



TEMASEK JUNIOR COLLEGE, SINGAPORE

JC 2
Preliminary Examination 2018
Higher 2

MATHEMATICS

9758/01

Paper 1

10 September 2018

Additional Materials: Answer paper

3 hours

List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your Civics group and name on all the work that you hand in.

Write in dark blue or black pen on both sides of the paper.

You may use a soft pencil for any diagrams or graphs.

Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

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You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [] at the end of each question or part question.

At the end of the examination, fasten all your work securely together.

This document consists of 6 printed pages including this page.



**TEMASEK
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TEMASEK JUNIOR COLLEGE, SINGAPORE
PASSION PURPOSE DRIVE



1 A cubic curve cuts the y -axis at $y=8$ and the x -axis at $x=-2$. The line L , with equation $y=6x-4$, intersects the curve at the point where $x=-3$. The tangent to the curve at the point where $x=-\frac{4}{3}$ is parallel to L . Find the equation of the curve. [5]

2 Sketch the graph of $y=3-|1-x^2|$, indicating clearly all the key features. [2]

Hence solve the inequality $3-|1-x^2| \geq \frac{x+5}{2}$. [3]

3 Find the exact volume of the solid of revolution formed when the region bounded by the curve $y=\frac{\sqrt{x}}{4-x^2}$, the y -axis and the line $y=\frac{1}{3}$ is rotated through 2π radians about the x -axis. [5]

4 A sequence u_1, u_2, u_3, \dots is such that $u_n = \frac{1}{1+a^n}$ and $u_{n+1} = u_n - \frac{a^n(a-1)}{(1+a^{n+1})(1+a^n)}$,

where a is constant such that $0 < a < 1$.

(i) Find $\sum_{n=1}^N \frac{a^n}{(1+a^{n+1})(1+a^n)}$ in terms of a and N . [3]

(ii) Determine whether $\sum_{n=1}^{\infty} \frac{a^n}{(1+a^{n+1})(1+a^n)}$ is a convergent series. [2]

5 The function f is defined by $f: x \mapsto (x^2-4)^2 - 24, x \in \mathbb{R}$.

(i) Sketch the graph of $y=f(x)$, indicating clearly all intercepts and stationary points. [2]

(ii) Explain why f^{-1} does not exist. [1]

(iii) The functions f_1 and f_2 are defined by

$$f_1: x \mapsto (x^2-4)^2 - 24, x \in \mathbb{R}, x \leq k,$$

$$f_2: x \mapsto (x^2-4)^2 - 24, x \in \mathbb{R}, x > k,$$

where k is a real number. State the range of values of k for which f_1^{-1} exists and

f_2^{-1} does not exist. [1]

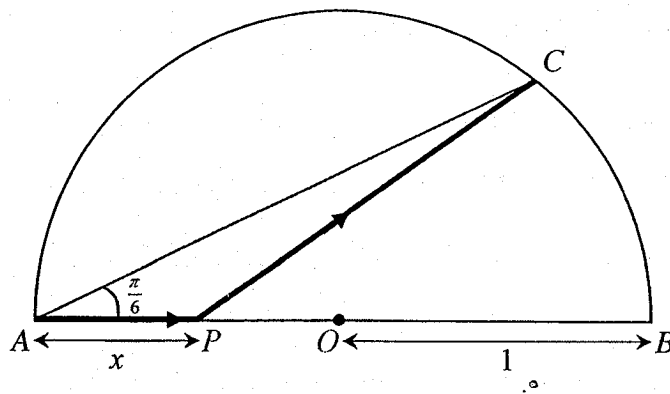
(iv) Using the largest possible value of k found in (iii), find f_1^{-1} in a similar form. [4]

6 It is given that

$$g(x) = \begin{cases} x+2 & \text{for } 0 < x \leq 2, \\ 2 + \sqrt{4 - (x-2)^2} & \text{for } 2 < x \leq 4, \end{cases}$$

and that $g(x) = g(x+4)$ for all real values of x .

- (i) Sketch the graph of $y = g(x)$ for $-3 \leq x \leq 13$. [3]
- (ii) Without using integration, find the exact value of $\int_{-3}^{13} g(x) dx$. [2]
- (iii) Solve the equation $g(x) = x$ exactly. [3]
- 7 The figure shows a plot of rough terrain that is in the shape of a semicircle, centred at O with a radius of 1 km. A group of explorers arrives at the point A and needs to get to their final destination at the point C where the chord AC makes an angle of $\frac{\pi}{6}$ with the diameter AB . To do so, they decide to walk x km along the diameter AB to a variable point P before cutting across the rough terrain in the direction PC . It is known to them that OB , a portion of the path AB , was destroyed by an earthquake and impassable.



- (i) Write down the angle COA . Hence show that $CP = \sqrt{3 - 3x + x^2}$. [3]
- (ii) The explorers need to reach C as soon as possible as it is getting dark and dangerous. They can walk along AB at a speed of 5 km/h but will need to slow down to a speed of 3 km/h when crossing the rough terrain. Show that the time taken for the explorers to reach point C is shortest when x satisfies the equation

$$64x^2 - 192x + 117 = 0.$$

Hence find this shortest time.

[6]

- 8 Kesamet Bank offers its customers a housing loan and charges a monthly interest such that t months after taking the loan, the amount of money owed to the bank, $\$x$, increases at a rate proportional to the amount of money owed. Both x and t are taken to be continuous variables. Customers are required to make a fixed monthly repayment of $\$r$ until the loan is fully repaid.

Given that the monthly interest and the repayment amount are the same when $x = a$, show that

$$\frac{dx}{dt} = \frac{r}{a}(x - a). \quad [3]$$

Given also that the loan taken from the bank is $\$A$, find x in terms of t , r , a and A . [5]

Sketch the graph of x against t in the case where the loan can be repayable within a finite duration and state the range of values of a in terms of A in this case. [2]

Hock Yau takes a loan of $\$250000$ to finance the purchase of his new apartment and he decides to repay a fixed amount of $\$5000$ monthly. Given that the monthly interest is the same as the repayment amount of $\$5000$ when he owes the bank $\$255000$, determine the duration, to the nearest month, he needs to repay his loan fully. [2]

9 Do not use a calculator in answering this question.

(a) It is given that one of the roots of the equation $z^4 + pz^3 + 7z^2 + 6z - 30 = 0$ is $1 + 3i$.

(i) Evaluate $(1 + 3i)^4$, showing your calculations clearly. [2]

(ii) Show that $p = -2$. [2]

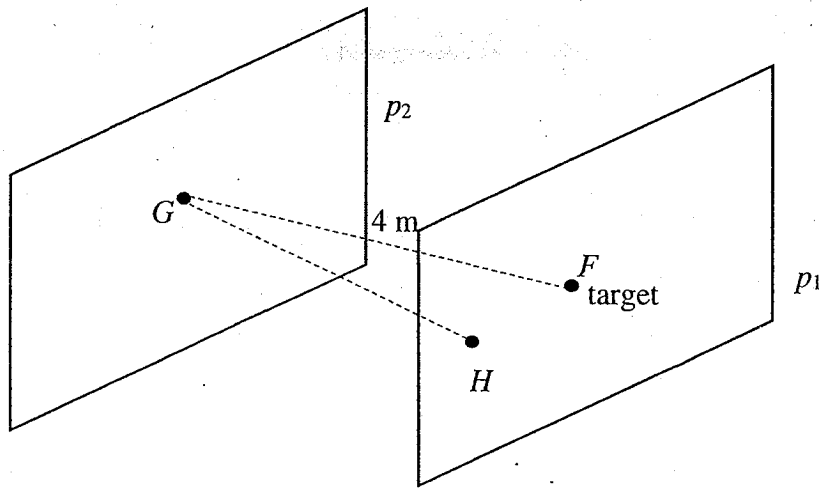
(iii) Find the other roots of the equation in exact form. [5]

(b) The complex number z is given by $z = (\sqrt{2})e^{i\left(\frac{5}{12}\pi\right)}$.

(i) Given that $w = \frac{z^3}{z^*}$, find w in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [3]

(ii) Find the smallest positive integer n such that w^n is a real number. [3]

10



The diagram shows two parallel boards p_1 and p_2 modelled by planes with equations $4x - 3z = -2$ and $ax + z = b$ respectively. p_1 and p_2 are placed vertically on a horizontal ground, with a distance of 4 m apart.

State the value of a . [1]

Robin places a gun through a hole at point G on p_2 and aims to shoot in a direction perpendicular to p_1 at a target F on p_1 , with coordinates $\left(-\frac{1}{2}, 1, 0\right)$. It can be assumed that all shots travel in a straight line.

Show that a set of possible coordinates of G are $\left(-\frac{37}{10}, 1, \frac{12}{5}\right)$. [3]

For the rest of this question, take the coordinates of G to be $\left(-\frac{37}{10}, 1, \frac{12}{5}\right)$.

(i) Find the value of b . [1]

The bullet from G travels in a path parallel to the vector $-\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and ends up at a point H on p_1 instead.

(ii) Find the acute angle between the path travelled by the bullet and p_1 . Hence find the exact distance between H and F . [4]

Due to long term usage, p_1 becomes slanted such that the new slanted board p_3 is now modelled by a plane that passes through the origin and contains the line with equation

$$x + 2 = \frac{y - 1}{2} = \frac{3z + 6}{4}.$$

(iii) Find a vector equation of p_3 , in scalar product form. [2]

(iv) Find a vector equation of the line of intersection between p_1 and p_3 . [2]

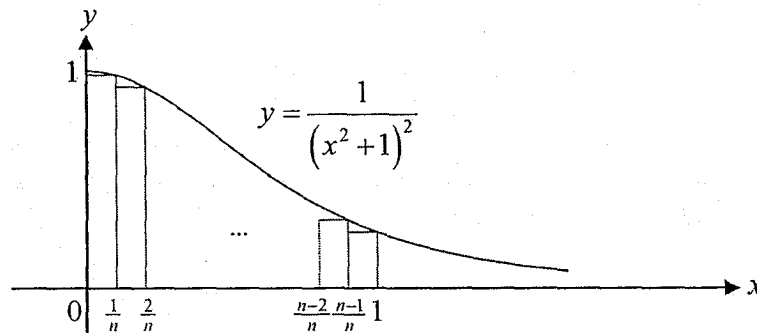
11 (a) Find $\int \ln(x+1) dx$. [3]

Hence find in terms of k , the area bounded by the curve $y = \ln(x+1)$, the x -axis and the lines $x = -k$ and $x = k$, where $0 < k < 1$. [3]

(b) (i) By using the substitution $x = \tan \theta$, find the exact value of $\int_0^1 \frac{1}{(x^2+1)^2} dx$.

[6]

(ii)



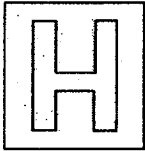
The figure above shows the graph of $y = \frac{1}{(x^2+1)^2}$ for $x \geq 0$. Rectangles,

each of width $\frac{1}{n}$, are drawn under the curve. Show that the total area of all

n rectangles can be written as $\sum_{r=1}^n \frac{n^3}{(r^2+n^2)^2}$. [2]

(iii) Hence state the exact value of $\lim_{n \rightarrow \infty} \left[\sum_{r=1}^n \frac{n^3}{(r^2+n^2)^2} \right]$. [1]

- End of Paper -



TEMASEK JUNIOR COLLEGE, SINGAPORE

JC 2
Preliminary Examinations 2018
Higher 2

MATHEMATICS

9758/02

Paper 2

12 September 2018

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3 hours

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PASSION PURPOSE DRIVE



Section A: Pure Mathematics [40 marks]

- 1 A curve C has parametric equations

$$x = 2e^t, \quad y = t^3 - t,$$

where $-1 \leq t \leq \frac{3}{2}$.

- (i) Find $\frac{dy}{dx}$ in terms of t . Hence find the exact equations of the normals to the curve which are parallel to the y -axis. [3]
- (ii) Sketch the curve C . [1]
- (iii) Find the equation of the tangent to C at the point $(2e^a, a^3 - a)$. Find the possible exact values of a such that the tangent cuts the y -axis at $y = 1$. [4]

- 2 A convergent geometric progression has first term a and common ratio r , where $a > 0$ and $r > 0$. An arithmetic progression has positive first term b and non-zero common difference d . It is given that the 6th, 9th and 16th terms of the geometric progression are equal to the 7th, 10th and 13th terms of the arithmetic progression respectively.

- (a) Find d in terms of b and r . Hence show that the arithmetic progression is a decreasing sequence. [4]
- (b) (i) Show that r satisfies the equation

$$r^{10} - 2r^3 + 1 = 0.$$

Hence find the value of r , giving your answer correct to 3 significant figures.

[4]

- (ii) Using the value of r found in (i), find the least value of n for which the sum to infinity of the geometric series exceeds the sum of the first n terms by less than $0.006a$. [3]

3 Let $f(x) = \sqrt{3 + \sin x + \cos x}$.

(i) Given that $y = f(x)$, show that $2y \frac{d^2y}{dx^2} + 2 \left(\frac{dy}{dx} \right)^2 + y^2 = 3$. [3]

Hence obtain the Maclaurin's expansion of y up to and including the term in x^2 .

[3]

(ii) If x is sufficiently small for terms in x^3 and higher powers of x to be neglected, show that the same result in (i) may be obtained using suitable standard series expansions. [3]

(iii) Denote the answer to (i) by $g(x)$. On the same set of axes, sketch the graphs of $y = f(x)$ and $y = g(x)$ for $-5 \leq x \leq 5$. Explain why the graphs are similar only for a certain range of values of x . [3]

4 (a) Given that \mathbf{p} and \mathbf{q} are two non-zero vectors, show that $|\mathbf{p} \times \mathbf{q}|^2 = |\mathbf{p}|^2 |\mathbf{q}|^2 - (\mathbf{p} \cdot \mathbf{q})^2$. [2]

(b) The points O , A and B are distinct and non-collinear. A and B have position vectors \mathbf{a} and \mathbf{b} respectively. It is given that $\angle AOB = 90^\circ$ and $|\mathbf{a}| = \alpha |\mathbf{b}|$ where α is a constant. Point M is the midpoint of AB and point T lies on OM produced such that $3OM = MT$.

(i) Write down a vector equation of the line AB in terms of \mathbf{a} and \mathbf{b} . [1]

(ii) Show that the position vector of the foot of perpendicular from point T to line

$$AB \text{ is given by } \frac{1}{1+\alpha^2} [(2\alpha^2 - 1)\mathbf{a} + (2 - \alpha^2)\mathbf{b}]. \quad [6]$$

Section B: Statistics [60 marks]

5 5 French, 3 Croatian and 2 Belgian dancers were invited to perform at a charity event, where a dinner was held to welcome them. In how many ways can these dancers be seated at a round table if those of the same nationality must sit next to one another? [2]

For the opening item, the dancers were randomly divided into two groups of 3 dancers each and one group of 4 dancers. Find the number of ways this can be done. [2]

At the end of the performance, 5 of the 10 dancers were chosen to pose for photographs. Find the number of ways the dancers can be arranged in a row, if exactly 3 of the chosen dancers are of the same nationality. [3]

- 6 A study on the toxicity of Rotenone, an ingredient in insecticide, on Chrysanthemum aphids was conducted. Insecticides containing different concentrations of Rotenone were sprayed on 5 separate groups of Chrysanthemum aphids. The following data was obtained:

Rotenone concentration (x mg/L)	10.2	7.7	5.1	3.8	2.6
Percentage of Chrysanthemum aphids killed ($y\%$)	88.0	85.7	52.2	33.3	12

- (i) Draw a scatter diagram for the data. [1]

It is thought that the effect of Rotenone concentration on Chrysanthemum aphids can be modelled by one of the following equations:

$$(A) y = a + b\sqrt{x}$$

$$(B) y = c + d \ln x$$

where a , b , c and d are constants.

- (ii) Find the product moment correlation coefficient between

(a) \sqrt{x} and y

(b) $\ln x$ and y

[2]

- (iii) Explain which of model (A) or (B) is the better model, and find the equation of the least-squares regression line for this model. [3]

An insecticide manufacturer claims its brand of insecticide which contains 11.25 mg/L of Rotenone is effective in killing at least 99.9% of Chrysanthemum aphids.

- (iv) Explain whether the model chosen in (iii) can be used to support the manufacturer's claim. [1]

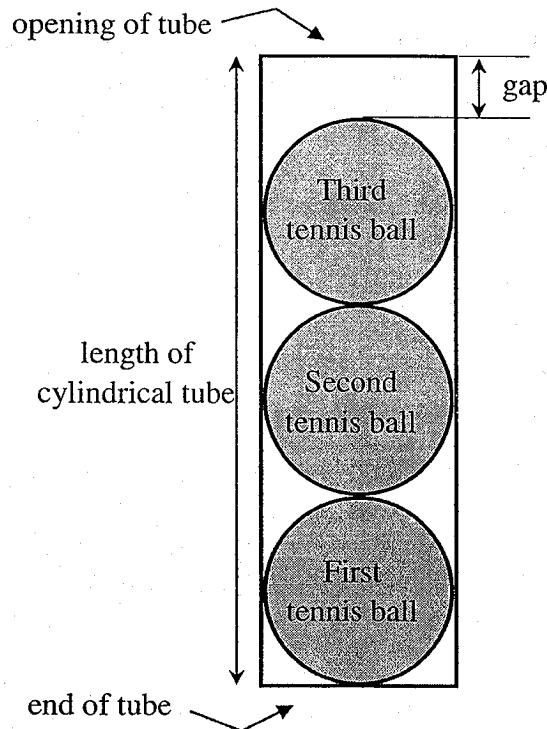
7 A company manufactures tennis balls with radii that are normally distributed with mean 3.3 cm and standard deviation 0.2 cm.

- (i) Find the probability that the radius of a randomly selected tennis ball lies between 3.135 cm and 3.465 cm.

Without any further calculation, explain with the aid of a diagram how the answer obtained would compare with the probability that the radius lies between 3.465 cm and 3.795 cm. [3]

- (ii) 3 tennis balls are randomly selected. Find the probability that the largest tennis ball has radius less than 3.4 cm. [2]

The tennis balls are packed into cylindrical tubes for sale. The cylindrical tubes have lengths that are normally distributed with mean 20 cm and standard deviation 0.3 cm. 3 tennis balls are randomly selected and packed into a randomly selected cylindrical tube such that the first tennis ball is in contact with the end of the tube and each subsequent ball is in contact with its neighbouring ball as shown in the diagram below.



- (iii) The probability that a gap exists between the third tennis ball and the opening of the tube and that the gap is at least k cm long is at least 0.15. Assuming that the centres of all the tennis balls are vertically aligned, find the possible range of values of k . [4]

- 8 Box A contains five cards numbered 1, 1, 2, 2 and 3 and Box B contains three cards numbered 4, 4 and 5. Cards numbered 1 and 5 are red, whereas cards numbered 2, 3 and 4 are blue.

A card is drawn from each of the two boxes. If both cards are the same colour then the score will be the product of the numbers on the two cards. If both cards are different in colour then the score will be the sum of the numbers on the two cards.

Let X be the score obtained. The probability distribution of X is given below:

x	5	7	8	12
$P(X = x)$	h	k	$\frac{1}{3}$	k

- (i) Show that $h = \frac{2}{5}$. [2]
- (ii) Find k , $E(X)$ and $\text{Var}(X)$. [4]
- (iii) Find the probability that the mean score of 60 independent observations of X lies between 6.5 and 7.5. [3]

In a game, a player pays \$20 and draws a card each from Box A and Box B. She receives, in dollars, an amount equal to three times the score obtained. Find the variance of her winnings per game. [2]

- 9 In an industrial process, a machine is programmed to manufacture metal pieces of length 8 cm. An automated quality-control function in the machine takes measurements from a random sample of 48 metal pieces and collates them. X is the random variable that denotes the length of one metal piece.

During a routine inspection, the quality-control manager retrieves one set of readings from the machine and obtains the following information:

$$\sum x = 390, \quad \sum x^2 = 3181.$$

Based on this set of data, test, at the 4% level of significance, whether the mean length of metal pieces produced by the machine is indeed 8 cm. [6]

Explain, in the context of the question, the meaning of "at the 4% level of significance". [1]

A series of 12 routine inspections was carried out over a year. For each inspection, a test, at 4% level of significance, was performed. All the tests concluded that the mean length of metal pieces produced is not 8 cm. Find the expected number of inspections that made an incorrect conclusion. [2]

After a few months of operations, it is found that the length of metal pieces produced by the machine have a standard deviation of 0.4 cm. It is also known that the use of oversized metal pieces in subsequent manufacturing processes is more likely to result in equipment failure and production line disruption.

The quality-control manager is asked to test, at the 3% level of significance, whether the machine is producing oversized metal pieces. If that is the case, the machine will be replaced. A sample of size 80 is used and the sample mean length is denoted by \bar{x} cm. Find the range of values of \bar{x} for which the machine will not be replaced. [4]

- 10 Mrs Gat's confectionery produces a large number of sweets every day. On average, 20% of the sweets are wasabi-flavoured and the rest are caramel-flavoured.

A random sample of n sweets is chosen. If the probability that there are fewer than three wasabi-flavoured sweets in the sample is at most 0.3, find the least possible value of n .

[3]

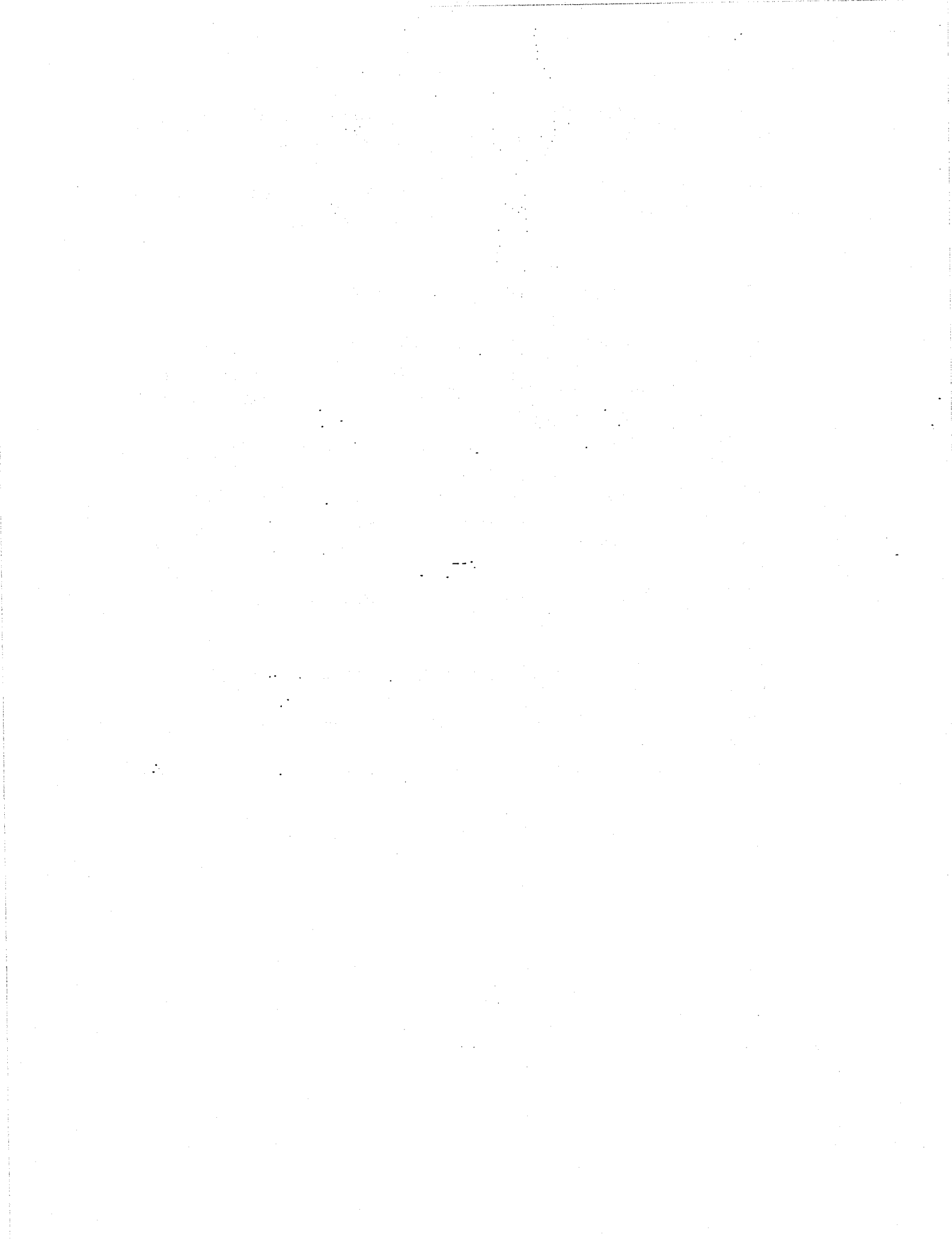
Mrs Gat decides to put the sweets randomly into packets of 20.

- (i) A customer selects packets of 20 sweets at random from a large consignment until she finds a packet with exactly 12 wasabi-flavoured sweets. Give a reason, in the context of the question, why a binomial distribution is not an appropriate model for the number of packets she selects. [1]
- (ii) Find the probability that a randomly chosen packet of sweets contains at least 3 wasabi-flavoured sweets. [2]
- (iii) Find the probability that a randomly chosen packet of sweets contains fewer wasabi-flavoured sweets than caramel-flavoured sweets, given that it contains at least 3 wasabi-flavoured sweets. [3]

Mrs Gat then packs the packets of sweets into boxes. Each box contains 10 packets of sweets.

- (iv) Find the probability that each packet in a randomly chosen box contains at least 3 wasabi-flavoured sweets. [1]
- (v) Find the probability that there are at least 30 wasabi-flavoured sweets in a randomly chosen box. [2]
- (vi) Explain why the answer to (v) is greater than the answer to (iv). [1]

End of Paper



2018 TJC JC2 H2 Mathematics Prelim Exam Paper 1 Solution

- 1 A cubic curve cuts the y -axis at $y=8$ and the x -axis at $x=-2$. The line L , with equation $y=6x-4$, intersects the curve at the point where $x=-3$. The tangent to the curve at the point where $x=-\frac{4}{3}$ is parallel to L . Find the equation of the curve. [5]

[Solution]

Let the equation of the cubic curve be $y = ax^3 + bx^2 + cx + d$

When $x=0, y=8, \Rightarrow d=8$

When $x=-2, y=0, \Rightarrow -8a+4b-2c=-8$ ---- (1)

When $x=-3, y=6(-3)-4=-22$, we get

$-27a+9b-3c+8=-22 \Rightarrow -27a+9b-3c=-30$ ---- (2)

$$\frac{dy}{dx} = 3ax^2 + 2bx + c$$

When $x=-\frac{4}{3}, \frac{dy}{dx}=6$, we get

$3a\left(-\frac{4}{3}\right)^2 + 2b\left(-\frac{4}{3}\right) + c = 6 \Rightarrow \frac{16}{3}a - \frac{8}{3}b + c = 6$ ---- (3)

Using GC, $a=1, b=-1, c=-2$.

Equation of the curve is $y = x^3 - x^2 - 2x + 8$.

Comment

Question is well done by students who recognized that a general cubic equation can be expressed as $y = ax^3 + bx^2 + cx + d$ and then formulated 4 linear equations to solve via GC.

Some students wrongly assumed that the cubic expression is completely factorisable i.e. $y = (x-a)(x-b)(x-c)$ or $y = a(x-b)(x-c)(x-d)$, which are both not true.

Some students also assumed that the coefficient of x^3 is 1 i.e. $y = (x+2)(x^2 + bx + c)$, which is not given in the question.

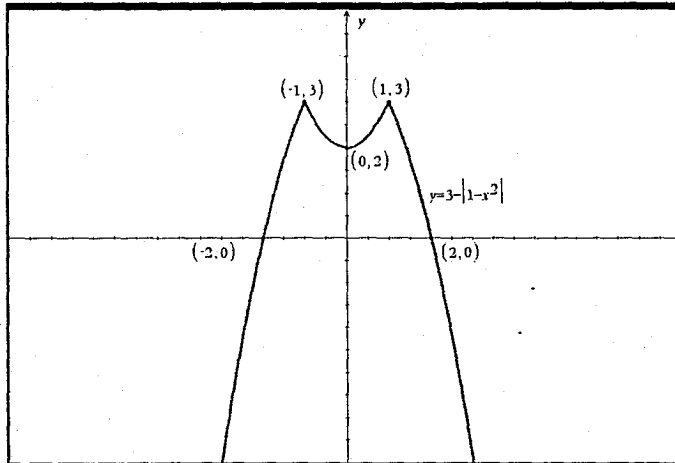
Reminder that the following are not "equations of the curve":

- $x^3 - x^2 - 2x + 8 = 0$
- equation = $x^3 - x^2 - 2x + 8$

2 Sketch the graph of $y = 3 - |1 - x^2|$, indicating clearly all the key features. [2]

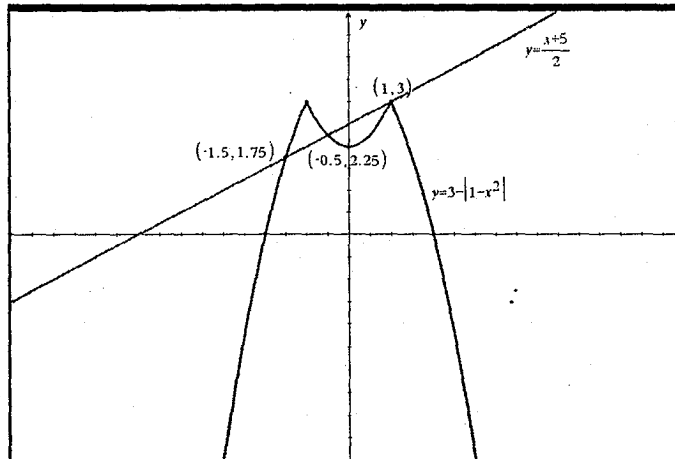
Hence solve the inequality $3 - |1 - x^2| \geq \frac{x+5}{2}$. [3]

[Solution]



[Shape must include sharp points][Show x, y intercepts and (-1,3), (1,3)]

Add the graph $y = \frac{x+5}{2}$ to the same set of axes:



From GC: intersection points are at $(-1.5, 1.75)$, $(-0.5, 2.25)$ and $(1, 3)$.

Hence for $3 - |1 - x^2| \geq \frac{x+5}{2}$, $-1.5 \leq x \leq -0.5$ or $x = 1$.

Comment

The first part on sketching of the curve of $y = 3 - |1 - x^2|$ was generally well done. However, many students fell short of obtaining full credit due to

- Failure to identify the pointed ends at $(-1, 3)$ and $(1, 3)$ as a key feature

- Failure to observe proportionality of the x - and y - scales. The graph needs to look visually proportional in terms of the values on each axis. Note that x - and y - scales need not be the same.
- Failure to observe symmetrical sketching.

The second part was generally well done. However, there were students who gave the presentation

“From GC, $-1.5 \leq x \leq -0.5$ or $x=1$ ”

without any indication of what did they do on the GC. A sketch of the graphs that they used were also not presented. This leads to insufficient working to qualify for award of marks.

A significant group of students missed out on $x=1$.

There were students who went through the entire algebraic working to find the intersection points of the two graphs before referring to the graph to determine the correct ranges. This leads to unnecessary lost of time which could be better used on other questions. A GC could be used to find the intersections since there was no requirement for exact values.

Students who gave the entire algebraic working but did not make specific reference to the graphical sketches when determining the correct ranges were not awarded any marks as it was unclear how the “HENCE” requirement was fulfilled.

- 3 Find the exact volume of the solid of revolution formed when the region bounded by the curve $y = \frac{\sqrt{x}}{4-x^2}$, the y -axis and the line $y = \frac{1}{3}$ is rotated through 2π radians about the x -axis. [5]

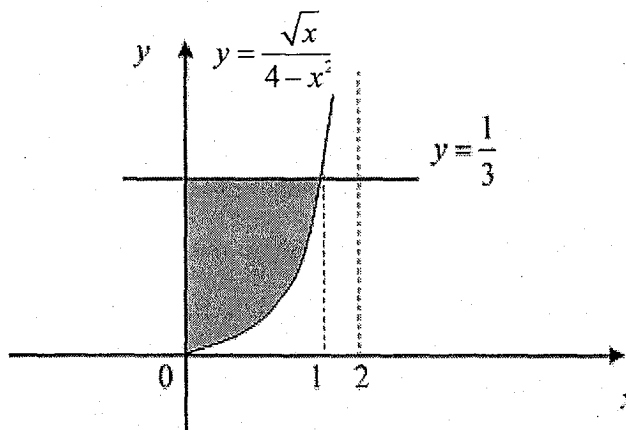
[Solution]

$$\text{When } y = \frac{1}{3}, \frac{\sqrt{x}}{4-x^2} = \frac{1}{3}$$

From GC, $x = 1$

Required volume

$$\begin{aligned} &= \pi \left(\frac{1}{3}\right)^2 (1) - \pi \int_0^1 \left[\frac{\sqrt{x}}{4-x^2}\right]^2 dx \\ &= \frac{1}{9}\pi - \pi \int_0^1 \frac{x}{(4-x^2)^2} dx \\ &= \frac{1}{9}\pi + \frac{1}{2}\pi \int_0^1 (-2x)(4-x^2)^{-2} dx \\ &= \frac{1}{9}\pi - \frac{1}{2}\pi \left[\frac{1}{4-x^2}\right]_0^1 \\ &= \frac{1}{9}\pi - \frac{1}{2}\pi \left(\frac{1}{3} - \frac{1}{4}\right) \\ &= \frac{5}{72}\pi \text{ units}^3 \end{aligned}$$



Comment

This question was in general well done, with quite a number of students scoring full marks. Some students wasted time solving for the intersection point $(1, \frac{1}{3})$.

Common mistakes seen were:

- Found the volume of revolution of the region under the curve from $x = 0$ to $x = 1$ instead.
- Mistook the height of the cylinder as the radius.
- Forgot to square the expression $\frac{\sqrt{x}}{4-x^2}$ in the integral and/or forgot to multiply by π .
- Failed to see that the formula for integrating $\int f'(x)[f(x)]^n dx$ could be applied. Some students resorted to finding the partial fractions of $\frac{x}{(4-x^2)^2}$ which was very tedious and most either got the partial fractions wrong or gave up half way.

- 4 A sequence u_1, u_2, u_3, \dots is such that $u_n = \frac{1}{1+a^n}$ and $u_{n+1} = u_n - \frac{a^n(a-1)}{(1+a^{n+1})(1+a^n)}$, where a is constant such that $0 < a < 1$.

(i) Find $\sum_{n=1}^N \frac{a^n}{(1+a^{n+1})(1+a^n)}$ in terms of a and N . [3]

(ii) Determine whether $\sum_{n=1}^{\infty} \frac{a^n}{(1+a^{n+1})(1+a^n)}$ is a convergent series. [2]

[Solution]

(i) $\sum_{n=1}^N \frac{a^n}{(1+a^{n+1})(1+a^n)}$

$$= \frac{1}{a-1} \sum_{n=1}^N (u_n - u_{n+1})$$

$$= \frac{1}{a-1} [u_1 - u_2 + u_2 - u_3 + \dots + u_{N-1} - u_N + u_N - u_{N+1}]$$

$$= \frac{1}{a-1} [u_1 - u_{N+1}]$$

$$= \frac{1}{a-1} \left(\frac{1}{1+a} - \frac{1}{1+a^{N+1}} \right)$$

Given $u_{n+1} = u_n - \frac{a^n(a-1)}{(1+a^{n+1})(1+a^n)}$

$$\Rightarrow \frac{a^n}{(1+a^{n+1})(1+a^n)} = \frac{1}{a-1} (u_n - u_{n+1})$$

Given $u_n = \frac{1}{1+a^n}$, $u_{N+1} = \frac{1}{1+a^{N+1}}$

- (ii) Recall that a series $\sum_{r=1}^{\infty} u_r$ converges if as $n \rightarrow \infty$, $\sum_{r=1}^n u_r \rightarrow$ a finite number

As $N \rightarrow \infty$, $a^{N+1} \rightarrow 0$ ($\because 0 < a < 1$), and so $\frac{1}{1+a^{N+1}} \rightarrow 1$

$$\sum_{n=1}^N \frac{a^n}{(1+a^{n+1})(1+a^n)} = \frac{1}{a-1} \left(\frac{1}{1+a} - \frac{1}{1+a^{N+1}} \right) \rightarrow \frac{1}{a-1} \left(\frac{1}{1+a} - 1 \right) \text{ which is a finite number}$$

Thus $\sum_{n=1}^{\infty} \frac{a^n}{(1+a^n)(1+a^{n-1})} = \frac{1}{a-1} \left(\frac{1}{1+a} - 1 \right)$ is a convergent series.

Marker's comments:

Half of the cohort did not realise that the sum of the series could be obtained using method of difference. Some students mistaken the series as geometric.

Common mistakes made were as follows:

- As $n \rightarrow \infty$, $a^{n+1} \rightarrow 0$, $\frac{1}{1+a^{n+1}} \rightarrow 0$ (should be N instead of n ; also $\frac{1}{1+a^{N+1}} \rightarrow 1$, not 0)
- As $N \rightarrow \infty$, $a^{N+1} \rightarrow \infty$ (did not use given info that $0 < a < 1$)

Good to know:

As $n \rightarrow \infty$, $u_n \rightarrow 0$ does not imply that $\sum_{n=1}^{\infty} u_n$ is convergent.

A counter-example is $\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots$ (called the harmonic series)

See that $n \rightarrow \infty$, $\frac{1}{n} \rightarrow 0$ i.e. the sequence converges to zero.

However,

$$\begin{aligned} & \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \dots \\ & > \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \dots \\ & = \frac{1}{1} + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \left(\frac{1}{16} + \dots + \frac{1}{16}\right) + \dots \\ & = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots \text{ which is infinite} \end{aligned}$$

i.e. $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent series

5 The function f is defined by $f : x \mapsto (x^2 - 4)^2 - 24, x \in \mathbb{R}$.

(i) Sketch the graph of $y = f(x)$, indicating clearly all intercepts and stationary points. [2]

(ii) Explain why f^{-1} does not exist. [1]

(iii) The functions f_1 and f_2 are defined by

$$f_1 : x \mapsto (x^2 - 4)^2 - 24, x \in \mathbb{R}, x \leq k,$$

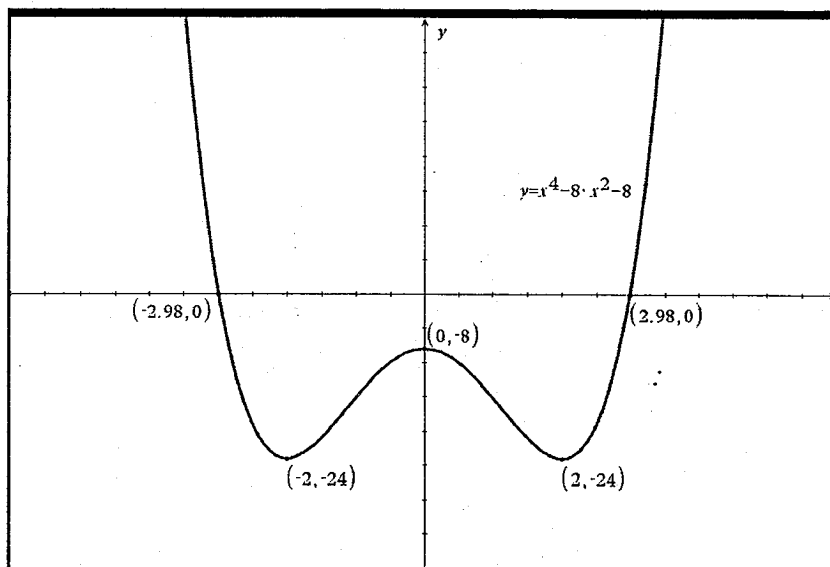
$$f_2 : x \mapsto (x^2 - 4)^2 - 24, x \in \mathbb{R}, x > k,$$

where k is a real number. State the range of values of k for which f_1^{-1} exists and f_2^{-1} does not exist. [1]

(iv) Using the largest possible value of k found in (iii), find f_1^{-1} in a similar form. [4]

[Solution]

(i)



(ii) The line $y = 0$ cuts the graph of $y = f(x)$ more than once. Hence f is not 1-1 and thus f^{-1} does not exist.

(iii) $k \leq -2$

(iv) Let $y = (x^2 - 4)^2 - 24$

$$(x^2 - 4)^2 = y + 24 \Rightarrow x^2 - 4 = \sqrt{y + 24} \text{ since } x^2 - 4 \geq 0$$

$$\Rightarrow x^2 = 4 + \sqrt{y + 24} \Rightarrow x = -\sqrt{4 + \sqrt{y + 24}} \text{ since } x \leq -2$$

$$f_1^{-1} : x \mapsto -\sqrt{4 + \sqrt{x + 24}}, x \in \mathbb{R}, x \geq -24$$

Comment

- Students found this question manageable, but are making several technical errors.
- A significant number of students labelled the x -intercepts as 3 instead of 2.98.
- Many students are still unable to phrase their explanation for why the inverse does not exist properly.
- It is surprising that a majority of students do not recognise that part (iii) asks for a **single** range of values of k that addresses both conditions.
- Find inverse in (iv): students are mostly able to manipulate the expression correctly, but are unable to justify their choice of + or – clearly, especially in justifying
$$(x^2 - 4)^2 = y + 24 \Rightarrow x^2 - 4 = \sqrt{y + 24}$$
- A good number of students wrote the range of f_1 as $(\infty, -24]$ instead of $[-24, \infty)$ and ended up writing the domain of the inverse function as $x \leq -24$.

6 It is given that

$$g(x) = \begin{cases} x+2 & \text{for } 0 < x \leq 2, \\ 2 + \sqrt{4 - (x-2)^2} & \text{for } 2 < x \leq 4, \end{cases}$$

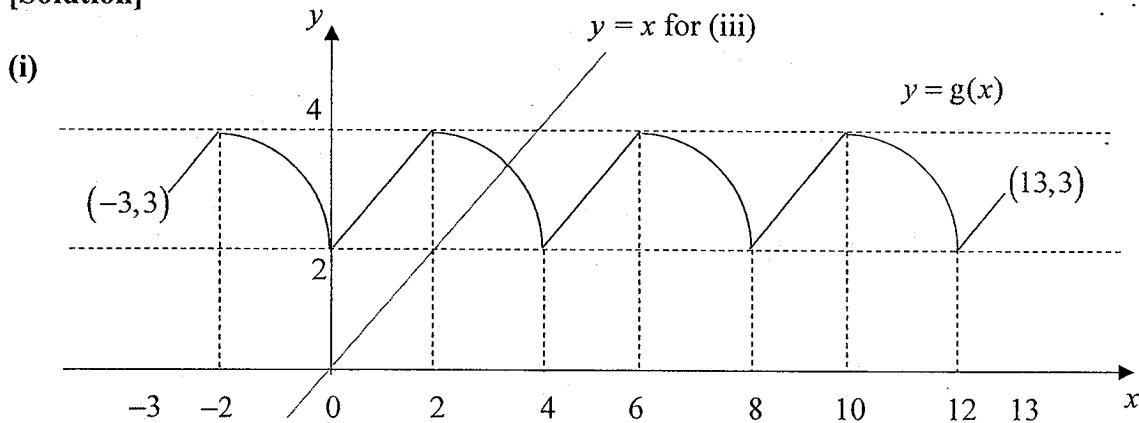
and that $g(x) = g(x+4)$ for all real values of x .

(i) Sketch the graph of $y = g(x)$ for $-3 \leq x \leq 13$. [3]

(ii) Without using integration, find the exact value of $\int_{-3}^{13} g(x) dx$. [2]

(iii) Solve the equation $g(x) = x$ exactly. [3]

[Solution]



(ii) $\int_{-3}^{13} g(x) dx = \text{Area} = [13 - (-3)](2) + 4 \left[\frac{1}{2}(2)(2) \right] + \pi(2)^2$
 $= (40 + 4\pi) \text{ unit}^2$

(iii) Using graph in (i), the line $y = x$ cuts the graph of $y = g(x)$ in the interval where $2 < x < 4$,

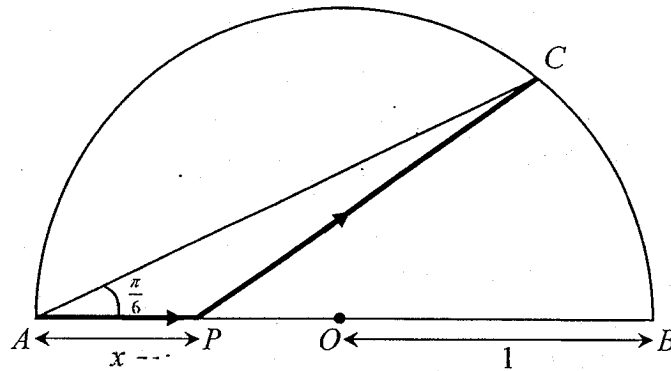
$$\begin{aligned} \bullet \quad 2 + \sqrt{4 - (x-2)^2} &= x \\ \Rightarrow x - 2 &= \sqrt{4 - (x-2)^2} \\ \Rightarrow (x-2)^2 &= 4 - (x-2)^2 \\ \Rightarrow (x-2)^2 &= 2 \\ \Rightarrow x &= 2 + \sqrt{2} \text{ since } x > 2. \end{aligned}$$

Comment

- Common errors – endpoints not labelled, giving poor reasons for rejecting the answer in (ii).
- Many students had the idea for (ii) but were not careful, leaving out one section (or many sections) of the region to be calculated.

Reasoning in (iii) for which function to start the equation with was poor. Ideally, a sketch graph will justify the choice of $2 + \sqrt{4 - (x-2)^2} = x$ but most students simply did not explain why they started that way.

- 7 The figure shows a plot of rough terrain that is in the shape of a semicircle, centred at O with a radius of 1 km. A group of explorers arrives at the point A and needs to get to their final destination at the point C where the chord AC makes an angle of $\frac{\pi}{6}$ with the diameter AB . To do so, they decide to walk x km along the diameter AB to a variable point P before cutting across the rough terrain in the direction PC . It is known to them that OB , a portion of the path AB , was destroyed by an earthquake and impassable.



- (i) Write down the angle COA . Hence show that $CP = \sqrt{3-3x+x^2}$. [3]
- (ii) The explorers need to reach C as soon as possible as it is getting dark and dangerous. They can walk along AB at a speed of 5 km/h but will need to slow down to a speed of 3 km/h when crossing the rough terrain. Show that the time taken for the explorers to reach point C is shortest when x satisfies the equation

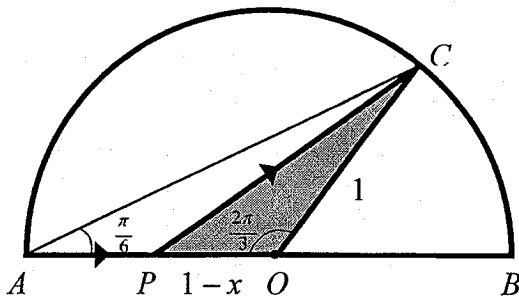
$$64x^2 - 192x + 117 = 0.$$

Hence find this shortest time.

[6]

[Solution]

(i)



$$\angle COA = \pi - 2\left(\frac{\pi}{6}\right) = \frac{2\pi}{3}$$

Consider triangle COP

$$CP^2$$

$$= OP^2 + OC^2 - 2(OP)(OC) \cos \angle COA$$

$$= (1-x)^2 + 1 - 2(1-x)(1) \cos \frac{2\pi}{3}$$

$$= 1 - 2x + x^2 + 1 - 2(1-x) \left(-\frac{1}{2}\right)$$

$$= 3 - 3x + x^2$$

$$CP = \sqrt{3 - 3x + x^2}$$

(since $CP > 0$) (shown)

(ii) Time taken to travel from point A to point C ,

$$T = \frac{x}{5} + \frac{\sqrt{3-3x+x^2}}{3}$$

$$\frac{dT}{dx} = \frac{1}{5} + \frac{1}{6} \frac{2x-3}{\sqrt{3-3x+x^2}}$$

$$\frac{dT}{dx} = 0 \Rightarrow \frac{1}{5} + \frac{1}{6} \frac{2x-3}{\sqrt{3-3x+x^2}} = 0$$

$$-5(2x-3) = 6\sqrt{3-3x+x^2}$$

$$25(4x^2 - 12x + 9) = 36(3 - 3x + x^2)$$

$$64x^2 - 192x + 117 = 0$$




Solving by GC:

$$x = 0.85048$$

$$\approx 0.850 \text{ (3 s.f.)} \quad \text{or } x = 2.1495 \text{ (reject since } 0 < x < 1)$$

$$T = \frac{0.85048}{5} + \frac{\sqrt{3-3(0.85048)+(0.85048)^2}}{3} \approx 0.531 \text{ (3 s.f.)}$$

[Accept use of graph of T against x to find the shortest T value and verify min point]

x	0.85048^-	0.85048	0.85048^+
$\frac{dT}{dx}$	- ve	0	+ ve
slope			

Using first derivative test, shortest time taken is 0.531 hours when $x = 0.850$ km.

Comment

(i) Generally well done. Students who could not show the length of CP quoted the cosine rule wrongly (the most common error is students put a + $2(OP)(OC)\cos\angle COA$)

(ii) Common errors are listed below:

- Some students need to read carefully from the question that time taken minimised. Some assume that to minimise time is to minimise distance which may not be true.
- Many students wrote the expression for time wrongly, for example $T = 5x + 3\sqrt{3-3x+x^2}$.
- A number of students made errors such as $T = \frac{x}{5} + \frac{\sqrt{3-3x+x^2}}{3} \Rightarrow T = \frac{x^2}{25} + \frac{3-3x+x^2}{9}$.
- Many students did not reject the other solution.

Many students did not verify minimum point using the first/second derivative test.

- 8 Kesamet Bank offers its customers a housing loan and charges a monthly interest such that t months after taking the loan, the amount of money owed to the bank, $\$x$, increases at a rate proportional to the amount of money owed. Both x and t are taken to be continuous variables. Customers are required to make a fixed monthly repayment of $\$r$ until the loan is fully repaid.

Given that the monthly interest and the repayment amount are the same when $x = a$, show that

$$\frac{dx}{dt} = \frac{r}{a}(x - a). \quad [3]$$

Given also that the loan taken from the bank is $\$A$, find x in terms of t, r, a and A . [5]

Sketch the graph of x against t in the case where the loan can be repayable within a finite duration and state the range of values of a in terms of A in this case. [2]

Hock Yau takes a loan of $\$250000$ to finance the purchase of his new apartment and he decides to repay a fixed amount of $\$5000$ monthly. Given that the monthly interest is the same as the repayment amount of $\$5000$ when he owes the bank $\$255000$, determine the duration, to the nearest month, he needs to repay his loan fully. [2]

[Solution]

$$\frac{dx}{dt} = kx - r, \quad k > 0$$

Rate of increase in $x = kx$
Rate of decrease in $x = r$
 \therefore Net rate of change in $x = kx - r$

$$\text{At } x = a, \quad \frac{dx}{dt} = 0$$

$$\Rightarrow ka - r = 0 \Rightarrow k = \frac{r}{a}$$

$$\therefore \frac{dx}{dt} = \frac{r}{a}x - r$$

$$= \frac{r}{a}(x - a) \quad (\text{shown})$$

$$\int \frac{1}{x-a} dx = \int \frac{r}{a} dt$$

$$\ln|x-a| = \frac{r}{a}t + C$$

modulus is necessary

$$|x-a| = e^{\frac{r}{a}t + C}$$

$$x-a = \pm e^{\frac{r}{a}t + C}$$

$$x = a + Be^{\frac{r}{a}t}, \quad B = \pm e^C$$

Do not use A here as A is the bank loan in the question

$$\text{At } t=0, \quad x = A$$

Substitute into equation without modulus, not earlier

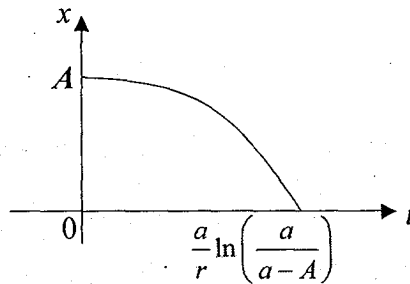
$$\Rightarrow A = a + B \Rightarrow B = A - a$$

$$\therefore x = a + (A - a)e^{\frac{r}{a}t}$$

$$x = a + (A - a)e^{\frac{r}{a}t}$$

Since a and r are positive, $\frac{r}{a} > 0$ and so $e^{\frac{r}{a}t}$ is an increasing function.

For the loan to be repayable within a finite duration, x has to be a **decreasing** function (i.e. x decreases to zero). Thus $A - a < 0 \Rightarrow a > A$



Let $x \leq 0$ (or $x = 0$)

For $A = 250000$, $r = 5000$ and $a = 255000$ (using earlier info)

$$(250000 - 255000)e^{\frac{5000t}{255000}} + 255000 \leq 0$$

$$-5000e^{\frac{t}{51}} + 255000 \leq 0$$

$$-5000e^{\frac{t}{51}} \leq -255000 \quad (\text{or use GC to solve this inequality})$$

$$e^{\frac{t}{51}} \geq 51$$

$$\frac{t}{51} \geq \ln 51$$

$$t \geq 51 \ln 51$$

$$t \geq 200.52$$

He will need 201 months.

Marker's comments:

Near one-third of the students could not formulate the differential equation, giving wrong DEs

such as $\frac{dx}{dt} = k(x - r)$ or $\frac{dx}{dt} = kx - rt$.

Most students were able to solve the DE almost correctly – some missed out the modulus or \pm sign while some did not realise they should use B (and not A) to denote the arbitrary constant.

The second half of the question was very badly answered.

For the loan to be repayable in a finite duration, less than half of the students correctly sketched a decreasing solution curve (and many did not indicate A when $t = 0$). Only a small proportion of students were able to state that $a > A$.

Many students did not realise they could use the earlier info (“*monthly interest and the repayment amount are the same when $x = a$* ”) to deduce that $a = 255000$. These students misinterpreted the question and many attempted to solve for a by substituting x as 225000.

9 Do not use a calculator in answering this question.

(a) It is given that one of the roots of the equation $z^4 + pz^3 + 7z^2 + 6z - 30 = 0$ is $1 + 3i$.

(i) Evaluate $(1 + 3i)^4$, showing your calculations clearly. [2]

(ii) Show that $p = -2$. [2]

(iii) Find the other roots of the equation in exact form. [5]

(b) The complex number z is given by $z = (\sqrt{2})e^{i\left(\frac{5}{12}\pi\right)}$.

(i) Given that $w = \frac{z^3}{z^*}$, find w in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [3]

(ii) Find the smallest positive integer n such that w^n is a real number. [3]

[Solution]

(a)(i) $(1 + 3i)^2 = -8 + 6i$

$$(1 + 3i)^4 = (-8 + 6i)^2$$

$$= 28 - 96i$$

(ii) Since one of the roots of the equation is $1 + 3i$

$$\Rightarrow (1 + 3i)^4 + p(1 + 3i)^3 + 7(1 + 3i)^2 + 6(1 + 3i) - 30 = 0$$

$$\Rightarrow 28 - 96i + p(-26 - 18i) + 7(-8 + 6i) + 6(1 + 3i) - 30 = 0$$

$$\Rightarrow -52 - 26p - 36i - 18ai = 0$$

$$p(-26 - 18i) = 52 + 36i$$

$$p = \frac{52 + 36i}{-26 - 18i}$$

$$p = -2$$

[Note that p is not given as real here]

(iii) As the coefficients are all real, complex roots occur in conjugate pairs,

\therefore another root is $1 - 3i$.

$$z^4 - 2z^3 + 7z^2 + 6z - 30 = 0$$

$$[z - (1 + 3i)][z - (1 - 3i)](z^2 + bz + c) = 0$$

$$[(z - 1) - 3i][(z - 1) + 3i](z^2 + bz + c) = 0$$

$$[(z - 1)^2 + 9](z^2 + bz + c) = 0$$

$$(z^2 - 2z + 10)(z^2 + bz + c) = 0$$

By comparing coefficients: $z^4 - 2z^3 + 7z^2 + 6z - 30 = (z^2 - 2z + 10)(z^2 + bz + c)$

Comparing constant term: $10c = -30$, $\therefore c = -3$.

Comparing coefficient of z : $6 = -2c + 10b$, $\therefore b = 0$.

Or by long division, the other quadratic factor is $z^2 - 3$.

$$z^2 - 3 = 0$$

$$z = \sqrt{3} \text{ or } -\sqrt{3}$$

The other 3 roots are $1-3i$, $\sqrt{3}$ and $-\sqrt{3}$.

$$\begin{aligned} \text{(b)(i)} \quad w &= \frac{z^3}{z^*} \\ &= \frac{(\sqrt{2})^3 e^{i3\left(\frac{5}{12}\pi\right)}}{(\sqrt{2}) e^{-i\left(\frac{5}{12}\pi\right)}} \\ &= 2e^{\frac{i5\pi}{3}} \\ &= 2e^{-i\frac{\pi}{3}} \end{aligned}$$

(ii) **Method 1**

$$\text{For } w^n \text{ to be real, } \sin\left(-\frac{n\pi}{3}\right) = 0 \Rightarrow \sin\left(\frac{n\pi}{3}\right) = 0$$

$$\frac{n\pi}{3} = 0, \pi, 2\pi, \dots$$

$$n = 3, 6, 9, \dots$$

Smallest positive integer $n = 3$

OR

Method 2

$$-\frac{n\pi}{3} = k\pi, \text{ where } k \in \mathbb{Z}^-$$

$$n = -3k$$

Smallest positive integer $n = 3$ when $k = -1$

Comment

(a)(i) Generally well done. However, a significant group of students expanded $(1+3i)^4 = (1+6i-9)^2 = \dots$ without simplifying $(1+6i-9)^2$ to $(-8+6i)^2$. This seem to indicate a lack of numerical sensitivity amongst students.

(a)(ii) Very poorly done. Many student arrived at $-26p-18pi = 52+36i$ and went ahead to compare real and imaginary parts. Note that this is only possible if the question defined p to be real, which in this case, there was no definition.

The basis of comparing real and imaginary parts is premised on the assumption that the coefficients are well defined to be real numbers.

(a)(iii) Generally well done. However, many students quoted $1 - 3i$ to be a root without explaining why the complex conjugate root theorem is applicable.

(b)(i) This part was not very well done. Many students thought that the conversion to principal range involves subtraction (or addition) in multiples of π . It should be 2π .

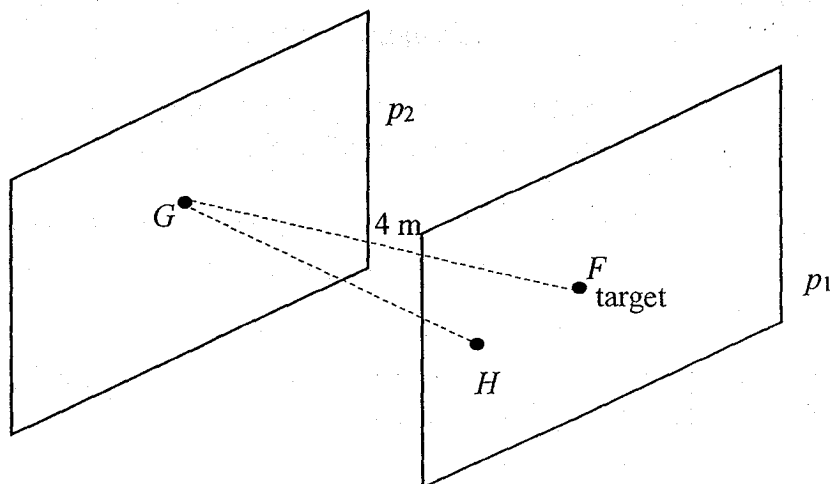
A significant number of students failed to apply the range of validity and left the argument as $\frac{5}{3}\pi$.

(b)(ii) Most students were unable to give a complete presentation of the solution to include both positive and negative integral multiples of π

i.e. $\frac{n\pi}{3} \dots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi \dots$. Those that were able to generalize the possible

arguments as $\frac{n\pi}{3} = k\pi$ did not define k to be an integer or wrongly defined k to be real.

10



The diagram shows two parallel boards p_1 and p_2 modelled by planes with equations $4x - 3z = -2$ and $ax + z = b$ respectively. p_1 and p_2 are placed vertically on a horizontal ground, with a distance of 4 m apart.

State the value of a . [1]

Robin places a gun through a hole at point G on p_2 and aims to shoot in a direction perpendicular to p_1 at a target F on p_1 , with coordinates $\left(-\frac{1}{2}, 1, 0\right)$. It can be assumed that all shots travel in a straight line.

Show that a set of possible coordinates of G are $\left(-\frac{37}{10}, 1, \frac{12}{5}\right)$. [3]

For the rest of this question, take the coordinates of G to be $\left(-\frac{37}{10}, 1, \frac{12}{5}\right)$.

(i) Find the value of b . [1]

The bullet from G travels in a path parallel to the vector $-\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and ends up at a point H on p_1 instead.

(ii) Find the acute angle between the path travelled by the bullet and p_1 . Hence find the exact distance between H and F . [4]

Due to long term usage, p_1 becomes slanted such that the new slanted board p_3 is now modelled by a plane that passes through the origin and contains the line with equation

$$x + 2 = \frac{y - 1}{2} = \frac{3z + 6}{4}.$$

(iii) Find a vector equation of p_3 , in scalar product form. [2]

(iv) Find a vector equation of the line of intersection between p_1 and p_3 . [2]

[Solution]

$$\text{Equation of } p_2: \vec{r} \cdot \begin{pmatrix} a \\ 0 \\ 1 \end{pmatrix} = b \quad \text{and equation of } p_1: \vec{r} \cdot \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} = -2$$

$$\text{Since } p_2 \text{ parallel to } p_1, \begin{pmatrix} a \\ 0 \\ 1 \end{pmatrix} = k \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} \Rightarrow k = -\frac{1}{3} \therefore a = -\frac{4}{3}$$

Comment:

The method of using cross product of parallel vectors is zero vector is an alternative method but is more tedious.

$$\vec{OF} = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{OG} = \vec{OF} + \alpha \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$$

$$\text{Since } \left| \alpha \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} \right| = 4 \Rightarrow \alpha = \pm \frac{4}{5}$$

$$\vec{OG} = \vec{OF} + \alpha \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} \pm \frac{4}{5} \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 27/10 \\ 1 \\ -12/5 \end{pmatrix} \text{ or } \begin{pmatrix} -37/10 \\ 1 \\ 12/5 \end{pmatrix}$$

A set of possible coordinates of G is $\left(-\frac{37}{10}, 1, \frac{12}{5}\right)$. (Shown)

Alternative:

$$\vec{OG} = \vec{OF} \pm 4\mathbf{n}$$

where \mathbf{n} is unit vector parallel to \vec{GF} .

$$= \vec{OF} \pm \frac{4}{5} \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 27/10 \\ 1 \\ -12/5 \end{pmatrix} \text{ or } \begin{pmatrix} -37/10 \\ 1 \\ 12/5 \end{pmatrix}$$

There must be explanation of how $-\frac{4}{5}$ is

obtained to show that G is $\left(-\frac{37}{10}, 1, \frac{12}{5}\right)$.

There must be evidence of making use of $|\vec{GF}| = 4$ in the proof. By substituting in

coordinates of G to see that you obtain

$\alpha = -\frac{4}{5}$ is not a "show" but is purely "verifying".

(i) G is a point on p_2

[Make sure you use equation of p_2 where $\vec{r} \cdot \begin{pmatrix} -4 \\ 3 \\ 0 \\ 1 \end{pmatrix} = b$. If you use normal as $\begin{pmatrix} -4 \\ 0 \\ 3 \end{pmatrix}$ then

equation of p_2 becomes $\vec{r} \cdot \begin{pmatrix} -4 \\ 0 \\ 3 \end{pmatrix} = 3b$ (That is, RHS of equation of p_2 is no longer b)

$$\begin{pmatrix} -37 \\ 10 \\ 1 \\ \frac{12}{5} \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 3 \\ 0 \\ 1 \end{pmatrix} = b \Rightarrow b = \frac{22}{3}$$

(ii) Let θ (which is $\angle GHF$) be the required acute angle between path along GH and p_1 .
(Note that $\angle GFH = 90^\circ$, $\angle GFH$ is NOT the angle between path travelled by bullet and p_1)

$$\begin{aligned} \cos(90^\circ - \theta) &= \frac{\left| \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right|}{5\sqrt{6}} = \frac{2}{\sqrt{6}} \\ \theta &= 90^\circ - 35.264^\circ \\ &= 54.736^\circ \\ &\approx 54.7^\circ \text{ (1 d.p.)} \end{aligned}$$

Alternative:

$$\sin \theta = \frac{\left| \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right|}{5\sqrt{6}} = \frac{2}{\sqrt{6}}$$

In above, we are using $\cos(90^\circ - \theta) = \sin \theta$

Thus it is still dot, not cross!

To find exact length of HF :

[Note that in the sentence " G travels in a path parallel to the vector $-\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and ends

up at a point H on p_1 ", it only means that \overline{GH} is parallel to vector $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ but $\overline{GH} \neq \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$.

$\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ is also NOT \overline{OH} . Thus those who tried to find the length of projection of $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

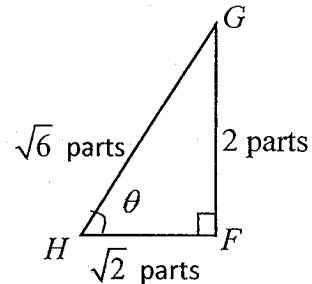
onto the normal vector $\begin{pmatrix} -4 \\ 0 \\ 3 \end{pmatrix}$ fails to find the answer. Furthermore, question stated

“Hence find....” So result from part (ii) must be used]

$$\text{From (ii), } \sin \theta = \frac{2}{\sqrt{6}} \Rightarrow \tan \theta = \frac{2}{\sqrt{2}}$$

$$\tan \theta = \frac{FG}{HF} = \frac{4}{HF}$$

$$\therefore HF = 2\sqrt{2} \text{ units}$$



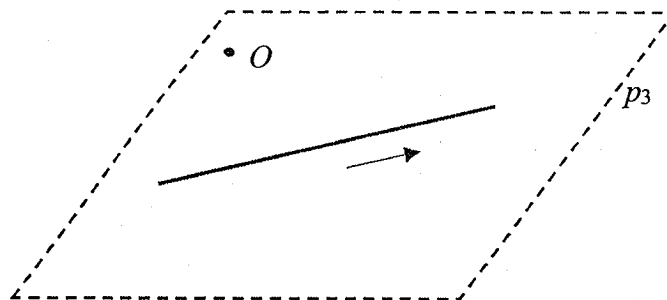
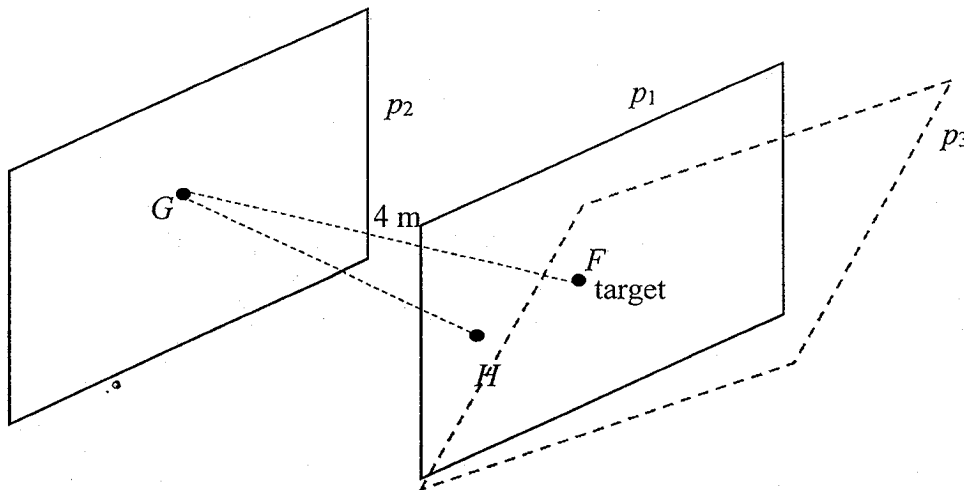
Comment:

By writing $\tan 54.7^\circ = \frac{4}{HF} \Rightarrow HF = 2\sqrt{2}$ lacks sufficient working to show how this exact form is obtained.

(ii) “ p_1 becomes slanted such that the new slanted board p_3 is now modelled by a plane that passes through the origin and contains the line with equation $x + 2 = \frac{y - 1}{2} = \frac{3z + 6}{4}$,”

Things to note:

- Points such as F and H which were on p_1 CANNOT be assumed to be points on p_3
- p_3 passes through origin O means that its equation must be of the form $\mathbf{r} \cdot \mathbf{n} = 0$



Solution to (ii)

$$\text{Equation of given line } x+2 = \frac{y-1}{2} = \frac{3z+6}{4}$$

$$\text{Let } x+2 = \frac{y-1}{2} = \frac{3z+6}{4} = \lambda$$

$$x = -2 + \lambda$$

$$y = 1 + 2\lambda$$

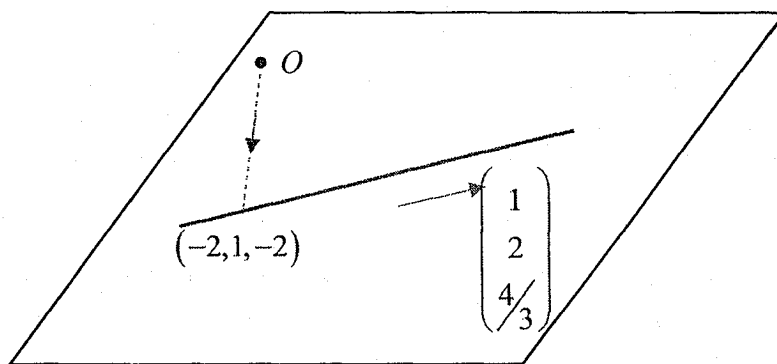
$$z = -2 + \frac{4}{3}\lambda$$

\therefore vector equation form of given line (Note that this is NOT equation of p_3 !) is

$$r = \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ \frac{4}{3} \end{pmatrix}, \lambda \in \mathbb{R}$$

$$n_3 = \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ \frac{4}{3} \end{pmatrix} = \begin{pmatrix} \frac{16}{3} \\ \frac{2}{3} \\ -5 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 16 \\ 2 \\ -15 \end{pmatrix}$$

$$\text{Equation of } p_3 \text{ is } r \cdot \begin{pmatrix} 16 \\ 2 \\ -15 \end{pmatrix} = 0$$



[Scalar Product Form]

Note :

$$r = \mu \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 2 \\ \frac{4}{3} \end{pmatrix} \text{ where } \mu, \alpha \in \mathbb{R} \text{ is the parametric form of the equation of } p_3$$

(iv)

$$p_1: 4x - 3z = -2$$

$$p_3: 16x + 2y - 15z = 0$$

Using GC, equation of line of intersection is:

$$r = \begin{pmatrix} -\frac{1}{2} \\ 4 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} \frac{3}{4} \\ \frac{3}{2} \\ 1 \end{pmatrix}, \beta \in \mathbb{R}$$

Other method without using GC:

Equation of p_3 using $\underline{r} = \begin{pmatrix} -2\mu + \alpha \\ \mu + 2\alpha \\ -2\mu + \frac{4}{3}\alpha \end{pmatrix}$, and substitute into $\underline{r} \cdot \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} = -2$,

$$\begin{pmatrix} -2\mu + \alpha \\ \mu + 2\alpha \\ -2\mu + \frac{4}{3}\alpha \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} = -2, \text{ we obtain } -8\mu + 4\alpha + 6\mu - 4\alpha = -2$$

$\therefore \mu = 1$ and subst. back into equation of p_3 :

$$\underline{r} = \begin{pmatrix} -2 + \alpha \\ 1 + 2\alpha \\ -2 + \frac{4}{3}\alpha \end{pmatrix} \Rightarrow \underline{r} = \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 2 \\ \frac{4}{3} \end{pmatrix}, \quad \alpha \in \mathbb{R}$$

General Comment:

Q10 is very badly done.

- Many students are still very confused with the concept of “parallel vectors” and “equal vectors”, not knowing when to use them.

For example, some students tried very hard to find \overline{GH} and \overline{GF} so that they can find the acute angle $\angle GHF$, without realising that the formula $\cos \theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}| |\underline{b}|}$ can be applied

using vectors that are parallel to \overline{GH} and \overline{GF} .

On the other hand when finding length of HF , many students attempt to use

$$\frac{\left| \overline{GH} \cdot \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} \right|}{\sqrt{4^2 + 3^2}} = 4 \text{ but did not realise that } \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \text{ cannot be taken as } \overline{GH}.$$

- Conceptual meaning of \underline{r} in vector equation of line is still very poor among many students. \underline{r} is either missing from equation of line or students do not understand what

it meant by position vector of point on line. Quite a handful of students did not realise

$$\text{that } \overrightarrow{GF} \neq \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}.$$

- Students are generally very careless in :

Finding cross product

Conversion from Cartesian form of equation of line to vector equation form

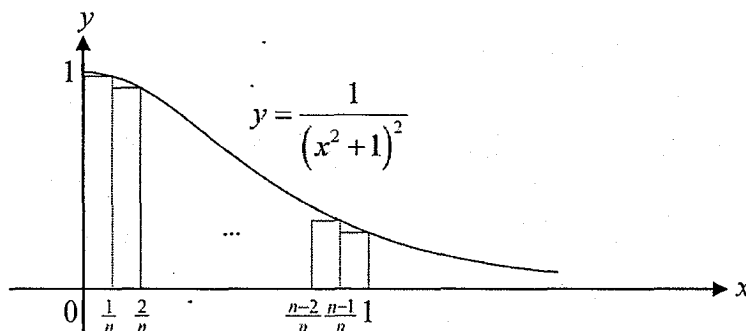
Algebraic manipulation of the scalar when finding $a = -\frac{4}{3}$

- 11 (a) Find $\int \ln(x+1) dx$. [3]

Hence find in terms of k , the area bounded by the curve $y = \ln(x+1)$, the x -axis and the lines $x = -k$ and $x = k$, where $0 < k < 1$. [3]

- (b) (i) By using the substitution $x = \tan \theta$, find the exact value of $\int_0^1 \frac{1}{(x^2+1)^2} dx$. [6]

(ii)



The figure above shows the graph of $y = \frac{1}{(x^2+1)^2}$ for $x \geq 0$. Rectangles,

each of width $\frac{1}{n}$, are drawn under the curve. Show that the total area of all

n rectangles can be written as $\sum_{r=1}^n \frac{n^3}{(r^2+n^2)^2}$. [2]

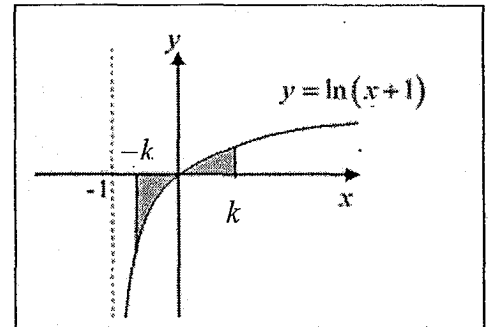
- (iii) Hence state the exact value of $\lim_{n \rightarrow \infty} \left[\sum_{r=1}^n \frac{n^3}{(r^2+n^2)^2} \right]$. [1]

[Solution]

$$\begin{aligned}
 \text{(a)} \quad & \int \ln(x+1) \, dx \\
 &= x \ln(x+1) - \int \frac{x}{x+1} \, dx \\
 &= x \ln(x+1) - \int \left(1 - \frac{1}{x+1}\right) \, dx \\
 &= x \ln(x+1) - x + \ln(x+1) + C \\
 &= (x+1) \ln(x+1) - x + C
 \end{aligned}$$

Required area

$$\begin{aligned}
 &= - \int_{-k}^0 \ln(x+1) \, dx + \int_0^k \ln(x+1) \, dx \\
 &= -[0 - (1-k) \ln(1-k) + k] + [(k+1) \ln(k+1) - k - 0] \\
 &= (1-k) \ln(1-k) + (k+1) \ln(k+1)
 \end{aligned}$$



$$\text{(b) (i) By letting } x = \tan \theta, \quad \frac{dx}{d\theta} = \sec^2 \theta$$

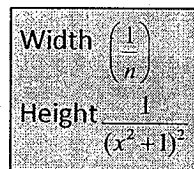
$$\text{When } x=1, \theta = \tan^{-1} 1 = \frac{\pi}{4}$$

$$\text{When } x=0, \theta = \tan^{-1} 0 = 0$$

$$\begin{aligned}
 \int_0^1 \frac{1}{(x^2+1)^2} \, dx &= \int_0^{\frac{\pi}{4}} \frac{1}{(\tan^2 \theta + 1)^2} \sec^2 \theta \, d\theta \\
 &= \int_0^{\frac{\pi}{4}} \frac{1}{(\sec^2 \theta)^2} \sec^2 \theta \, d\theta \\
 &= \int_0^{\frac{\pi}{4}} \frac{1}{\sec^2 \theta} \, d\theta \\
 &= \int_0^{\frac{\pi}{4}} \cos^2 \theta \, d\theta \\
 &= \frac{1}{2} \int_0^{\frac{\pi}{4}} (1 + \cos 2\theta) \, d\theta \\
 &= \frac{1}{2} \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^{\frac{\pi}{4}} \\
 &= \frac{\pi}{8} + \frac{1}{4}
 \end{aligned}$$

(ii) Area of n rectangles

$$= \left(\frac{1}{n}\right) \left\{ \left[\frac{1}{\left(\frac{1}{n}\right)^2 + 1} \right]^2 \right\} + \left(\frac{1}{n}\right) \left\{ \left[\frac{1}{\left(\frac{2}{n}\right)^2 + 1} \right]^2 \right\} + \dots + \left(\frac{1}{n}\right) \left\{ \left[\frac{1}{\left(\frac{n}{n}\right)^2 + 1} \right]^2 \right\}$$



$$= \left(\frac{1}{n}\right) \left[\frac{1}{\left(\frac{1+n^2}{n^2}\right)^2} \right] + \left(\frac{1}{n}\right) \left[\frac{1}{\left(\frac{2^2+n^2}{n^2}\right)^2} \right] + \dots + \left(\frac{1}{n}\right) \left[\frac{1}{\left(\frac{n^2+n^2}{n^2}\right)^2} \right]$$

$$= \frac{n^3}{(1+n^2)^2} + \left[\frac{n^3}{(2^2+n^2)^2} \right] + \dots + \left[\frac{n^3}{(n^2+n^2)^2} \right]$$

$$= \sum_{r=1}^n \frac{n^3}{(r^2+n^2)^2}$$

$$(iii) \quad \lim_{n \rightarrow \infty} \left[\sum_{r=1}^n \frac{n^3}{(r^2+n^2)^2} \right] = \int_0^1 \frac{1}{(x^2+1)^2} dx = \frac{\pi}{8} + \frac{1}{4}$$

Comment

(a) Generally the first part (finding indefinite integral) was well done with only a handful of students who were unsure how to handle $\int \frac{x}{x+1} dx$

Second part involving the area was badly attempted.

For the portion of the area below the x -axis, students either

- ignored the “negative area”, wrongly giving the expression as $\int_{-k}^k \ln(x+1) dx$
- forgot to negate the area below the x -axis, i.e. $\int_{-k}^0 \ln(x+1) dx + \int_0^k \ln(x+1) dx$
- correctly used $\left| \int_{-k}^0 \ln(x+1) dx \right|$ but did not know how to handle the “modulus” sign

A significant number of students simply ignored “0” in the limits, wrongly assuming that it would not affect the answer when it actually does (see solution above).

Many students also did not simplify their final expression involving 4 to 6 terms.

(b)(i) Question very well done. A handful of students were careless or did not know how to simplify $\tan^2 \theta + 1 = \sec^2 \theta$ or simplify $\frac{1}{\sec^2 \theta} = \cos^2 \theta$.

(b)(ii) Badly attempted.

Students either left this part blank or gave an inadequate presentation of steps for a “show” type of question:

- did not give the area of the first rectangle (hence $r = 1$ in lower limit of summation)
- did not give the area of the last rectangle (hence $r = n$ in lower limit of summation)
- wrongly took the width of the rectangles as $\frac{r}{n}$ instead of $\frac{1}{n}$
- did not show how they simplify the complicated fraction for the area of rectangles to the simple form given in the question

(b)(iii) Badly attempted with answers such as 0, ∞ , or 1.

Students failed to recognize that $\lim_{n \rightarrow \infty} \left[\sum_{r=1}^n \frac{n^3}{(r^2 + n^2)^2} \right]$ gives the area under the curve

bounded by the x -axis, $x = 0$ and $x = 1$.

1 A curve C has parametric equations

$$x = 2e^t, \quad y = t^3 - t,$$

where $-1 \leq t \leq \frac{3}{2}$.

- (i) Find $\frac{dy}{dx}$ in terms of t . Hence find the exact equations of the normals to the curve which are parallel to the y -axis. [3]
- (ii) Sketch the curve C . [1]
- (iii) Find the equation of the tangent to C at the point $(2e^a, a^3 - a)$. Find the possible exact values of a such that the tangent cuts the y -axis at $y = 1$. [4]

[Solution]

(i) $x = 2e^t \Rightarrow \frac{dx}{dt} = 2e^t$
 $y = t^3 - t \Rightarrow \frac{dy}{dt} = 3t^2 - 1$

$$\frac{dy}{dx} = \frac{3t^2 - 1}{2e^t}$$

$$\text{Gradient of normal} = -\frac{2e^t}{3t^2 - 1}$$

For normals to be parallel to y -axis, $3t^2 - 1 = 0$

$$\Rightarrow t = -\frac{1}{\sqrt{3}} \text{ or } \frac{1}{\sqrt{3}}$$

$$\Rightarrow x = 2e^{-\frac{1}{\sqrt{3}}} \text{ or } x = 2e^{\frac{1}{\sqrt{3}}}$$

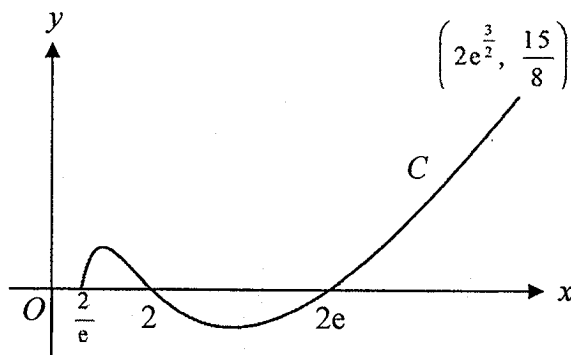
\therefore Equations of normals parallel to the y -axis are $x = 2e^{-\frac{1}{\sqrt{3}}}$ and $x = 2e^{\frac{1}{\sqrt{3}}}$

[Note that normal line parallel to y -axis has equation of the form $x = \text{constant}$, cannot be found using $y - y_1 = m(x - x_1)$]

Normal is parallel to y -axis

\Rightarrow Gradient of normal is undefined.

(ii)



$$x = 2e^t, \quad y = t^3 - t$$

Note that $-1 \leq t \leq \frac{3}{2}$, thus need to

sketch graph from $x = \frac{2}{e}$ (when $t = -1$)

to $x = 2e^{\frac{3}{2}}$ (when $t = \frac{3}{2}$)

Must indicate coordinates of these end-points.

(iii) Equation of the tangent at $t = a$: $y - (a^3 - a) = \frac{3a^2 - 1}{2e^a}(x - 2e^a)$

When $x = 0, y = 1$, $1 - (a^3 - a) = \frac{3a^2 - 1}{2e^a}(0 - 2e^a)$

$$a(a^2 - 3a - 1) = 0$$

$$\therefore a = 0 \quad \text{or} \quad a = \frac{3 - \sqrt{13}}{2} \quad \text{or} \quad a = \frac{3 + \sqrt{13}}{2} \quad (\text{rejected as } -1 \leq t \leq \frac{3}{2})$$

Marker's comments:

This question is generally well done. Some common errors include:

- (i) Wrongly assume that $y = 0$ when normal is parallel to y -axis.
Unable to find equation of the normal, did not realise that $y - y_1 = m(x - x_1)$ fails to work when the normal is parallel to y -axis.
- (ii) Very badly done. Many did not indicate coordinates of end points or they did not include the portion of the curve where $t < 0$.
- (iii) Many fail to check that $-1 \leq t \leq \frac{3}{2}$ and did not reject $\frac{3 + \sqrt{13}}{2}$. Some just conveniently "cancel" a from the equation $a(a^2 - 3a - 1) = 0$ and did not include $a = 0$ as one of the solution. Many careless mistakes especially when handling bracket and negative one.

- 2 A convergent geometric progression has first term a and common ratio r , where $a > 0$ and $r > 0$. An arithmetic progression has positive first term b and non-zero common difference d . It is given that the 6th, 9th and 16th terms of the geometric progression are equal to the 7th, 10th and 13th terms of the arithmetic progression respectively.

(a) Find d in terms of b and r . Hence show that the arithmetic progression is a decreasing sequence. [4]

(b) (i) Show that r satisfies the equation

$$r^{10} - 2r^3 + 1 = 0.$$

Hence find the value of r , giving your answer correct to 3 significant figures. [4]

(ii) Using the value of r found in (i), find the least value of n for which the sum to infinity of the geometric series exceeds the sum of the first n terms by less than $0.006a$. [3]

[Solution]

$$(a)(i) \quad b + 6d = ar^5 \quad \dots (1)$$

$$b + 9d = ar^8 \quad \dots (2)$$

$$b + 12d = ar^{15} \quad \dots (3)$$

$$\frac{(2)}{(1)}: \quad r^3 = \frac{b+9d}{b+6d} \Rightarrow br^3 + 6r^3d = b+9d$$

$$d(6r^3 - 9) = b(1 - r^3)$$

$$\therefore d = \frac{b(1 - r^3)}{6r^3 - 9}$$

$$\text{OR } \frac{(3)}{(1)} \Rightarrow d = \frac{b(1 - r^{10})}{6r^{10} - 12}$$

$$\frac{(3)}{(2)} \Rightarrow d = \frac{b(1 - r^7)}{9r^7 - 12}$$

with Similar arguments to deduce $d < 0$

Since the GP is convergent, $|r| < 1$.

$$0 < r < 1 \text{ and } b > 0$$

$$\Rightarrow b(1 - r^3) > 0 \text{ and } 6r^3 - 9 < 0. \text{ Hence } d < 0.$$

Thus the arithmetic progression is a decreasing sequence.

$$(b)(i) \quad (2) - (1): \quad 3d = ar^8 - ar^5 = ar^5(r^3 - 1)$$

$$(3) - (2): \quad 3d = ar^{15} - ar^8 = ar^8(r^7 - 1)$$

$$\text{Equating, } ar^5(r^3 - 1) = ar^8(r^7 - 1)$$

$$\text{Since } r \neq 0, \quad r^3 - 1 = r^3(r^7 - 1)$$

$$r^{10} - 2r^3 + 1 = 0 \quad (\text{shown})$$

Using GC, $r = 0.83518 = 0.835$ (3 s.f.) or $r = 1$ (rejected since $0 < r < 1$)

$$(ii) \quad 0 < S_\infty - S_n < 0.006a$$

$$\frac{a}{1-r} - \frac{a(1-r^n)}{1-r} < 0.006a \Rightarrow \frac{0.83518^n}{1-0.83518} < 0.006$$

$$0.83518^n < 0.0098892$$

$$n \ln 0.83518 < \ln 0.0098892$$

$$n > 38.4$$

Marker's Comment:

Performance on this question varied much from candidate to candidate. Many candidates were able to obtain one form of d in terms of r and b in part (a), but failed to apply the essential condition $0 < r < 1$ to prove that $d < 0$. For part (b)(i), there were a lot of tedious algebraic manipulations seen in proving the result. The solution $r = 1$ was often omitted or not rejected. Part (ii) was generally well done among candidates who managed to obtain $r = 0.835$. There were a number of candidates who mistook the sum to n terms as that of the AP.

In general, candidates seemed to have spent too much time answering this question because of the tedious methods applied.

3 Let $f(x) = \sqrt{3 + \sin x + \cos x}$.

(i) Given that $y = f(x)$, show that $2y \frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + y^2 = 3$. [3]

Hence obtain the Maclaurin's expansion of y up to and including the term in x^2 . [3]

(ii) If x is sufficiently small for terms in x^3 and higher powers of x to be neglected, show that the same result in (i) may be obtained using suitable standard series expansions. [3]

(iii) Denote the answer to (i) by $g(x)$. On the same set of axes, sketch the graphs of $y = f(x)$ and $y = g(x)$ for $-5 \leq x \leq 5$. Explain why the graphs are similar only for a certain range of values of x . [3]

[Solution]

$$\begin{aligned} \text{(i) } y = \sqrt{3 + \sin x + \cos x} &\Rightarrow y^2 = 3 + \sin x + \cos x \\ &\Rightarrow 2y \left(\frac{dy}{dx}\right) = \cos x - \sin x \\ &\Rightarrow 2y \left(\frac{d^2y}{dx^2}\right) + 2\left(\frac{dy}{dx}\right)^2 = -\sin x - \cos x = 3 - y^2 \\ \therefore 2y \frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + y^2 &= 3 \end{aligned}$$

When $x = 0$, $y = \sqrt{3 + 0 + 1} = 2$

$$2(2) \left(\frac{dy}{dx}\right) = 1 - 0 \Rightarrow \frac{dy}{dx} = \frac{1}{4}$$

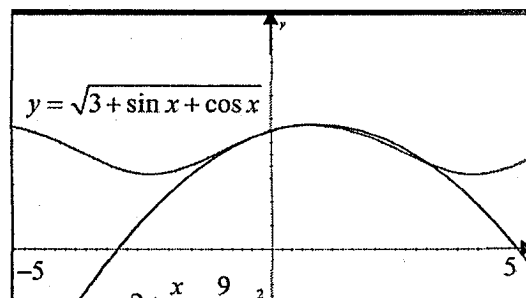
$$2(2) \frac{d^2y}{dx^2} + 2\left(\frac{1}{4}\right)^2 + 2^2 = 3 \Rightarrow \frac{d^2y}{dx^2} = -\frac{9}{32}$$

$$\text{Hence } y = 2 + \frac{1}{4}x + \frac{\left(-\frac{9}{32}\right)}{2!}x^2 + \dots = 2 + \frac{1}{4}x - \frac{9}{64}x^2 + \dots$$

(ii) For small values of x , $\sin x \approx x$ and $\cos x \approx 1 - \frac{1}{2}x^2$

$$\begin{aligned} y &\approx \left(4 + x - \frac{1}{2}x^2\right)^{\frac{1}{2}} = 2 \left(1 + \frac{x}{4} - \frac{x^2}{8}\right)^{\frac{1}{2}} = 2 \left[1 + \frac{1}{2} \left(\frac{x}{4} - \frac{x^2}{8}\right) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!} \left(\frac{x}{4} - \frac{x^2}{8}\right)^2 + \dots\right] \\ &= \left(2 + \left(\frac{x}{4} - \frac{x^2}{8}\right) + \left(-\frac{1}{4}\right)\left(\frac{x}{4}\right)^2 + \dots\right) \\ &= 2 + \frac{1}{4}x - \frac{9}{64}x^2 \text{ (shown)} \end{aligned}$$

(iii)



The graphs are similar for small values of x close to 0, since terms in higher powers of x in the expansion are

Marker's comments:

- (i) Students have done well in this part. Majority of them are able to prove in a more efficient way, i.e., squaring both sides first.
- (ii) The question is generally well done. They are able to apply the small angle approximation and use Binomial Expansion.

Common error made by students:

$$y = \sqrt{3} \left(1 + \frac{1}{3} \sin x + \frac{1}{3} \cos x \right)^{\frac{1}{2}} = \sqrt{3} \left[1 + \frac{1}{2} \left(\frac{\sin x + \cos x}{3} \right) + \frac{1}{2} \left(\frac{-1}{2} \right) \left(\frac{\sin x + \cos x}{3} \right)^2 + \dots \right]$$

[Can you see that higher powers for example $\left(\frac{\sin x + \cos x}{3} \right)^5 \approx \left(\frac{x+1-\frac{x^2}{2}}{3} \right)^5 = \left(\frac{x}{3} + \frac{1}{3} - \frac{x^2}{6} \right)^5$

also includes a constant term?]

- (iii) The drawing of the graphs is well done. The explanation of why the two graphs are similar only for a certain range of values of x is badly done. In fact, most students have the idea, but they are not able to express the idea clearly.

- 4 (a) Given that \mathbf{p} and \mathbf{q} are two non-zero vectors, show that $|\mathbf{p} \times \mathbf{q}|^2 = |\mathbf{p}|^2 |\mathbf{q}|^2 - (\mathbf{p} \cdot \mathbf{q})^2$. [2]
- (b) The points O , A and B are distinct and non-collinear. A and B have position vectors \mathbf{a} and \mathbf{b} respectively. It is given that $\angle AOB = 90^\circ$ and $|\mathbf{a}| = \alpha |\mathbf{b}|$ where α is a constant. Point M is the midpoint of AB and point T lies on OM produced such that $3OM = MT$.
- (i) Write down a vector equation of the line AB in terms of \mathbf{a} and \mathbf{b} . [1]
- (ii) Show that the position vector of the foot of perpendicular from point T to line

$$AB \text{ is given by } \frac{1}{1+\alpha^2} [(2\alpha^2-1)\mathbf{a} + (2-\alpha^2)\mathbf{b}]. \quad [6]$$

[Solution]

(a) $|\mathbf{p} \times \mathbf{q}|^2 = |\mathbf{p}|^2 |\mathbf{q}|^2 \sin^2 \theta$

$$= |\mathbf{p}|^2 |\mathbf{q}|^2 (1 - \cos^2 \theta)$$

$$= |\mathbf{p}|^2 |\mathbf{q}|^2 - |\mathbf{p}|^2 |\mathbf{q}|^2 \cos^2 \theta$$

$$= |\mathbf{p}|^2 |\mathbf{q}|^2 - (\mathbf{p} \cdot \mathbf{q})^2$$

(b) (i) Equation of line AB is $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$ where $\lambda \in \mathbb{R}$.

(ii) $\overline{OM} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$

$$\overline{OT} = 4\overline{OM} = 2(\mathbf{a} + \mathbf{b})$$

Let F be the foot of perpendicular from T to line AB .

$$\overline{OF} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a}) = (1-\lambda)\mathbf{a} + \lambda\mathbf{b} \text{ for some } \lambda \in \mathbb{R}.$$

$$\overline{TF} = (1-\lambda)\mathbf{a} + \lambda\mathbf{b} - 2(\mathbf{a} + \mathbf{b})$$

$$= (\lambda-2)\mathbf{b} - (1+\lambda)\mathbf{a}$$

Since TF is perpendicular to AB , $\overline{TF} \cdot \overline{AB} = 0$

$$[(\lambda-2)\mathbf{b} - (1+\lambda)\mathbf{a}] \cdot (\mathbf{b} - \mathbf{a}) = 0$$

$$(\lambda-2)|\mathbf{b}|^2 + (1+\lambda)|\mathbf{a}|^2 = 0 \quad (\text{since } \angle AOB = 90^\circ \Rightarrow \mathbf{a} \cdot \mathbf{b} = 0)$$

$$(\lambda-2)|\mathbf{b}|^2 + (1+\lambda)\alpha^2|\mathbf{b}|^2 = 0$$

$$\lambda = \frac{2-\alpha^2}{1+\alpha^2}$$

$$\therefore \overline{OF} = \frac{2-\alpha^2}{1+\alpha^2}\mathbf{b} + \left(1 - \frac{2-\alpha^2}{1+\alpha^2}\right)\mathbf{a}$$

$$= \frac{1}{1+\alpha^2} [(2\alpha^2-1)\mathbf{a} + (2-\alpha^2)\mathbf{b}] \quad (\text{shown})$$

Marker's comments:

- (a) This part of the question was poorly attempted.

Many students applied cross multiplication on either of the following

$$(\mathbf{p} \times \mathbf{q})^2 = (\mathbf{p} \times \mathbf{q})(\mathbf{p} \times \mathbf{q}) = \mathbf{p} \times \mathbf{p} + \mathbf{p} \times \mathbf{q} + \mathbf{q} \times \mathbf{p} + \mathbf{q} \times \mathbf{q} \text{ or}$$

$$(\mathbf{p} \cdot \mathbf{q})^2 = (\mathbf{p} \cdot \mathbf{q})(\mathbf{p} \cdot \mathbf{q}) = \mathbf{p} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{q} + \mathbf{q} \cdot \mathbf{p} + \mathbf{q} \cdot \mathbf{q}$$



or other similar forms.

Such working is mathematically incoherent and arises largely due to the confusion with algebraic expansion of the form

$$(\mathbf{p} + \mathbf{q})^2 = (\mathbf{p} + \mathbf{q})(\mathbf{p} + \mathbf{q}) = \mathbf{p} \times \mathbf{p} + \mathbf{p} \times \mathbf{q} + \mathbf{q} \times \mathbf{p} + \mathbf{q} \times \mathbf{q}$$

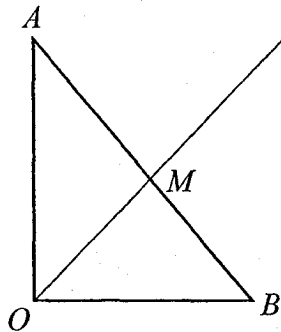
Students need to be very careful of the difference. In general, if there was no “+” or “-” sign to begin with, there should be no “+” or “-” sign in the resultant product.

Advice

Attempt more practice questions involving the application of dot and cross product to hone your proficiency in applying them to derive various results.

- (b) About half the cohort were familiar to the various approaches of finding the foot of the perpendicular. However, not many managed to arrive at the required result. This is largely due to very fundamental errors. In particular the misunderstanding that

$\overline{OM} = \frac{1}{2}(\mathbf{b} - \mathbf{a})$ because M was the midpoint of segment AB was prevalent. This could have been avoided with familiarity in the use of ratio theorem and the aid of a diagram.



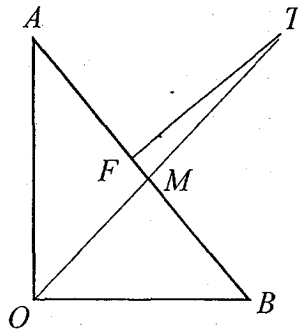
By ratio theorem, $\overline{OM} = \frac{\overline{OA} + \overline{OB}}{2} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$

Tip: When working on questions involving vectors, it is always good to sketch a diagram for clarity where possible.

Another prevalent weakness leading to failure in obtaining the result is the lack of competency in manipulation of terms. Students should learn to deal with the arithmetic operations involving vectors, especially when expressions get complicated.

A significant number of students were still unable to correctly identify the vectors to apply dot product to. After definition the foot of the perpendicular, F , students immediately use the position vector \overline{OF} in the dot product as

$$\overline{OF} \cdot \overline{AB} = 0$$



Tip: When considering the choice of vector to apply for dot product, we can always use a diagram to help us identify the right angled triangle of interest, which in this case is triangle MFT. The vector perpendicular to line AB (which MF is on) should be the correct vector to select i.e

$$\overline{TF} \cdot \overline{AB} = 0$$



Some students were unable to simplify their solution gave up midway due to failure to recognise that the dot product of \mathbf{a} and \mathbf{b} was 0 as they were perpendicular. Students ought to be familiar with the special results involving dot and cross products.

- 5 5 French, 3 Croatian and 2 Belgian dancers were invited to perform at a charity event, where a dinner was held to welcome them. In how many ways can these dancers be seated at a round table if those of the same nationality must sit next to one another? [2]

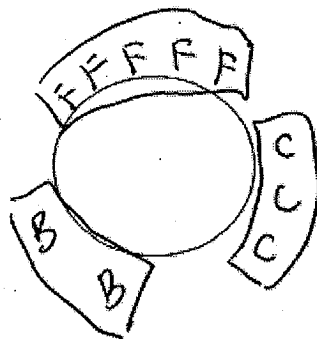
For the opening item, the dancers were randomly divided into two groups of 3 dancers each and one group of 4 dancers. Find the number of ways this can be done. [2]

At the end of the performance, 5 of the 10 dancers were chosen to pose for photographs. Find the number of ways the dancers can be arranged in a row, if exactly 3 of the chosen dancers are of the same nationality. [3]

[Solution]

$$\text{Number of ways to sit the dancers} = 2! \times 5! \times 3! \times 2! = 2880$$

Arrange the 3 groups at a round table



$$\text{Number of ways to divide the dancers} = \frac{{}^{10}C_3 \times {}^7C_3 \times {}^4C_4}{2!} = 2100$$

$$\text{Number of ways of choosing 5 dancers with 3 of same nationality} = {}^5C_3 \times {}^5C_2 + {}^3C_3 \times {}^7C_2 = 121$$

$$\text{Required number of ways of arrangements} = 121 \times 5! = 14520$$

Marker's comments:

This question is generally well done with many students getting full mark. Some common errors include:

- (i) $3! \times 5! \times 3! \times 2!$ (Forgot to arrange the 3 groups **in a circle**)
- (ii) Many did not recognise that ${}^{10}C_3 \times {}^7C_3 \times {}^4C_4$ introduces arrangement of the two groups of 3 and hence need to $\div 2!$ to remove this arrangement. Some common mistakes are $\times \frac{3!}{2!}$ or $\div 3!$.
- (iii) About one third of students do not understand the meaning of "exactly 3 of the chosen dancers are of the same nationality". Many thought that the other 2 dancers must be of different nationality (such as 3F1C1B), thus they did not include cases such as 3F2C, 3F2B, etc. There are also a handful of students who thought that the arrangement involves same nationality and wrongly $\times \frac{5!}{3!}$ instead of $\times 5!$ only.

- 6 A study on the toxicity of Rotenone, an ingredient in insecticide, on Chrysanthemum aphids was conducted. Insecticides containing different concentrations of Rotenone were sprayed on 5 separate groups of Chrysanthemum aphids. The following data was obtained:

Rotenone concentration (x mg/L)	10.2	7.7	5.1	3.8	2.6
Percentage of Chrysanthemum Aphids killed (y %)	88.0	85.7	52.2	33.3	12

- (i) Draw a scatter diagram for the data. [1]

It is thought that the effect of Rotenone concentration on Chrysanthemum aphids can be modelled by one of the following equations:

$$(A) y = a + b\sqrt{x}$$

$$(B) y = c + d \ln x$$

where a , b , c and d are constants.

- (ii) Find the product moment correlation coefficient between

(a) \sqrt{x} and y

(b) $\ln x$ and y

[2]

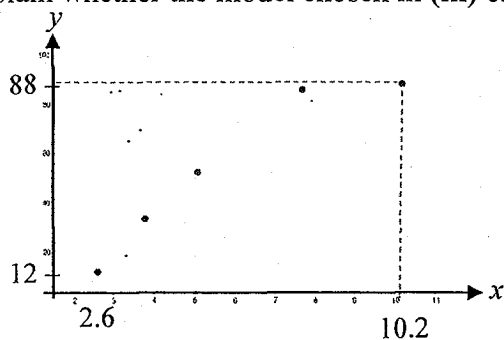
- (iii) Explain which of model (A) or (B) is the better model, and find the equation of the least-squares regression line for this model. [3]

An insecticide manufacturer claims its brand of insecticide which contains 11.25 mg/L of Rotenone is effective in killing at least 99.9% of Chrysanthemum aphids.

- (iv) Explain whether the model chosen in (iii) can be used to support the manufacturer's

[Solution]

- (i)

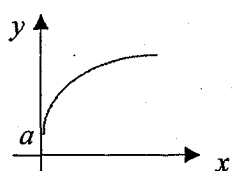


- (ii) From GC,

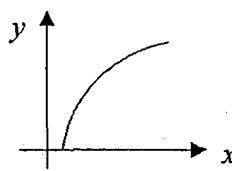
(a) $r_{\sqrt{x}, y} = 0.976$

(b) $r_{\ln x, y} = 0.987$

- (iii) Model A



Model B



The 2 models both show that as x increases, y increases at a decreasing rate. Since the correlation coefficient of model B is closer to 1 than that of model A, model B is the better model.

Equation of least squares regression of better model: $y = -44.8 + 59.8 \ln x$

- (iv) Since $x = 11.25$ is outside the data range $[2.6, 10.2]$, the model cannot be used to support the manufacturer's claim

Marker's comments:

The question is generally well done. Some students lost one mark as no scale or marking of coordinates is shown on the scatter diagram. Most students did not comment on the shape of the 2 models, resulting in one mark deducted.

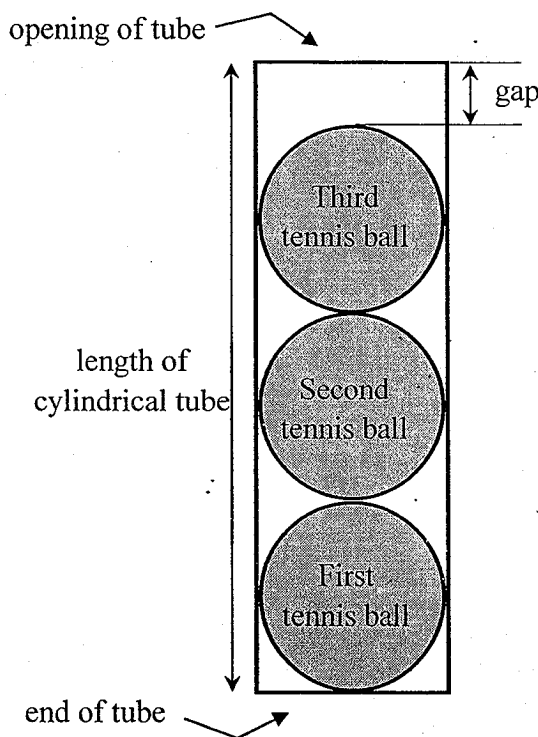
7 A company manufactures tennis balls with radii that are normally distributed with mean 3.3 cm and standard deviation 0.2 cm.

(i) Find the probability that the radius of a randomly selected tennis ball lies between 3.135 cm and 3.465 cm.

Without any further calculation, explain with the aid of a diagram how the answer obtained would compare with the probability that the radius lies between 3.465 cm and 3.795 cm. [3]

(ii) 3 tennis balls are randomly selected. Find the probability that the largest tennis ball has radius less than 3.4 cm. [2]

The tennis balls are packed into cylindrical tubes for sale. The cylindrical tubes have lengths that are normally distributed with mean 20 cm and standard deviation 0.3 cm. 3 tennis balls are randomly selected and packed into a randomly selected cylindrical tube such that the first tennis ball is in contact with the end of the tube and each subsequent ball is in contact with its neighbouring ball as shown in the diagram below.

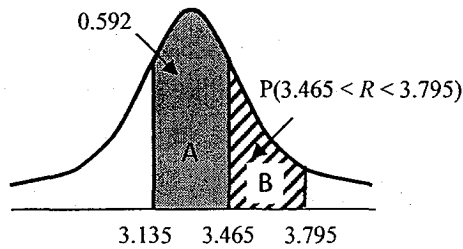


(iii) The probability that a gap exists between the third tennis ball and the opening of the tube and that the gap is at least k cm long is at least 0.15. Assuming that the centres of all the tennis balls are vertically aligned, find the possible range of values of k . [4]

[Solution]

Let R be the radius of a randomly chosen tennis ball. $R \sim N(3.3, 0.2^2)$

(i) $P(3.135 < R < 3.465) = 0.59063 = 0.591$ (3 s.f.)



From diagram, $P(3.135 < R < 3.465) > P(3.465 < R < 3.795)$

since area A is larger than area B given that the widths of the two intervals $3.135 < R < 3.465$ and $3.465 < R < 3.795$ are the same.

(ii) Required probability = $[P(R < 3.4)]^3 = 0.331$ (3 sf)

(iii) Let H be the height of a randomly chosen cylindrical tube. $H \sim N(20, 0.3^2)$

$$\text{Gap, } G = H - 2(R_1 + R_2 + R_3)$$

$$G \sim N(20 - 2(3.3 \times 3), 0.3^2 + 2^2(0.2^2 \times 3))$$

$$\text{i.e. } G \sim N(0.2, \sqrt{0.57})$$

$$P(G \geq k) \geq 0.15$$

Using GC, $0 < k \leq 0.982$

Marker's comments:

(i) Generally straightforward although some students still included “=” instead of giving strict inequality for “between 3.135 cm and 3.465 cm”.

In comparing $P(3.135 < R < 3.465)$ and $P(3.465 < R < 3.795)$, most students drew the normal distribution curve and required regions correctly but missed out

- the important fact that the widths of the intervals were the same when they compared the two areas
- that 3.135 and 3.465 were symmetrical about the mean 3.3

A few students missed out that the question's instruction for an explanation “without any further calculations, and with the aid of a diagram”.

(ii) Many students were unable to deduce correctly that “all balls” must be smaller than 3.4 cm, i.e. $[P(R < 3.4)]^3$. Common mistakes include

- $P(R < 3.4)$ (just one ball is smaller than 3.4 cm)

- $P\left(\bar{R} = \frac{R_1 + R_2 + R_3}{3} < 3.4\right)$ or $P(R_1 + R_2 + R_3 < 10.2)$

- $P(R < 3.4) \times P(R_1 > R_2) \times P(R_1 > R_3)$

(iii) Many students needed to be more careful in reading the question and applying their formulas / problem solving technique so as not to avoid the following common errors:

- forgot to use the diameter instead of radius in computing the gap i.e. $H - (R_1 + R_2 + R_3)$ instead of the correct $H - 2(R_1 + R_2 + R_3)$
- variance wrongly computed as $\text{Var}(H - 2(R_1 + R_2 + R_3)) = 0.3^2 + 2 \times 0.2^2 \times 3$ instead of $\text{Var}(H - 2(R_1 + R_2 + R_3)) = 0.3^2 + 2^2 \times 0.2^2 \times 3$
- “at least” wrongly written translated to “>” instead of “ \geq ”
- use table (assuming k must be integer) instead of “invNorm” to solve for k in $P(G \geq k) \geq 0.15$

- 8 Box A contains five cards numbered 1, 1, 2, 2 and 3 and Box B contains three cards numbered 4, 4 and 5. Cards numbered 1 and 5 are red, whereas cards numbered 2, 3 and 4 are blue. A card is drawn from each of the two boxes. If both cards are the same colour then the score will be the product of the numbers on the two cards. If both cards are different in colour then the score will be the sum of the numbers on the two cards.

Let X be the score obtained. The probability distribution of X is given below:

x	5	7	8	12
$P(X=x)$	h	k	$\frac{1}{3}$	k

- (i) Show that $h = \frac{2}{5}$. [2]
- (ii) Find k , $E(X)$ and $\text{Var}(X)$. [4]
- (iii) Find the probability that the mean score of 60 independent observations of X lies between 6.5 and 7.5. [3]

In a game, a player pays \$20 and draws a card each from Box A and Box B . She receives, in dollars, an amount equal to three times the score obtained. Find the variance of her winnings per game. [2]

[Solution]

Need to list the cases as this is a "show" question.

- (i) $h = P(X=5) = P(1 \text{ from box A, 4 from box B}) + P(1 \text{ from box A, 5 from box B})$

$$= \frac{2}{5} \times \frac{2}{3} + \frac{2}{5} \times \frac{1}{3}$$

$$= \frac{2}{5} \text{ (shown)}$$

- (ii) Total probability = 1, $\frac{2}{5} + k + \frac{1}{3} + k = 1 \Rightarrow k = \frac{2}{15}$

$$E(X) = 5 \times \frac{2}{5} + 7 \times \frac{2}{15} + 8 \times \frac{1}{3} + 12 \times \frac{2}{15} = \frac{36}{5}$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= \frac{856}{15} - \left(\frac{36}{5}\right)^2 = \frac{392}{75}$$

- (iii) **Mean score**, $\bar{X} = \frac{X_1 + X_2 + \dots + X_{60}}{60}$

Since sample size 60 is large, by **Central Limit Theorem**,

$$\bar{X} \sim N\left(\frac{36}{5}, \frac{\left(\frac{392}{75}\right)}{60}\right) = N\left(\frac{36}{5}, \frac{98}{1125}\right) \text{ approximately.}$$

$$P(6.5 < \bar{X} < 7.5) = \underline{0.836} \text{ (3 s.f.)}$$

$$E(\bar{X}) = E(X)$$

$$\text{Var}(\bar{X}) = \frac{\text{Var}(X)}{n}$$

Note that X is discrete and cannot be normal, not even with CLT!

Let W be the player's winnings per game. $W = 3X - 20$

$$\text{Var}(W) = 3^2 \text{Var}(X) + 0 = \frac{1176}{25} \text{ or } 47.04 \text{ (exact) or } 47.0 \text{ (3 s.f.)}$$

Marker's comments:

The question was generally well-answered.

(iv) Students would need to list the cases in order to get full marks.

(v) Only a few students did not get the full marks, mainly due to careless mistakes.

(vi) Several conceptual **mistakes** were made in finding $P(6.5 < \bar{X} < 7.5)$:

- assumed X is normally distributed even though X is clearly discrete
- used CLT to approximate X to be normal
- assumed the sample mean \bar{X} is normal without the use of CLT
- took the variance of sample mean as the same as the variance of X
- confused with the notations X , \bar{X} , $X_1 + X_2 + X_3 + \dots + X_{60}$ etc and their means and variances.

Students should try to use the earlier results to find the variance of the winning in a game, W . More than half of the cohort did not use $W = 3X - 20$ to find the variance of W . Instead they calculated tediously the mean and variance of W using the pdf of W .

Some students did not realise that they needed to subtract the initial payment of \$20 to get the correct amount won.

- 9 In an industrial process, a machine is programmed to manufacture metal pieces of length 8 cm. An automated quality-control function in the machine takes measurements from a random sample of 48 metal pieces and collates them. X is the random variable that denotes the length of one metal piece.

During a routine inspection, the quality-control manager retrieves one set of readings from the machine and obtains the following information:

$$\sum x = 390, \quad \sum x^2 = 3181.$$

Based on this set of data, test, at the 4% level of significance, whether the mean length of metal pieces produced by the machine is indeed 8 cm. [6]

Explain, in the context of the question, the meaning of “at the 4% level of significance”. [1]

A series of 12 routine inspections was carried out over a year. For each inspection, a test, at 4% level of significance, was performed. All the tests concluded that the mean length of metal pieces produced is not 8 cm. Find the expected number of inspections that made an incorrect conclusion. [2]

After a few months of operations, it is found that the length of metal pieces produced by the machine have a standard deviation of 0.4 cm. It is also known that the use of oversized metal pieces in subsequent manufacturing processes is more likely to result in equipment failure and production line disruption.

The quality-control manager is asked to test, at the 3% level of significance, whether the machine is producing oversized metal pieces. If that is the case, the machine will be replaced. A sample of size 80 is used and the sample mean length is denoted by \bar{x} cm. Find the range of values of \bar{x} for which the machine will not be replaced. [4]

[Solution]

Let μ be the population mean length of metal pieces.

$$\bar{x} = \frac{390}{48} = 8.125 \quad \text{and} \quad s^2 = \frac{1}{47} \left(3181 - \frac{390^2}{48} \right) = 0.26064 \quad (5\text{sf})$$

$$H_0: \mu = 8$$

$$H_1: \mu \neq 8$$

Level of significance: 4%

Under H_0 , since the sample size $n = 48$ is large, by Central Limit Theorem,

$$\text{the test statistic: } Z = \frac{\bar{X} - 8}{\left(\frac{\sqrt{0.26064}}{\sqrt{48}} \right)} \sim N(0,1) \text{ approximately.}$$

$$\text{From GC, } z_{\text{cal}} = 1.696$$

$$p\text{-value} = 0.0898 > 0.04$$

Since the p -value is more than the level of significance, we do not reject H_0 .

Hence there is **insufficient** evidence at the 4% level of significance to conclude that the mean length of the square metal pieces produced by the machine is **not** 8 cm.

“At the 4% level of significance” means there is a probability of 0.04 that we wrongly conclude that the mean length of metal pieces produced by the machine is not 8 cm when it is indeed 8 cm.

$$\text{Expected number of inspections that draw wrong conclusion} = 12 \times 0.04 = 0.48$$

$$H_0: \mu = 8$$

$$H_1: \mu > 8$$

Given level of significance = 3%, $\sigma = 0.4$ and $n = 80$

Under H_0 , since the sample size $n = 80$ is large, by Central Limit Theorem,

the test statistic: $Z = \frac{\bar{X} - 8}{\left(\frac{0.4}{\sqrt{80}}\right)} \sim N(0,1)$ approximately.

At 3% level of significance, the critical region is $\{z: z \geq 1.8808\}$

Since the machine will not be replaced, H_0 is not rejected.

$$\frac{\bar{x} - 8}{\left(\frac{0.4}{\sqrt{80}}\right)} < 1.8808$$

$$\bar{x} < 8 + 1.8808 \left(\frac{0.4}{\sqrt{80}}\right)$$

Hence $0 < \bar{x} < 8.08$

Marker's Comments:

There was a significant improvement in the performance on hypothesis testing in this paper as compared to that in the MYA.

Some common mistakes seen were:

- Failure to quote the Central Limit Theorem.
- Wrongly stating that X has a normal distribution by Central Limit Theorem.
- Wrongly stating the distribution of \bar{X} as $N(8.125, \frac{s^2}{48})$ i.e. substituting the value of \bar{x} for μ in the distribution.
- Substituting the value of \bar{x} in place of \bar{X} in the distribution of the test statistic. i.e. stating $\frac{8.125 - 8}{s/\sqrt{48}} \sim N(0, 1)$.
- Writing an incomplete conclusion with level of significance missing.
- Failure to state the conclusion or the definition of the level of significance in the context of the question. The term "mean length" was often missing in both.
- Stating the wrong conclusion such as:
"There is insufficient evidence at 4% level of significance to conclude that the mean length of metal pieces is indeed 8cm."
"There is sufficient evidence at 4% level of significance to conclude that the mean length of metal pieces is indeed 8cm." (i.e. Mistook not rejecting H_0 as equivalent to H_0 is true)
- Wrong interpretation of the expected number of inspection that made an incorrect conclusion.
- Wrong interpretation of "not rejecting H_0 " or giving the wrong critical region in the last part. " \leq " was often used instead of " $<$ " for the non-rejection region. Most candidate stated the answer as " $x < 8.08$ " instead of " $0 < x < 8.08$ " but no mark was deducted

- 10 Mrs Gat's confectionery produces a large number of sweets every day. On average, 20% of the sweets are wasabi-flavoured and the rest are caramel-flavoured.

A random sample of n sweets is chosen. If the probability that there are fewer than three wasabi-flavoured sweets in the sample is at most 0.3, find the least possible value of n . [3]

Mrs Gat decides to put the sweets randomly into packets of 20.

- (i) A customer selects packets of 20 sweets at random from a large consignment until she finds a packet with exactly 12 wasabi-flavoured sweets. Give a reason, in the context of the question, why a Binomial Distribution is not an appropriate model for the number of packets she selects. [1]
- (ii) Find the probability that a randomly chosen packet of sweets contains at least 3 wasabi-flavoured sweets. [2]
- (iii) Find the probability that a randomly chosen packet of sweets contains fewer wasabi-flavoured sweets than caramel-flavoured sweets, given that it contains at least 3 wasabi-flavoured sweets. [3]

Mrs Gat then packs the packets of sweets into boxes. Each box contains 10 packets of sweets.

- (iv) Find the probability that each packet in a randomly chosen box contains at least 3 wasabi-flavoured sweets. [1]
- (v) Find the probability that there are at least 30 wasabi-flavoured sweets in a randomly chosen box. [2]
- (vi) Explain why the answer to (v) is greater than the answer to (iv). [1]

[Solution]

Let X be the number of wasabi-flavoured sweets (out of n sweets).

$$X \sim B(n, 0.2)$$

$$P(X < 3) = P(X \leq 2) \leq 0.3$$

n	$P(X \leq 2)$
17	0.30962
18	0.27134

\therefore least n is 18

- (i) The binomial distribution is not an appropriate model as the number of packets to be selected (in order to find a packet with exactly 12 wasabi-flavoured sweets) is not fixed.

- (ii) Let Y be the number of wasabi-flavoured sweets (out of 20).

$$Y \sim B(20, 0.2)$$

$$P(Y \geq 3) = 1 - P(Y \leq 2) = 0.7939152811 = 0.794 \text{ (3 s.f.)}$$

- (iii) $P(Y \leq 9 | Y \geq 3) = \frac{P(3 \leq Y \leq 9)}{P(Y \geq 3)}$

$$= \frac{P(Y \leq 9) - P(Y \leq 2)}{P(Y \geq 3)}$$

$$= 0.997$$

- (iv) Probability = $[P(Y \geq 3)]^{10} = (0.7939153)^{10} = 0.0995$
- (v) Let W be the number of wasabi-flavoured sweets (out of 200).
 $W \sim B(200, 0.2)$
 $P(W \geq 30) = 1 - P(W \leq 29) = 0.972$
- (vi) The events in part (iv) is a subset of the events in part (v).
Hence the answer to part (v) is greater than the answer to part (iv).

Marker's comments:

- Students do not define their Binomial random variables properly. Bad examples include:
 - "Let X be the event..."
 - "Let X be the probability..."
 - "Let X be a sweet chosen..."

Students also keep using the same variable throughout the question (e.g. W for all parts) This leads to a loss of marks in , for example, part (ii), where a different distribution needs to be defined.

- There are many errors in computing n for part (i) despite using the GC. Students are not clear about the pdf and cdf functions in the GC, or whether $P(W \leq 3)$ or $P(W \leq 2)$ should be used.
- A failure to use 5sf in calculations cost many students the mark in part (iv).
- Answer to (v) is poorly given despite the question mirroring a tutorial question (on the mass of a chicken and a turkey). There are many long-winded explanations that miss the point.

