordinates $(-4,\ 9,\ 24)$. **[3]** **(v)** State the vector equation of the drill line. **[1]** **(vi)** If the computer program continues drilling through the cuboid along the same line as in part **(v)**, determine the side of the cuboid that the drill exits from. Justify your answer. **[4]**

– End Of Paper –

ANNEX B

SAJC H2 Math JC2 Preliminary Examination Paper 1

QN	Topic Set	Answers
1	Differentiation & Applications	$2\lambda\left(\frac{\pi}{15}\right)^{\frac{1}{3}}\ \text{cm}^2/\text{s}$
2	Maclaurin series	-
3	Functions	$\frac{\pi}{4}ab$
4	Graphs and Transformation	i) 1. Scale by a factor of $\frac{1}{\sqrt{a}}$ parallel to the x-axis, 2. Translate the resulting curve by b units in the negative y-direction.
5	Vectors	ii) $\frac{\sqrt{3}}{4}$ units^2 iii) $\frac{\sqrt{3}}{12}$ units^3
6	Application of Integration	a) i) $\left(\frac{n^2}{1+n^2}\right)\left[e^x\left(\frac{\sin nx}{n}\right)+\frac{1}{n}\left(\frac{e^x\cos nx}{n}\right)\right]+c$ ii) $\left(\frac{1}{1+n^2}\right)\left(e^{2\pi}+e^{\pi}\right)$ b) $\frac{5\pi}{288}$ units^3
7	Complex numbers	ii) $z=2e^{i\left(-\frac{\pi}{3}\right)}$ iii) $w=\frac{1}{2}-\frac{\sqrt{3}}{2}i$ or $w=\frac{1}{2}+\frac{\sqrt{3}}{2}i$
8	AP and GP	i) $n=24$ ii) $\left(3^n-1\right)+58n$, 5614 m away from the coach once he finishes 8 km.
9	Application of Integration	i) $y=2x+\frac{3}{2}$ ii) Q (0.421, 2.34) iii) 2.77 units^2

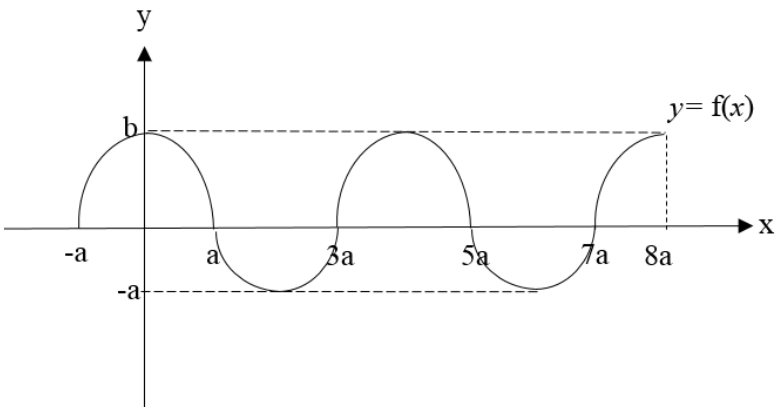
10	Differential Equations	iii) $x = 40e^{-0.981e^{-0.116t}}$ iv) 40
11	Vectors	ii) $3x + 6y + 2z = 90$ iii) $\theta = 73.4^0$ iv) $\mathbf{r} = \begin{pmatrix} -4 \\ 9 \\ 24 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix},\ \lambda \in \mathbb{R}$. v) The drill line will not exit from the side *GCBF*.

H2 Mathematics 2017 Prelim Exam Paper 1 Solution

1	$V=\frac{4}{3}\pi r^3$ $\frac{\mathrm{d}V}{\mathrm{d}t}=4\pi r^2\frac{\mathrm{d}r}{\mathrm{d}t}$ When V=20, $20=\frac{4}{3}\pi r^3$ $r=\left(\frac{15}{\pi}\right)^{\frac{1}{3}}$ When $r=\left(\frac{15}{\pi}\right)^{\frac{1}{3}}$, $\frac{\mathrm{d}V}{\mathrm{d}t}=\lambda$. $\lambda=4\pi\left(\frac{15}{\pi}\right)^{\frac{2}{3}}\frac{\mathrm{d}r}{\mathrm{d}t}$ $\frac{\mathrm{d}r}{\mathrm{d}t}=\frac{\lambda}{4\pi}\left(\frac{\pi}{15}\right)^{\frac{2}{3}}$ Surface Area, $A=4\pi r^2$ $\frac{\mathrm{d}A}{\mathrm{d}t}=8\pi r\frac{\mathrm{d}r}{\mathrm{d}t}$ When $r=\left(\frac{15}{\pi}\right)^{\frac{1}{3}}$, $\frac{\mathrm{d}r}{\mathrm{d}t}=\frac{\lambda}{4\pi}\left(\frac{\pi}{15}\right)^{\frac{2}{3}}$. $\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{\mathrm{d}A}{\mathrm{d}r}\times\frac{\mathrm{d}r}{\mathrm{d}t}$ $=8\pi\left(\frac{15}{\pi}\right)^{\frac{1}{3}}\frac{\lambda}{4\pi}\left(\frac{\pi}{15}\right)^{\frac{2}{3}}$ $=2\lambda\left(\frac{\pi}{15}\right)^{\frac{1}{3}}\ \mathrm{cm}^2/\mathrm{s}$

2	(i) $BC = BD + DC$ $= \dfrac{h}{\tan\frac{\pi}{3}} + \dfrac{h}{\tan\left(\frac{\pi}{4}+x\right)}$ (ii) BC $= \dfrac{h}{\sqrt{3}} + \dfrac{h}{\dfrac{\tan\frac{\pi}{4}+\tan x}{1-\tan\frac{\pi}{4}\tan x}}$ $= \dfrac{h\sqrt{3}}{3} + \dfrac{h(1-\tan x)}{1+\tan x}$ $\approx \dfrac{h\sqrt{3}}{3} + \dfrac{h(1-x)}{1+x}$ $= \dfrac{h\sqrt{3}}{3} + h(1-x)(1+x)^{-1}$ $= \dfrac{h\sqrt{3}}{3} + h(1-x)[1+(-1)x+\dfrac{(-1)(-2)}{2!})x^2+...]$ $= \dfrac{h\sqrt{3}}{3} + h(1-x)[1-x+x^2+...]$ $= \dfrac{h\sqrt{3}}{3} + h(1-2x+2x^2+...)$ $= h\left(1+\dfrac{\sqrt{3}}{3} - 2x + 2x^2\right)$

3	(i)



(ii)

$$\int_{3a}^{4a} \mathrm{f}(x)\,\mathrm{d}x$$

$$= \int_{-a}^{0} b\sqrt{1-\frac{x^2}{a^2}}\,\mathrm{d}x$$

$$= b\int_{\pi}^{\frac{\pi}{2}} \sqrt{1-\frac{a^2\cos^2\theta}{a^2}}\,(-a\sin\theta)\,\mathrm{d}\theta$$

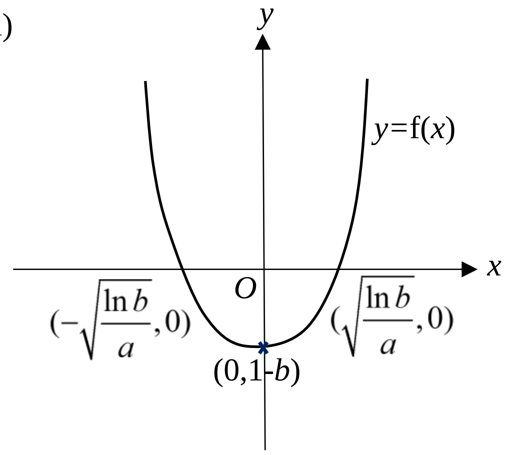
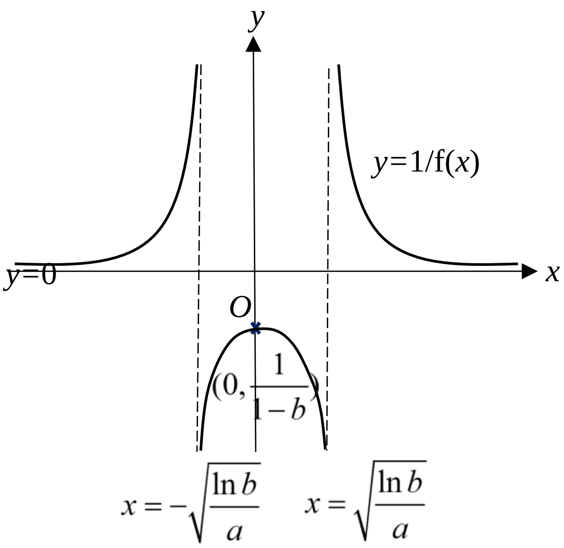
$$= ab\int_{\frac{\pi}{2}}^{\pi} \sin^2\theta\,\mathrm{d}\theta$$

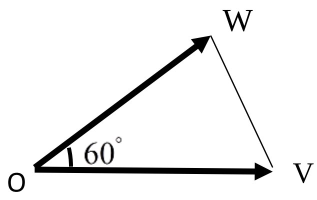
$$= ab\int_{\frac{\pi}{2}}^{\pi} \frac{1-\cos 2\theta}{2}\,\mathrm{d}\theta$$

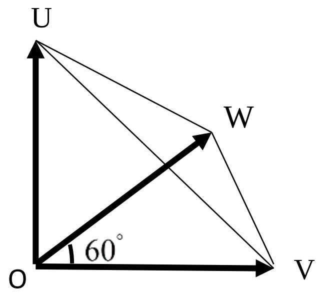
$$= \frac{ab}{2}\left[\theta-\frac{\sin 2\theta}{2}\right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{ab}{2}\left[\pi-\frac{\pi}{2}\right]$$

$$= \frac{\pi}{4}ab$$

4	(i) $y = e^{ax^2} - b = e^{(\sqrt{a}x)^2} - b$ If $f(x) = e^{x^2}$, then $f(\sqrt{a}x) = e^{(\sqrt{a}x)^2}$ and so $y = f(x) \to y = f(\sqrt{a}x) \to y = f(\sqrt{a}x) + b$ Hence the sequence of transformations are: 1. Scale by a factor of $\frac{1}{\sqrt{a}}$ parallel to the x-axis, 2. Translate the resulting curve by b units in the negative y-direction. (ii)  

5	**(i)** Since $\mathbf{u}+\mathbf{v}-\mathbf{w}$ is perpendicular to $\mathbf{u}-\mathbf{v}+\mathbf{w}$, $(\mathbf{u}+\mathbf{v}-\mathbf{w})\cdot(\mathbf{u}-\mathbf{v}+\mathbf{w})=0$ $\mathbf{u}\cdot\mathbf{u}-\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$ $+\mathbf{v}\cdot\mathbf{u}-\mathbf{v}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{w}$ $-\mathbf{w}\cdot\mathbf{u}+\mathbf{w}\cdot\mathbf{v}-\mathbf{w}\cdot\mathbf{w}=0$ Since $\mathbf{u}\cdot\mathbf{u}=\lvert\mathbf{u}\rvert^2, \mathbf{v}\cdot\mathbf{v}=\lvert\mathbf{v}\rvert^2, \mathbf{w}\cdot\mathbf{w}=\lvert\mathbf{w}\rvert^2$, and $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}, \mathbf{u}\cdot\mathbf{w}=\mathbf{w}\cdot\mathbf{u}, \mathbf{v}\cdot\mathbf{w}=\mathbf{w}\cdot\mathbf{v}$, $\lvert\mathbf{u}\rvert^2-\lvert\mathbf{v}\rvert^2-\lvert\mathbf{w}\rvert^2+2\mathbf{v}\cdot\mathbf{w}=0$ Since $\mathbf{u},\mathbf{v},\mathbf{w}$ are unit vectors, $\lvert\mathbf{u}\rvert=1, \lvert\mathbf{v}\rvert=1, \lvert\mathbf{w}\rvert=1$, $1-1-1+2\mathbf{v}\cdot\mathbf{w}=0$ $\mathbf{v}\cdot\mathbf{w}=\frac{1}{2}$ $\lvert\mathbf{v}\rvert\lvert\mathbf{w}\rvert\cos\theta=\frac{1}{2}$ $\cos\theta=\frac{1}{2}$ Hence, $\theta=60^\circ$ **(ii)**  Area of $\triangle$OVW $=\left(\frac{1}{2}(OV)(OW)\sin 60^\circ\right)$ $=\left(\frac{1}{2}\right)(1)(1)\left(\frac{\sqrt{3}}{2}\right)$ $=\frac{\sqrt{3}}{4}$ units^2 **(iii)**



	Since **u** and $\mathbf{v}\times\mathbf{w}$ are parallel, we have $OU \perp OV, OU \perp OW$. Volume of $OUVW$ $=\frac{1}{3}(\text{Area of } \triangle \text{OVW})(OU)$ $=\frac{1}{3}\left(\frac{\sqrt{3}}{4}\right)(1)$ $=\frac{\sqrt{3}}{12}$ units^3

6	**(a)**

(i) Using integration by parts,

$$\int e^x \cos nx \, dx$$

$u = e^x$	$\frac{dv}{dx} = \cos nx$
$\frac{du}{dx} = e^x$	$v = \frac{\sin nx}{n}$

$$= e^x\left(\frac{\sin nx}{n}\right) - \int \frac{e^x}{n}\sin nx \, dx$$

$u = e^x$	$\frac{dv}{dx} = \sin nx$
$\frac{du}{dx} = e^x$	$v = -\frac{\cos nx}{n}$

$$= e^x\left(\frac{\sin nx}{n}\right) - \frac{1}{n}\left[-\frac{e^x \cos nx}{n} + \int \frac{e^x \cos nx}{n} dx\right]$$

$$= e^x\left(\frac{\sin nx}{n}\right) + \frac{1}{n}\left(\frac{e^x \cos nx}{n}\right) - \frac{1}{n^2}\int e^x \cos nx \, dx$$

Rearranging,

$$\left(1+\frac{1}{n^2}\right)\int e^x \cos nx \, dx = e^x\left(\frac{\sin nx}{n}\right) + \frac{1}{n}\left(\frac{e^x \cos nx}{n}\right)$$

$$\int e^x \cos nx \, dx = \left(\frac{n^2}{1+n^2}\right)\left[e^x\left(\frac{\sin nx}{n}\right) + \frac{1}{n}\left(\frac{e^x \cos nx}{n}\right)\right] + c$$ where c is a constant

(ii) $$\int_{\pi}^{2\pi} e^x \cos nx \, dx = \left(\frac{n^2}{1+n^2}\right)\left[e^x\left(\frac{\sin nx}{n}\right) + \frac{1}{n}\left(\frac{e^x \cos nx}{n}\right)\right]_{\pi}^{2\pi}$$

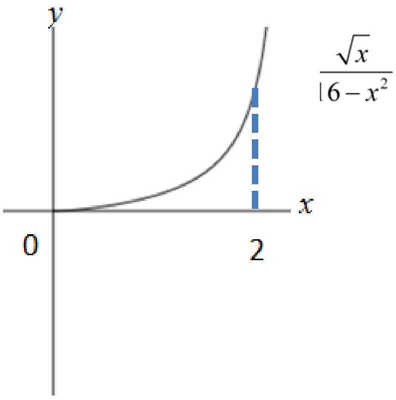
$$= \left(\frac{n^2}{1+n^2}\right)\left\{e^{2\pi}\left[\left(\frac{\sin 2n\pi}{n}\right) + \frac{1}{n}\left(\frac{\cos 2n\pi}{n}\right)\right] - e^{\pi}\left[\left(\frac{\sin n\pi}{n}\right) + \frac{1}{n}\left(\frac{\cos n\pi}{n}\right)\right]\right\}$$

For any positive integer n, $\sin 2n\pi = 0$ and $\cos 2n\pi = 1$

If n is odd, $\sin n\pi = 0$ and $\cos n\pi = -1$

$$\int_{\pi}^{2\pi} e^x \cos nx \, dx = \left(\frac{n^2}{1+n^2}\right)\left[e^{2\pi}\left(0+\frac{1}{n^2}\right) - e^{\pi}\left(0-\frac{1}{n^2}\right)\right] = \left(\frac{1}{1+n^2}\right)\left(e^{2\pi}+e^{\pi}\right)$$ (Ans)

(b)



$$y=\frac{\sqrt{x}}{16-x^2} \Rightarrow y^2=\frac{x}{\left(16-x^2\right)^2}$$

Hence volume required

$$=\pi r^2 h-\pi\int_0^2 y^2 \mathrm{d}x$$

$$=\pi\left(\frac{\sqrt{2}}{12}\right)^2(2)-\pi\int_0^2\frac{x}{\left(16-x^2\right)^2}\mathrm{d}x$$

$$=\pi\left(\frac{\sqrt{2}}{12}\right)^2(2)-\frac{\pi}{(-2)}\int_0^2\frac{-2x}{\left(16-x^2\right)^2}\mathrm{d}x$$

$$=\pi\left(\frac{\sqrt{2}}{12}\right)^2(2)+\frac{\pi}{2}\left[\frac{\left(16-x^2\right)^{-1}}{-1}\right]_0^2$$

$$=\frac{4}{144}\pi+\frac{\pi}{2}\left[-\frac{1}{12}+\frac{1}{16}\right]$$

$$=\frac{5\pi}{288}\text{ units}^3$$

7	(i) $\frac{re^{i\theta}}{re^{i\theta}-r}$ $=\frac{e^{i\theta}}{e^{i\left(\frac{\theta}{2}\right)}\left(e^{i\left(\frac{\theta}{2}\right)}-e^{-i\left(\frac{\theta}{2}\right)}\right)}$ $=\frac{e^{i\left(\frac{\theta}{2}\right)}}{2i\sin\left(\frac{\theta}{2}\right)}$ $=\frac{\cos\left(\frac{\theta}{2}\right)+i\sin\left(\frac{\theta}{2}\right)}{2i\sin\left(\frac{\theta}{2}\right)}$ $=\frac{1}{2}+\frac{1}{2i}\cot\left(\frac{\theta}{2}\right)$ $=\frac{1}{2}-\frac{1}{2}\left(\cot\left(\frac{\theta}{2}\right)\right)i$ (ii) $z=2e^{i\left(-\frac{\pi}{3}\right)}$ (iii) $\frac{4w^2}{(w-1)^2}-\frac{4w}{w-1}+4=0$ $\left(\frac{2w}{w-1}\right)^2-2\left(\frac{2w}{w-1}\right)+4=0$ Let $z=\frac{2w}{w-1}$, then $z^2-2z+4=0$ From (ii) the solutions are $z=2e^{i\left(\frac{\pi}{3}\right)}$ or $z=2e^{-i\left(\frac{\pi}{3}\right)}$ Since $z=\frac{2w}{w-1}$ $zw-z=2w$ $w(z-2)=z$ $w=\frac{z}{z-2}$ Part (i) result can be used as $z=2e^{i\left(\frac{\pi}{3}\right)}$, where $r=2$ with $\theta=\frac{\pi}{3},\ \theta=-\frac{\pi}{3}$. $w=\frac{1}{2}-\frac{1}{2}\left(\cot\frac{\pi}{6}\right)i$ or $w=\frac{1}{2}-\frac{1}{2}i\cot\left(-\frac{\pi}{6}\right)$

	$w=\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}$ or $w=\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}$

8

(i)

Distance travelled per lap is in AP:

$a = 2(30) = 60$, $d = 2 \times 3 = 6$.

Given total distance travelled > 3000

$$\frac{n}{2}\left[\,2(60) + (n-1)6\right] > 3000$$

$$3n^2 + 57n - 3000 > 0$$

$$(n + 42.52)(n - 23.52) > 0$$

$$n < -42.52 \text{ or } n > 23.52$$

Since $n \in \mathbb{Z}^+$, least $n = 24$

(ii)

Distance of the coach from S just before the runner completes the rth lap

$$=30 + 2(3^0) + 2(3^1) + 2(3^2) + ... + 2(3^{r-2})$$

$$= 30 + 2\left(1 + 3 + 3^2 + ... + 3^{r-2}\right)$$

$$= 30 + 2\left(\frac{3^{r-1}-1}{3-1}\right)$$

$$= 30 + (3^{r-1} - 1)$$

$$= 3^{r-1} + 29$$

Distance covered by the athlete after n laps

$$= \sum_{r=1}^{n} 2\left(3^{r-1} + 29\right)$$

$$= 2\sum_{r=1}^{n} 3^{r-1} + \sum_{r=1}^{n}(58)$$

$$= 2\sum_{r=1}^{n} 3^{r-1} + 58n$$

$$= 2\left(\frac{3^n - 1}{3-1}\right) + 58n$$

$$= \left(3^n - 1\right) + 58n$$

When $D = 8000$m

$$8000 = \left(3^n - 1\right) + 58n$$

From GC,

$n = 8.1254$

Hence the athlete has run 8 complete laps.

The athlete has completed 7024 m

Hence he still have 8000-7024=976 m

On the 9th lap, the coach is $3^{9-1} + 29 = 6590$ m from S.

Hence the athlete would be 6590- 976 = 5614 m away from the coach once he finishes 8 km.

9

(i)

$$\frac{dx}{d\theta} = 2\cos\theta \ , \quad \frac{dy}{d\theta} = -\sqrt{3}\sin\theta$$

$$\frac{dy}{dx} = \frac{-\sqrt{3}\sin\theta}{2\cos\theta} = -\frac{\sqrt{3}}{2}\tan\theta$$

When $\theta = \frac{\pi}{6}$, $x = 2$, $y = \frac{11}{2}$, $\frac{dy}{dx} = -\frac{1}{2}$

Equation of normal : $y - \left(\frac{11}{2}\right) = 2(x-2)$

$$y = 2x + \frac{3}{2}$$

(ii)

$x = 1 + 2\sin\theta \cdots\cdots (1)$

$y = 4 + \sqrt{3}\cos\theta \cdots\cdots (2)$

Substitute equation (1) and (2) into $y = 2x + \frac{3}{2}$

$$4 + \sqrt{3}\cos\theta = 2(1 + 2\sin\theta) + \frac{3}{2}$$

$$\frac{1}{2} + \sqrt{3}\cos\theta = 4\sin\theta$$

$$8\sin\theta - 2\sqrt{3}\cos\theta = 1$$

At Point Q, $\theta = \alpha$

$8\sin\alpha - 2\sqrt{3}\cos\alpha = 1$ (shown)

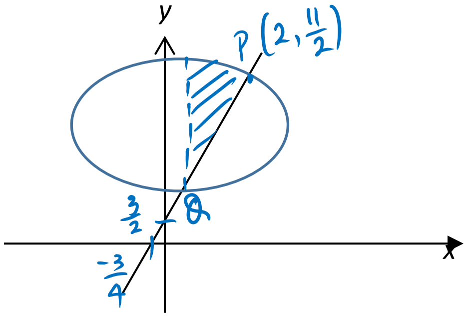
Using GC: $\alpha = -2.847916$ or $\alpha = 0.52359$ (Reject, same as $\frac{\pi}{6}$,point P)

Hence, using GC

coordinates of Q (0.42105, 2.3421)

Q (0.421, 2.34)

(iii)



when $x = 0.42105$

$0.42105 = 1 + 2\sin\theta$

$\sin\theta = -0.289475$

$\theta = -0.29368$ or -2.8479 (at point Q)

$$= \int_{0.42105}^{2} y_1 \, \mathrm{d}x - \int_{0.42105}^{2} y_2 \, \mathrm{d}x$$

Required Area $= \int_{-0.29368}^{\frac{\pi}{6}} \left(4 + \sqrt{3}\cos\theta\right) \left(2\cos\theta\right) \mathrm{d}\theta - \int_{0.42105}^{2} \left(2x + \frac{3}{2}\right) \mathrm{d}x$

$$= 8.9613 - 6.1911$$

$$= 2.7702 \approx 2.77 \text{ units}^2 \quad \text{(3 s.f.)}$$

10	(i) $\frac{dx}{dt} = cx\ln\left(\frac{40}{x}\right)$ $u = \ln\left(\frac{40}{x}\right)$ $= \ln(40) - \ln(x)$ $\frac{du}{dx} = -\frac{1}{x}$ $\frac{du}{dt} = \frac{du}{dx}\times\frac{dx}{dt}$ $= \left(-\frac{1}{x}\right)cx\ln\left(\frac{40}{x}\right)$ $= -cu$ (ii) $\frac{du}{dt} = -cu$ $\int \frac{1}{u}\,du = -\int c\ dt$ $\ln\lvert u\rvert = -ct + d$ $\lvert u\rvert = e^{-ct+d}$ $u = \pm e^{d}e^{-ct}$ $= Be^{-ct},\ B = \pm e^{d}$ Replace u with $\ln\left(\frac{40}{x}\right)$ $\ln\left(\frac{40}{x}\right) = Be^{-ct}$ $\frac{40}{x} = e^{Be^{-ct}}$ $x = \frac{40}{e^{Be^{-ct}}}$ $x = 40e^{-Be^{-ct}}$ (iii) When $t = 0,\ \ x = 15,$ $15 = 40\,e^{-B}$ $e^{-B} = \frac{3}{8}$ $B = \ln\left(\frac{8}{3}\right) = 0.98083 = 0.981$ When $t = 3,\ x = 20$

$$20 = 40e^{-Be^{-3c}}$$

$$e^{-Be^{-3c}} = \frac{1}{2}$$

$$-Be^{-3c} = \ln\frac{1}{2}$$

$$\ln\left(\frac{3}{8}\right)\left(e^{-3c}\right) = \ln\left(\frac{1}{2}\right)$$

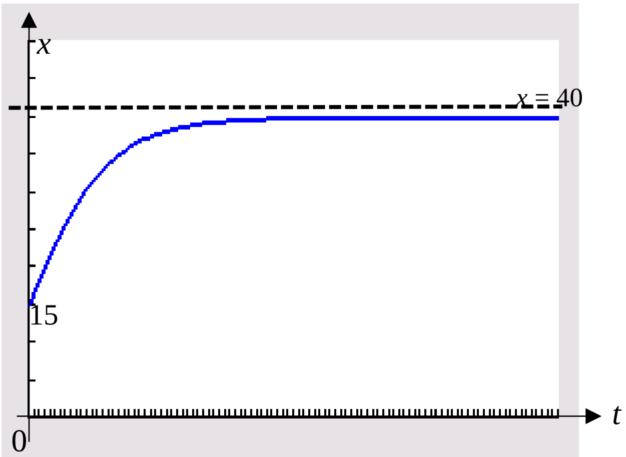
$$c = -\frac{1}{3}\ln\left(\frac{\ln\left(\frac{1}{2}\right)}{\ln\left(\frac{3}{8}\right)}\right) = 0.11572 = 0.116$$

$$x = 40e^{-0.981e^{-0.116t}}$$

(iv)

The population of foxes in the long run is 40.

(v)



11	(i) $\overrightarrow{OP}=\begin{pmatrix}0\\5\\30\end{pmatrix}$ $\overrightarrow{OQ}=\begin{pmatrix}0\\10\\15\end{pmatrix}$ $\overrightarrow{OR}=\begin{pmatrix}-10\\10\\30\end{pmatrix}$ $\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}=\begin{pmatrix}0\\10\\15\end{pmatrix}-\begin{pmatrix}0\\5\\30\end{pmatrix}=\begin{pmatrix}0\\5\\-15\end{pmatrix}$ $\overrightarrow{PR}=\overrightarrow{OR}-\overrightarrow{OP}=\begin{pmatrix}-10\\10\\30\end{pmatrix}-\begin{pmatrix}0\\5\\30\end{pmatrix}=\begin{pmatrix}-10\\5\\0\end{pmatrix}$ (ii) A normal to p $\begin{pmatrix}0\\1\\-3\end{pmatrix}\times\begin{pmatrix}-2\\1\\0\end{pmatrix}=\begin{pmatrix}3\\6\\2\end{pmatrix}$ Equation of plane $\mathbf{r}\cdot\begin{pmatrix}3\\6\\2\end{pmatrix}=\begin{pmatrix}0\\10\\15\end{pmatrix}\cdot\begin{pmatrix}3\\6\\2\end{pmatrix}$ $\mathbf{r}\cdot\begin{pmatrix}3\\6\\2\end{pmatrix}=90$ $3x+6y+2z=90$ Or any equivalent equation of plane (iii) A normal to the plane *EFGH* = $\begin{pmatrix}0\\0\\1\end{pmatrix}$ (or any equivalent vector) $\cos\theta=\dfrac{\left\|\begin{pmatrix}0\\0\\1\end{pmatrix}\bullet\begin{pmatrix}3\\6\\2\end{pmatrix}\right\|}{1\times\sqrt{9+36+4}}=\dfrac{\|2\|}{\sqrt{49}}$ $\theta=73.4^0$ (iv) $\overrightarrow{OS}=\frac{1}{2}\left[\overrightarrow{OQ}+\overrightarrow{OR}\right]=\frac{1}{2}\left[\begin{pmatrix}0\\10\\15\end{pmatrix}+\begin{pmatrix}-10\\10\\30\end{pmatrix}\right]=\begin{pmatrix}-5\\10\\22\frac{1}{2}\end{pmatrix}$

H2 Mathematics 2017 Prelim Exam Paper 1 Question
Answer all questions [100 marks].

1	Without the use of a calculator, find the complex numbers z and w which satisfy the simultaneous equations $$z - w\text{i} = 3$$ $$z^2 - w + 6 + 3\text{i} = 0$$ **[6]**
2	The function f is defined by $\text{f} : x \mapsto \dfrac{1}{x^2 - 1}, \quad x \in \mathbb{R}, \quad x > 1$. **(i)** Show that $\dfrac{2}{n-1} - \dfrac{3}{n} + \dfrac{1}{n+1} = \dfrac{An+B}{n^3 - n}$, where A and B are constants to be found. [3] **(ii)** Hence find $\displaystyle\sum_{r=2}^{n} \frac{2r+6}{r^3 - r}$. [4] **(iii)** Use your answer to part **(ii)** to find $\displaystyle\sum_{r=2}^{n} \frac{2r+10}{(r+1)(r+2)(r+3)}$. [1]
3	The function f is defined by $\text{f} : x \mapsto \dfrac{1}{x^2 - 1}, \quad x \in \mathbb{R}, \quad x > 1$. **(i)** Find $\text{f}^{-1}(x)$ and write down the domain of f^{-1}. [3] **(ii)** On the same diagram, sketch the graphs of $y = \text{f}(x)$, $y = \text{f}^{-1}(x)$ and $y = \text{f}^{-1}\text{f}(x)$ stating the equations of any asymptotes and showing the relationships between the graphs clearly. [4] **(iii)** State the set of values of x such that $\text{ff}^{-1}(x) = \text{f}^{-1}\text{f}(x)$. [1]
4	Referred to the origin O, the point A has position vector $-5\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and the point B has position vector $\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$. The plane π has equation: $$\mathbf{r} = (1 + \lambda - 2\mu)\mathbf{i} + (3 - 2\lambda)\mathbf{j} + (\mu - 2)\mathbf{k} \text{ where } \lambda, \mu \in \mathbb{R}$$ **(i)** Find the vector equation of plane π in scalar product form. [2] **(ii)** Find the position vector of the foot of perpendicular, C, from A to π. [3] The line l_1 passes through the points A and B. The line l_2 is the reflection of the line l_1 about the plane π. Find a vector equation of l_2. [3]

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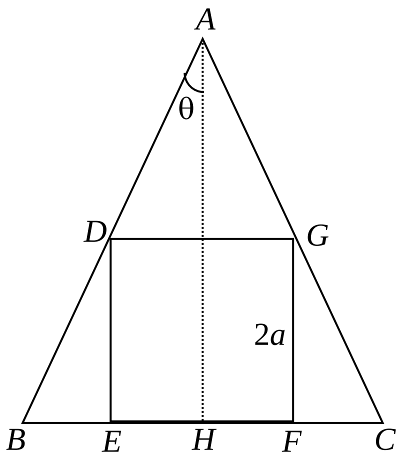

It is given that *DEFG* is a square with fixed side $2a$ cm and it is inscribed in the isosceles triangle *ABC* with height *AH* , where $AB = AC$ and angle $BAH = \theta$.

(i) Taking $t = \tan\theta$, show that the area of the triangle *ABC* is given by $S = a^2\left(4 + 4t + \frac{1}{t}\right)$ [3]

(ii) Find the minimum area of *S* in terms of *a* when *t* varies. [4]

(iii) Hence sketch the graph showing the area of the triangle *ABC* as θ varies. [3]

6

(a) There are three yellow balls, three red balls and three blue balls. Balls of each colour are numbered 1, 2, and 3. Find the number of ways of arranging the balls in a row such that adjacent balls do not sum up to two. **[2]**

(b) In a restaurant, there were two round tables available, a table for five and a table for six. Find the number of ways eleven friends can be seated if two particular friends are not seated next to each other. **[4]**

7

For the events *A* and *B*, it is given that

$$\mathrm{P}(A \cap B') = 0.6, \quad \mathrm{P}(A \cup B') = 0.83 \quad \text{and} \quad \mathrm{P}(A \mid B') = 0.83$$

Find,

(i) $\mathrm{P}(B)$ [2]

(ii) $\mathrm{P}(A \cap B)$ [2]

(iii) $\mathrm{P}(B \mid A')$ [2]

Hence determine whether *A* and *B* are independent. [1]

8

A fairground game involves trying to hit a moving target with a gunshot. A round consists of a **maximum** of 3 shots. Ten points are scored if a player hits the target. The **round** ends **immediately** if the player misses a shot. The probability that Linda hits the target in a single shot is 0.6. All shots taken are independent of one another.

(i) Find the probability that Linda scores 30 points in a round. [2]

The random variable *X* is the number of points Linda scores in a round.

(ii) Find the probability distribution of *X*. [3]

(iii) Find the mean and variance of *X*. [4]

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	(iv) A game consists of 2 rounds. Find the probability that Linda scores more points in round 2 than in round 1. [2]
9	Six cities in a certain country are linked by rail to city *O*. The rail company provides the information about the distance of each city to city *O* and the rail fare from that city to city *O* on its website. Charles copied the table below from the website, but he had copied one of the rail fares wrongly. (table below) **(i)** Give a sketch of the scatter diagram for the data as shown on your calculator. On your diagram, circle the point that Charles has copied wrongly. [2] For parts **(ii)**, **(iii)** and **(iv)** of this question you should **exclude** the point for which Charles has copied the rail fare value wrongly. **(ii)** Find, correct to 4 decimal places, the product moment correlation coefficient between **(a)** $\ln x$ and y, **(b)** x^2 and y. [2] **(iii)** Using parts **(i)** and **(ii)**, explain which of the cases in part **(ii)** is more appropriate for modelling the data. [2] **(iv)** By using the equation of a suitable regression line, estimate the rail fare when the distance is 210 km. Explain if your estimate is reliable. [3]
10	A factory manufactures round tables in two sizes: small and large. The radius of a small table, measured in cm, has distribution $\mathrm{N}(30, 2^2)$ and the radius of a large table, measured in cm, has distribution $\mathrm{N}(50, 5^2)$. **(i)** Find the probability that the sum of the radius of 5 randomly chosen small tables is less than 160 cm. [2] **(ii)** Find the probability that the sum of the radius of 3 randomly chosen small tables is less than twice the radius of a randomly chosen large table. [2] **(iii)** State an assumption needed in your calculation in part **(ii)**. [1] A shipment of 12 large tables is to be exported. Before shipping, a check is done and the shipment will be rejected if there are at least two tables whose radius is less than 40 cm. **(iv)** Find the probability that the shipment is rejected. [3] The factory decides now to manufacture medium sized tables. The radius of a medium sized table, measured in cm, has distribution $\mathrm{N}(\mu, \sigma^2)$. It is known that 20% of the medium sized tables have radius greater than 44 cm and 30% have radius of less than 40 cm. **(v)** Find the values of μ and σ. [4]
11	The Kola Company receives a number of complaints that the volume of cola in their cans are less than the stated amount of 500 ml. A statistician decides to sample 50 cola cans to investigate the complaints. He measures the volume of cola, *x* ml, in each can and summarised the results as follows: $\sum x = 24730$, $\sum x^2 = 12242631$.

Table for question 9:

City	*A*	*B*	*C*	*D*	*E*	*F*
Distance, *x* km	100	270	120	56	289	347
Rail fare, $*y*	11.1	17.1	6.44	7.62	17.9	18.8

(i) Find unbiased estimates of the population mean and variance correct to 2 decimal places and carry out the test at the 1% level of significance. [6]

(ii) One director in the company points out that the company should test whether the volume of cola in a can is 500 ml at the 1% significance level instead. Using the result of the test conducted in **(i)**, explain how the *p*-value of this test can be obtained from *p*-value in part **(i)** and state the corresponding conclusion. [2]

The head statistician agrees the company should test that the volume of cola in a can is 500 ml at the 1% level of significance. He intends to make a simple rule of reference for the production managers so that they will not need to keep coming back to him to conduct hypothesis tests. On his instruction sheet, he lists the following:

1. Collect a random sample of 40 cola cans and measure their volume.
2. Calculate the mean of your sample, $\bar{x}$ and the variance of your sample, s_x^2.
3. Conclude that the volume of cola differs from 500 ml if the value of $\bar{x}$ lies.....

(iii) Using the above information, complete the decision rule in step 3 in terms of s_x. [4]

A party organiser has *n* cans of cola and 2*n* packets of grape juice. Assume now that the volume of a can of cola has mean 500 ml and variance 144 ml^2, and the volume of a packet of grape juice has mean 250 ml and variance 25 ml^2. She mixes all the cola and grape juice into a mocktail, which she pours into a 120-litre barrel. Assume that *n* is sufficiently large and that the volumes of the cans of cola and packets of grape juice are independent.

(iv) Show that if the party organiser wants to be at least 95% sure that the barrel will not overflow, *n* must satisfy the inequality $1000n + 22.9\sqrt{n} - 120,000 \leq 0$. [4]

– End Of Paper –

ANNEX B

SAJC H2 Math JC2 Preliminary Examination Paper 2

QN	Topic Set	Answers
1	Complex numbers	$z=2\text{i}$ or $z=-3\text{i}$ $w=2+3i$ or $w=-3+3\text{i}$
2	Sigma Notation and Method of Difference	i) $\dfrac{n+3}{n^3-n}$ ii) $3-\dfrac{4}{n}+\dfrac{2}{n+1}$ iii) $\dfrac{5}{6}-\dfrac{4}{n+2}+\dfrac{2}{n+3}$
3	Functions	i) $\text{f}^{-1}(x)=\sqrt{1+\dfrac{1}{x}}$; $\text{D}_{\text{f}^{-1}}(x)=(0,\infty)$ iii) $x>1$
4	Vectors	i) $\mathbf{r}\cdot\begin{pmatrix}2\\1\\4\end{pmatrix}=-3$ ii) $\dfrac{1}{7}\begin{pmatrix}-37\\13\\10\end{pmatrix}$; l_2: $\mathbf{r}=\begin{pmatrix}1\\3\\-2\end{pmatrix}+t\begin{pmatrix}-46\\-9\\20\end{pmatrix}$, $t\in\mathbb{R}$
5	Differentiation & Applications	ii) $8a^2$
6	P&C, Probability	a) 151200 b) 1064448
7	P&C, Probability	i) 0.277 ii) 0.107 iii) 0.580; Events A & B are not independent.
8	DRV	i) 0.216 iii) 11.76, 137.7024 iv) 0.358
9	Correlation & Linear Regression	iia) $r=0.9996$ iib) $r=0.9514$ iv) 15.73
10	Normal Distribution	i) 0.987 ii) 0.828 iv) 0.0294 v) $\sigma\approx 2.93; \mu\approx 41.5$
11	Hypothesis Testing	i) $\bar{x}=494.60; s^2=228.02$ ii) p-value $=0.00572\le 0.01$, reject H_0.

H2 Mathematics 2017 Prelim Exam Paper 2 Solution

1

Method 1

$$z - w\mathrm{i} = 3$$

$$\Rightarrow \quad w = \frac{z-3}{\mathrm{i}} = 3\mathrm{i} - z\mathrm{i} \text{ --- (1)}$$

Substitute (1) into $z^2 - w + 6 + 3\mathrm{i} = 0$

$$z^2 - (3\mathrm{i} - z\mathrm{i}) + 6 + 3\mathrm{i} = 0$$

$$\Rightarrow \; z^2 + z\mathrm{i} + 6 = 0$$

$$\Rightarrow \; z = \frac{-\mathrm{i} \pm \sqrt{(\mathrm{i})^2 - 4(1)(6)}}{2} = \frac{-\mathrm{i} \pm \sqrt{-1-24}}{2}$$

$$= \frac{-\mathrm{i} \pm 5\mathrm{i}}{2}$$

$\therefore \; z = 2\mathrm{i}$ or $z = -3\mathrm{i}$

$\Rightarrow w = 3\mathrm{i} - (2\mathrm{i})\mathrm{i}$ $\qquad w = 3\mathrm{i} - (-3\mathrm{i})\mathrm{i}$

$= 2 + 3i$ $\qquad = -3 + 3\mathrm{i}$

Method 2

$$z - w\mathrm{i} = 3$$

$$\Rightarrow \quad z = 3 + w\mathrm{i} \; \ldots. \; (1)$$

Substitute (1) into $z^2 - w + 6 + 3\mathrm{i} = 0$

$$(3 + w\mathrm{i})^2 - w + 6 + 3\mathrm{i} = 0$$

$$\Rightarrow \quad 9 + 6w\mathrm{i} - w^2 - w + 6 + 3\mathrm{i} = 0$$

$$\Rightarrow \quad -w^2 - (1-6\mathrm{i})w + 15 + 3\mathrm{i} = 0$$

$$\Rightarrow \quad -w^2 - (1-6\mathrm{i})w + 15 + 3\mathrm{i} = 0$$

$$\Rightarrow \quad w^2 + (1-6\mathrm{i})w - 15 - 3\mathrm{i} = 0$$

$$\Rightarrow \quad w = \frac{-(1-6\mathrm{i}) \pm \sqrt{(1-6\mathrm{i})^2 - 4(1)(-15-3\mathrm{i})}}{2}$$

$$= \frac{-1 + 6\mathrm{i} \pm \sqrt{1 - 12\mathrm{i} - 36 + 60 + 12\mathrm{i}}}{2}$$

$$= \frac{-1 + 6\mathrm{i} \pm \sqrt{25}}{2}$$

$\therefore \; w = 2 + 3\mathrm{i}$ or $w = -3 + 3\mathrm{i}$

$\Rightarrow \; z = 3 + (2+3\mathrm{i})\mathrm{i} = 2\mathrm{i}$ $\qquad z = 3 + (-3+3\mathrm{i})\mathrm{i} = -3\mathrm{i}$

Method 3

$$w\mathrm{i} = z - 3$$

$$\Rightarrow w = -\mathrm{i}z + 3\mathrm{i}$$

$$\therefore \; z^2 - (-\mathrm{i}z + 3\mathrm{i}) + 6 + 3\mathrm{i} = 0$$

$$\Rightarrow \; z^2 + \mathrm{i}z + 6 = 0$$

Let $z = a + b\mathrm{i}$ where $a, b \in \mathbb{R}$

$$(a + b\mathrm{i})^2 + \mathrm{i}(a + b\mathrm{i}) + 6 = 0$$

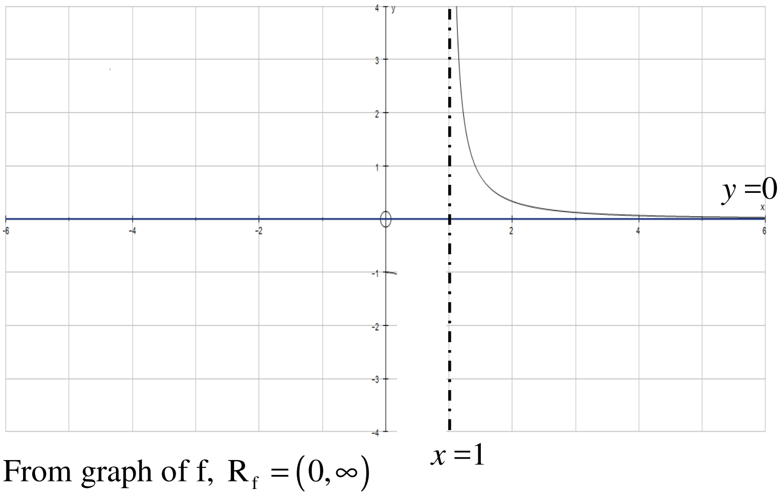
$$\Rightarrow \; a^2 - b^2 + 2ab\mathrm{i} + a\mathrm{i} - b + 6 = 0$$

$$\Rightarrow \; a^2 - b^2 - b + 6 + (2ab + a)\mathrm{i} = 0$$

By comparing the real and imaginary parts,

	$a^2 - b^2 - b + 6 = 0 \quad \ldots (1)$ $2ab + a = 0 \quad \ldots (2)$ From (2), $a = 0$ or $b = -\frac{1}{2}$ When $a = 0$, $b^2 + b - 6 = 0$ $(b-2)(b+3) = 0$ $b = 2$ or $b = -3$ Hence $z = 2\mathrm{i}, w = -\mathrm{i}(2\mathrm{i}) + 3\mathrm{i} = 2 + 3\mathrm{i}$ or $z = -3\mathrm{i}, w = -\mathrm{i}(-3\mathrm{i}) + 3\mathrm{i} = -3 + 3\mathrm{i}$ When $b = -\frac{1}{2}$, $a^2 = \frac{1}{4} - \frac{1}{2} - 6 = -\frac{25}{4}$ There is no real solution for *a*.
2	(i) $\frac{2}{n-1} - \frac{3}{n} + \frac{1}{n+1}$ $= \frac{2(n)(n+1) - 3(n-1)(n+1) + (n-1)(n)}{(n-1)(n)(n+1)}$ $= \frac{(2n^2 + 2n) - (3n^2 - 3) + (n^2 - n)}{n^3 - n}$ $= \frac{n+3}{n^3 - n}$ (ii) $\sum_{r=2}^{n} \frac{2r+6}{r^3 - r}$ $= 2\sum_{r=2}^{n} \frac{r+3}{r^3 - r}$ $= 2\sum_{r=2}^{n} \left(\frac{2}{r-1} - \frac{3}{r} + \frac{1}{r+1} \right)$ $= 2\left[\begin{array}{l} \frac{2}{1} - \frac{3}{2} + \frac{1}{3} \\ + \frac{2}{2} - \frac{3}{3} + \frac{1}{4} \\ + \frac{2}{3} - \frac{3}{4} + \frac{1}{5} \\ + \ldots \\ + \frac{2}{n-2} - \frac{3}{n-1} + \frac{1}{n} \\ + \frac{2}{n-1} - \frac{3}{n} + \frac{1}{n+1} \end{array} \right]$

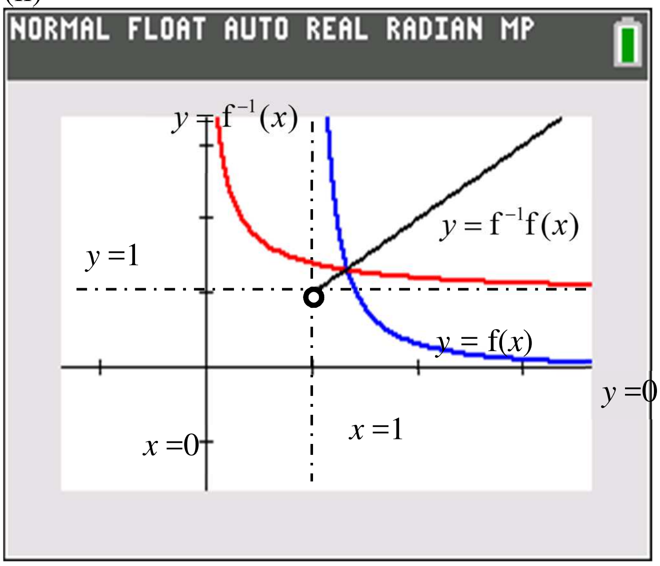
	$=2\left(\frac{2}{1}-\frac{3}{2}+\frac{2}{2}+\frac{1}{n}-\frac{3}{n}+\frac{1}{n+1}\right)$ $=2\left(\frac{3}{2}-\frac{2}{n}+\frac{1}{n+1}\right)$ $=3-\frac{4}{n}+\frac{2}{n+1}$ (iii) $\sum_{r=2}^{n}\frac{2r+10}{(r+1)(r+2)(r+3)}$ Let $r+2=p \Rightarrow r=p-2$ $=\sum_{p-2=2}^{p-2=n}\frac{2p+6}{(p-1)(p)(p+1)}$ $=\sum_{p=4}^{n+2}\frac{2p+6}{p^3-p}$ $=\sum_{p=2}^{n+2}\frac{2p+6}{p^3-p}-\sum_{p=2}^{3}\frac{2p+6}{p^3-p}$ $=\left(3-\frac{4}{n+2}+\frac{2}{n+3}\right)-\left(3-\frac{4}{3}+\frac{2}{4}\right)$ $=\frac{5}{6}-\frac{4}{n+2}+\frac{2}{n+3}$
3	(i) $\mathrm{f}: x \mapsto \frac{1}{x^2-1}$ Let $y=\frac{1}{x^2-1}$ $x^2=\frac{1}{y}+1$ $x=\pm\sqrt{1+\frac{1}{y}}$ Since $x>1$, $x=\sqrt{1+\frac{1}{y}}$ $\mathrm{f}^{-1}(x)=\sqrt{1+\frac{1}{x}}=\sqrt{\frac{1+x}{x}}$.



From graph of f, $R_f = (0, \infty)$

$\therefore D_{f^{-1}}(x) = (0, \infty)$.

(ii)



(iii)

Since $ff^{-1}(x) = f^{-1}f(x) = x$ have the same rule, we investigate the domain

$D_{f^{-1}f} = (1, \infty)$ $D_{ff^{-1}} = (0, \infty)$

Taking the intersection of these domains,

Range of values is $x > 1$.

4

(i)

Equation of plane is

$$\mathbf{r}=\begin{pmatrix}1\\3\\-2\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\0\end{pmatrix}+\mu\begin{pmatrix}-2\\0\\1\end{pmatrix},\ \lambda,\mu\in\mathbb{R}$$

A normal vector to plane is

$$\begin{pmatrix}1\\-2\\0\end{pmatrix}\times\begin{pmatrix}-2\\0\\1\end{pmatrix}=\begin{pmatrix}-2\\-1\\-4\end{pmatrix}$$

Hence vector equation of the plane is

$$\mathbf{r}\cdot\begin{pmatrix}2\\1\\4\end{pmatrix}=\begin{pmatrix}1\\3\\-2\end{pmatrix}\cdot\begin{pmatrix}2\\1\\4\end{pmatrix}$$

$$\mathbf{r}\cdot\begin{pmatrix}2\\1\\4\end{pmatrix}=-3$$

(ii)

$$l_{AC}:\ \ \mathbf{r}=\begin{pmatrix}-5\\2\\2\end{pmatrix}+s\begin{pmatrix}2\\1\\4\end{pmatrix},\ s\in\mathbb{R}$$

Thus $\overrightarrow{OC}=\begin{pmatrix}-5\\2\\2\end{pmatrix}+s\begin{pmatrix}2\\1\\4\end{pmatrix}$ for some $s\in\mathbb{R}$.

Since C lies on the plane:

$$\left[\begin{pmatrix}-5\\2\\2\end{pmatrix}+s\begin{pmatrix}2\\1\\4\end{pmatrix}\right]\cdot\begin{pmatrix}2\\1\\4\end{pmatrix}=-3$$

$$2(-5+2s)+(2+s)+4(2+4s)=-3$$

$$s=-\frac{3}{21}$$

Thus $$\overrightarrow{OC}=\begin{pmatrix}2\left(-\frac{3}{21}\right)-5\\\left(-\frac{3}{21}\right)+2\\4\left(-\frac{3}{21}\right)+2\end{pmatrix}=\frac{1}{7}\begin{pmatrix}-37\\13\\10\end{pmatrix}$$

(iii)

Using mid-point theorem

	$\overrightarrow{OA'} = 2\overrightarrow{OC} - \overrightarrow{OA}$ $= \frac{2}{7}\begin{pmatrix}-37\\13\\10\end{pmatrix} - \begin{pmatrix}-5\\2\\2\end{pmatrix} = \frac{1}{7}\begin{pmatrix}-39\\12\\6\end{pmatrix}$ B is the point of intersection of l_1 and π. $\overrightarrow{BA'} = \overrightarrow{OA'} - \overrightarrow{OB}$ $= \frac{1}{7}\begin{pmatrix}-39\\12\\6\end{pmatrix} - \begin{pmatrix}1\\3\\-2\end{pmatrix}$ $= \frac{1}{7}\begin{pmatrix}-46\\-9\\20\end{pmatrix}$ $l_2: \ \mathbf{r} = \frac{1}{7}\begin{pmatrix}-39\\12\\6\end{pmatrix} + t\begin{pmatrix}-46\\-9\\20\end{pmatrix}, \ t \in \mathbb{R}$ or $l_2: \ \mathbf{r} = \begin{pmatrix}1\\3\\-2\end{pmatrix} + t\begin{pmatrix}-46\\-9\\20\end{pmatrix}, \ t \in \mathbb{R}$
5	(i) The height of triangle ADG is $\frac{a}{\tan\theta} = \frac{a}{t}$. Hence $AH = 2a + \frac{a}{t} = a\left(2 + \frac{1}{t}\right)$. $BH = BE + EH = 2a\tan\theta + a = a(2t+1)$ Area $S = \frac{1}{2}(AH)(BC)$ $S = \frac{a}{2}\left(2 + \frac{1}{t}\right)\left(2a(2t+1)\right)$ $S = a^2\left(2 + \frac{1}{t}\right)(2t+1)$ $S = a^2\left(4 + 4t + \frac{1}{t}\right)$ (ii) $\frac{\mathrm{d}S}{\mathrm{d}t} = a^2\left(4 - \frac{1}{t^2}\right)$ When $\frac{\mathrm{d}S}{\mathrm{d}t} = 0$, $t^2 = \frac{1}{4}$ $\Rightarrow \ t = \pm\frac{1}{2}$

Reject $t = \tan\theta = -\frac{1}{2}$ as θ is acute

$\frac{d^2S}{dt^2} = a^2\left(\frac{2}{t^3}\right)$

When $t = \frac{1}{2}$, $\frac{d^2S}{dt^2} = a^3\left(\frac{2}{\left(\frac{1}{2}\right)^3}\right) = 16a > 0$.

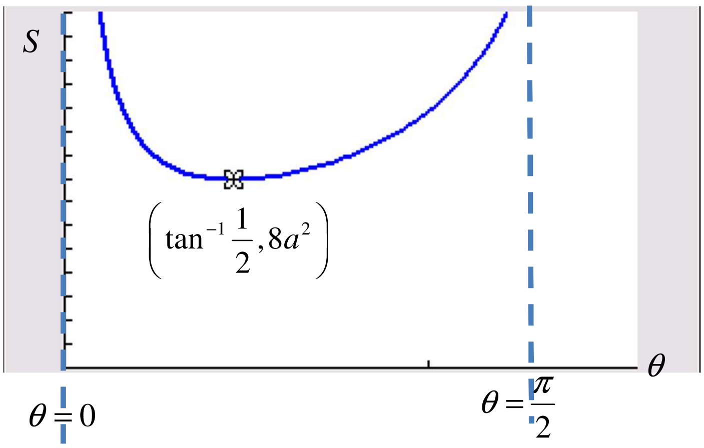
Hence the minimum value of S occurs when $t = \frac{1}{2}$.

Minimum $S = a^2(4+2+2) = 8a^2$.

(iii)

To sketch the graph of

$S = a^2\left(4 + 4\tan\theta + \frac{1}{\tan\theta}\right)$



6

(a)

Since adjacent balls do not sum up to two, balls numbered ‘1’ needs be separated.

Number of ways of arranging the other balls with no restriction = 6!

Slotting in the balls numbered ‘1’, permutation is done as balls are of different colour = ${}^7C_3 \times 3!$

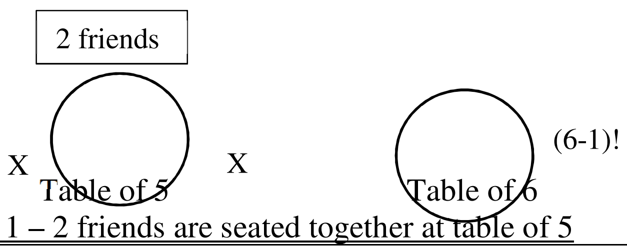
No of ways

$= 6! \times {}^7C_3 \times 3!$

$= 151200$

(b)

<u>Method 1</u>



<u>Case 1 – 2 friends are seated together at table of 5</u>

X

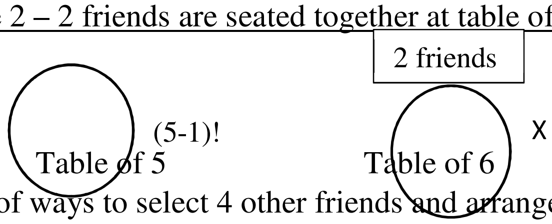
No. of ways to select 3 other friends and arrange them at the table of 5 = $^{9}C_{3}\times(4-1)!$

No. of ways to arrange the 2 friends = 2!

No. of ways to sit the remaining friends at the table of 6
= (6-1)! = 5! = 120

Total no. of ways = $^{9}C_{3}\times(4-1)!\times 2!\times 5! = 120960$

<u>Case 2 – 2 friends are seated together at table of 6</u>



No. of ways to select 4 other friends and arrange them at the table of 6 = $^{9}C_{4}\times(5-1)! =$ 3024

No. of ways to sit the 2 friends at the table of 6 = 2!

No. of ways to sit the remaining friends at the table of 5
= (5-1)! = 4! = 24

Total no. of ways = $^{9}C_{4}\times(5-1)!\times 2!\times 4! = 145152$

No of ways to arrange 11 friends without restrictions
= $^{11}C_{5}\times(5-1)!\times(6-1)! = 1330560$

Total no. of ways of arranging 11 people such that 2 particular friends are not seated together
= 1330560 – 120960 – 145152 = 1064448

<u>Method 2</u>

<u>Alternative Method</u>

Case 1: Two particular friends seated at table of 5

No of ways
$= {}^{9}C_{3}\times 2!\times 3\times 2\times 5!$
$= 120960$

$^{9}C_{3}$: Selection of friends to be seated at table of 5. This automatically selects friends to be seated at table of 6.

(3-1)!: Arranging the 3 other friends in table of 5.

$^{3}P_{2}$: Slotting in the 2 particular friends

5!: Arranging the 6 other friends in table of 6.

Case 2: Two particular friends seated at table of 6

No of ways
$= {}^{9}C_{4}\times 4!\times 3!\times 4\times 3$
$= 217728$

$^{9}C_{4}$: Selection of friends to be seated at table of 5. This automatically selects friends to be seated at table of 6.

(5-1)!: Arranging the 5 friends in table of 5.

4!: Arranging the 5 friends in table of 6.

${}^4P_2$: Slotting in the 2 particular friends

Case 3: Two particular friends seated at separate tables
No of ways

$= {}^9C_4 \times 4! \times 5! \times 2$

$= 725760$

${}^9C_4$: Selection of friends to be seated at table of 5. This automatically selects friends to be seated at table of 6.

(5-1)!: Arranging the 5 friends in table of 5.

(6-1)!: Arranging the 6 friends in table of 6.

x2: The 2 particular friends can switch tables

Total no. of ways

$= 120960 + 217728 + 725760$

$= 1064448$

7

(i)

Given $\mathrm{P}(A \mid B') = 0.83$

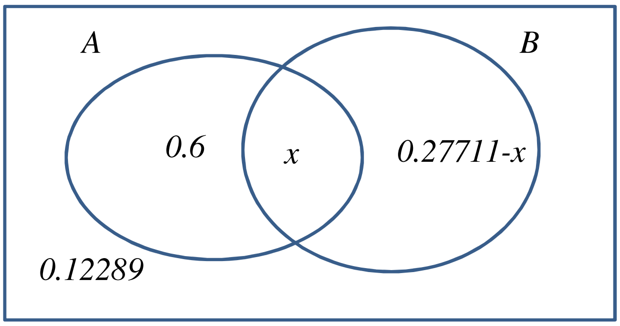
$$\Rightarrow \frac{\mathrm{P}(A \cap B')}{\mathrm{P}(B')} = 0.83$$

$$\Rightarrow \frac{0.6}{1 - \mathrm{P}(B)} = 0.83$$

$$\Rightarrow P(B) = 1 - 0.72289 = 0.27711 = 0.277$$

(ii)

Let $\mathrm{P}(A \cap B) = x$



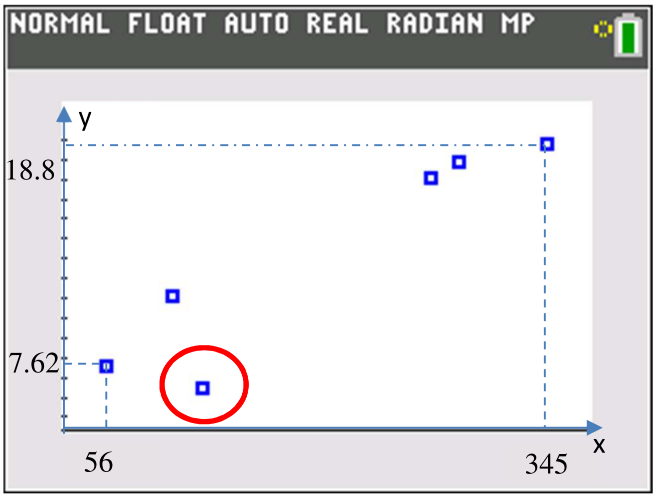
$$\mathrm{P}(A \cup B) = P(A \cap B') + P(B)$$

$$= 0.6 + x + 0.27711 - x$$

$$= 0.87711$$

	$P(A\cup B)'=1-0.87711=0.12289$ Since $P(A\cup B')=0.83$ $\therefore 0.6+x+0.12289=0.83$ $\Rightarrow x=0.10711$ $\therefore P(A\cap B)=0.107$. (iii) $P(B\mid A')=\dfrac{P(B\cap A')}{P(A')}$ $=\dfrac{0.27711-0.10711}{1-(0.6+0.10711)}$ $=\dfrac{0.17}{0.29289}$ $=0.58042$ $=0.580$ Since $P(B\mid A')\neq P(B)\Rightarrow B$ is not independent of A' $\therefore$ A and B are not independent.
8	(i) P(Linda scores 30 points) = $P(\{\text{hit, hit, hit}\})$ $= 0.6^3$ $= \frac{27}{125}$ (0.216) (ii) Let X be the number of points scored by Linda in a round. (see table below) (iii) $E(X) = 0\times0.4 + 10\times0.24 + 20\times0.144 + 30\times0.216$ $=11.76$ $E(X^2) = 0^2\times0.4 + 10^2\times0.24 + 20^2\times0.144 + 30^2\times0.216$ $= 276$ $\text{Var}(X) = E(X^2) - [E(X)]^2$ $= 276 - 11.76^2 = 137.7024$ (iv) Let X_1 be the number of points scored by Linda in Round 1 and let X_2 be the number of points scored by Linda in Round 2. P(Linda scores more in round 2 than in round 1)

X	0	10	20	30
$P(X=x)$	0.4	0.6×0.4 $=0.24$	$0.6^2\times0.4$ $=0.144$	0.216

	$= P(X_1 = 0 \ \& \ X_2 \geq 10)$ $+P(X_1 = 10 \ \& \ X_2 \geq 20)$ $+P(X_1 = 20 \ \& \ X_2 = 30)$ $= P(X_1 = 0)P(X_2 \geq 10)$ $+P(X_1 = 10)P(X_2 \geq 20)$ $+P(X_1 = 20)P(X_2 = 30)$ $= 0.4 \times (1 - 0.4)$ $+0.24 \times (0.144 + 0.216) + 0.144 \times 0.216$ $= 0.357504 = 0.358$ (3 s.f.)
9	(i)  (ii) (a) Product moment correlation coefficient , $r = 0.99959$ (b) Product moment correlation coefficient, $r = 0.95137$ (iii) From the scatter diagram, as x increases, the value of y increases at a decreasing rate, that seems to fit model (a) better. Also, the value of $\|r\|$ for model (a) is closer to 1 as compared to model (b). (iv) We use the regression line y on ln x $y = 6.1619(\ln x) - 17.223 \approx 6.16 \ln x - 17.2$ When $x = 210$, $y = 6.1619(\ln 210) - 17.223 = 15.725 \approx 15.7$ As the value of $\|r\|$ is close to 1 and $x = 210$ is within the given data range, the estimation may be reliable.
10	(i) Let S be the random variable "radius of a small table in cm'. Let L be the random variable "radius of a large table in cm'. $S \sim \mathrm{N}(30,\ 2^2)$ $L \sim \mathrm{N}(50,\ 5^2)$

	$S_1+S_2+S_3+S_4+S_5 \sim \text{N}(5\times 30,\ 5\times 2^2)$ $S_1+S_2+S_3+S_4+S_5 \sim \text{N}(150,\ 20)$ $\text{P}(S_1+S_2+S_3+S_4+S_5 < 160) = 0.98733 \approx 0.987$ (ii) $S_1+S_2+S_3-2L \sim \text{N}(3\times 30-2\times 50,\ 3\times 2^2+2^2\times 5^2)$ $S_1+S_2+S_3-2L \sim \text{N}(-10,\ 112)$ $P(S_1+S_2+S_3 < 2L) = P(S_1+S_2+S_3-2L<0) = 0.82765 \approx 0.828$ (iii) The radii of the large and small round tables are independent of one another. (iv) Let X be the random variable "number of large tables, out of 12, with radius less than 40 cm". $X \sim \text{B}(12,\ \text{P}(L<40))$ $X \sim \text{B}(12,\ 0.022750)$ $\text{P}(X \geq 2) = 1-\text{P}(X \leq 1)$ $=1-0.97064$ $=0.029357$ ≈ 0.0294 (v) Let Y be the random variable "radius of a medium sized table in cm" $\text{P}(Y \geq 44) = 0.20$ $\text{P}(Y < 44) = 0.80$ $P\left(Z<\frac{44-\mu}{\sigma}\right)=0.80$ $\frac{44-\mu}{\sigma}=0.84162$ $\mu = 44-0.84162\sigma ----(1)$ $\text{P}(Y<40)=0.30$ $P\left(Z<\frac{40-\mu}{\sigma}\right)=0.30$ $\frac{40-\mu}{\sigma}=-0.52440$ $\mu=40+0.52440\sigma ----(2)$ Solving (1) and (2), $44-0.84162\sigma=40+0.5244\sigma$ $4=1.3660\sigma$ $\sigma=2.9283\approx 2.93$ $\mu=41.535\approx 41.5$
11	(i) Unbiased estimate of population mean,

	$\overline{x} = \frac{24730}{50} = 494.60$ Unbiased estimate for population variance, $s^2 = \frac{1}{49}\left(12242631 - \frac{24730^2}{50}\right) = 228.02$ Let X be the volume of beer in one beer can in ml and μ be the population mean volume of beer of the beer cans. $H_0 : \mu = 500$ $H_1 : \mu < 500$ Under H_0 , since n = 50 is large, by the Central Limit Theorem, $\overline{X} \sim N\left(500, \frac{s^2}{50}\right)$ approximately. Use a left-tailed z-test at the 1% level of significance. Test statistic: $Z = \frac{\overline{X} - 500}{\frac{s}{\sqrt{50}}} \sim N(0,1)$. Reject H_0 if p-value ≤ 0.01. From the sample, $p - \text{value} = 0.0057248 = 0.00572$ Since p-value $= 0.00572 \leq 0.01$, we reject H_0. There is sufficient evidence at the 1% level of significance to conclude that the volume of cola in a can is less than 500 ml. (iii) Let X be the volume of cola in one can in ml and μ be the population mean volume of cola of the cans. $H_0 : \mu = 500$ $H_1 : \mu \neq 500$ Unbiased estimate of population variance, $s^2 = \frac{40}{39}(s_x)^2$ Under H_0 , since n = 40 is large, by the Central Limit Theorem, $\overline{X} \sim N\left(500, \frac{s_x^{\ 2}}{39}\right)$ approximately. Use a two-tailed z-test at the 1% level of significance. Test statistic: $Z = \frac{\overline{X} - 500}{\frac{s_x}{\sqrt{39}}} \sim N(0,1)$ Critical values: $z_{crit(1)} = -2.5758 \quad z_{crit(2)} = 2.5758$. Reject H_0 if $z_{cal} \leq -2.5758 \quad \text{or} \quad z_{cal} \geq 2.5758$. Since H_0 is rejected,

	$-2.5758 \le z_{cal}$ or $z_{cal} \ge 2.5758$ $-2.5758 \le \frac{\overline{x}-500}{\sqrt{\frac{s_x^{\ 2}}{39}}}$ or $\frac{\overline{x}-500}{\sqrt{\frac{s_x^{\ 2}}{39}}} \ge 2.5758$ $500-2.5758\sqrt{\frac{s_x^{\ 2}}{39}} \le \overline{x}$ or $\overline{x} \ge 500+2.5758\sqrt{\frac{s_x^{\ 2}}{39}}$ $500-0.41246s_x \le \overline{x}$ or $\overline{x} \ge 500+0.41246s_x$ $500-0.412s_x \le \overline{x}$ or $\overline{x} \ge 500+0.412s_x$ Hence the decision rule should read: Conclude that the volume of cola differs from 500 ml if the value of $\overline{x}$ lies within this range : $500-0.412s_x \le \overline{x}$ or $\overline{x} \ge 500+0.412s_x$. (iv) Let X be the volume of cola in one can in ml. since n is large, by the Central Limit Theorem, $X_1+X_2+....+X_n \sim N(500n,144n)$ approximately. Let Y be the volume of grape juice in one packet in ml. since $2n$ is large, by the Central Limit Theorem, $Y_1+Y_2+....+Y_{2n} \sim N(500n,50n)$ approximately. $X_1+X_2+....+X_n+Y_1+Y_2+....+Y_{2n} \sim N(1000n,194n)$ $P(X_1+X_2+....+X_n+Y_1+Y_2+....+Y_{2n} \le 120,000) \ge 0.95$ $P\left(Z \le \frac{120,000-1000n}{\sqrt{194n}}\right) \ge 0.95$ $\frac{120,000-1000n}{\sqrt{194n}} \ge 1.6449$ $120,000-1000n \ge 1.6449\sqrt{194n}$ $1000n+22.9\sqrt{n}-120,000 \le 0$