TAMPINES MERIDIAN JUNIOR COLLEGE

JC2 PRELIMINARY EXAMINATION

CANDIDATE NAME: ______________________________

CIVICS GROUP: ______________________________

H2 MATHEMATICS 9758

Paper 1

14 SEPTEMBER 2021

3 hours

Candidates answer on the question paper.

Additional material: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and Civics Group on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

For Examiners' Use	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
Total	

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

This document consists of 26 printed pages and 2 blank pages.

[Turn Over

1 A curve C has equation $y=a(2x+1)^2+bx+ce^{2x}$, where a, b and c are constants. It is given that C passes through the point $(1,\ 17-e^2)$ and the gradient of C at the point $(0,1)$ is 5. Find the equation of C. [5]

2 On the same axes, sketch the graphs of $y=\ln\left(\dfrac{4}{x-a}\right)$ and $y=\ln|x-a|$, where $a>1$.

Hence, or otherwise, solve the inequality $\ln\left(\dfrac{4}{x-a}\right)\geq \ln|x-a|$. [7]

3 Two curves C_1 and C_2 have equations $y=-\sqrt{\left(1-\dfrac{(x-2)^2}{4}\right)}$ and $y=-\dfrac{1}{4}x^2$ respectively.

(i) Sketch C_1 and C_2 on the same diagram, stating the coordinates of any axial intercepts and points of intersections between C_1 and C_2. [3]

(ii) The region R is bounded by C_1 and C_2. Find the volume of the solid of revolution formed when R is rotated through 2π radians about the x-axis. [3]

4 A quartic (degree four) polynomial $P(z)=z^4+az^3+bz^2+cz+d$ has real coefficients. The equation $P(z)=0$ has a root $re^{i\theta}$, where $r>0$ and $0<\theta<\pi$.

(i) Write down a second root in terms of r and θ, and hence show that a quadratic factor of $P(z)$ is $z^2-(2r\cos\theta)z+r^2$. [3]

(ii) Given that $\sqrt{3}e^{i\frac{\pi}{6}}$ and $\sqrt{2}e^{i\frac{\pi}{4}}$ are two roots of the equation $P(z)=0$, find an expression for $P(z)$ in the form $z^4+az^3+bz^2+cz+d$ where a, b, c and d are constants to be determined. [4]

5 It is given that

$$f(x)=\begin{cases} x, & \text{for } -2<x\le 3,\\ \sqrt{(x^2+kx+3)}+3, & \text{for } 3<x\le 5, \end{cases}$$

where k is a real constant such that f is continuous for $-2<x\le 5$.

(i) Show that $k=-4$. [1]

(ii) Sketch the graph of $y=f(x)$. Hence, justify that f has an inverse. [3]

(iii) Find $f^{-1}(x)$ in similar form. Leave your answer in the exact form. [4]

(iv) Write down the solution set for $f^{-1}(x)=f(x)$. [1]

6 The *folium of Descartes* is a curve given by the equation $x^3-9xy+y^3=0$.

(i) Find the equation of the tangent to the curve at the point $(4,2)$. [4]

(ii) Find the exact coordinates of the point, other than the origin, where the tangent to the curve is parallel to the y-axis. [4]

7 A geometric progression has first term a and common ratio r, and an arithmetic progression has first term b and common difference d, where a, b, d and r are non-zero real numbers. The first, third and eighth term of the geometric progression are equal to the second, third and fifth term of the arithmetic progression respectively.

(i) Show that $r^7-3r^2+2=0$. [2]

(ii) Find the values of r, giving your answer correct to 5 decimal places. Hence explain why the sum to infinity of the geometric progression exists. [2]

It is now given that $a=12$ and r is the positive value found in part **(ii)**.

(iii) Let E be the sum of the first n even-numbered terms of the arithmetic progression and S_∞ be the sum to infinity of the geometric progression. Find the largest value of n such that the difference between E and S_∞ is less than 1000. [6]

8 **You are not allowed to use a graphing calculator for this question.**

(i) By considering $z^3-8=(z-2)(z^2+2z+4)$, or otherwise, find the three roots of $z^3=8$ in **both** cartesian form $x+iy$ and exponential form $re^{i\theta}$ exactly, where $r>0$ and $-\pi<\theta\leq\pi$. [3]

(ii) Hence, or otherwise, find the six roots of the equation $z^6=64$ in exact exponential form. [3]

(iii) Let n be a positive integer. It is given that w is a root of the equation $z^n=2^n$, where $w\neq 2$. Let

$$f(w)=2^{n+1}+2^n w+2^{n-1}w^2+\ldots+2^2 w^{n-1}+2w^n+w^{n+1}.$$

Show that $f(w)=2^{n+1}\left(1+\frac{w}{2}\right)$. [3]

(iv) It is given that all the roots of the equation $z^n=2^n$ lie on a circle on the Argand diagram. For all values of n, the complex number $\frac{f(w)}{2^{n+1}}$ lie on another circle represented by the curve C. Find the cartesian equation of C. [2]

9 The line l passes through the point A with coordinates $(1,-2,3)$ and is parallel to the vector $\begin{pmatrix}4\\0\\-1\end{pmatrix}$.

The plane π_1 contains the point B with coordinates $(2,-1,3)$ and the line l.

(i) Show that the cartesian equation of π_1 is $x-y+4z=15$. [2]

Let F be the point on l which is closest to B.

(ii) Find the coordinates of F. [3]

Point C has coordinates $(8,1,-3)$.

(iii) Find the exact shortest distance from C to π_1. [2]

(iv) Hence, find the exact volume of the tetrahedron $CABF$. [3]

[Volume of tetrahedron $=\frac{1}{3}\times$ base area $\times$ perpendicular height]

(v) D is a general point on plane π_2 such that $DABF$ has the same volume as tetrahedron $CABF$. Given that π_2 does not contain C, find an equation of π_2. [2]

10 Two students, Andy and Ben, conduct a research on the population of Kawaii otters in Otterland. Initially, there are one thousand Kawaii otters in Otterland. After t years, the number of Kawaii otters in Otterland becomes x (in thousands).

Andy observes that the birth rate of the Kawaii otters is inversely proportional to the population (in thousands) of Kawaii otters. At the same time, the death rate of the Kawaii otters is proportional to the population (in thousands) of Kawaii otters. Andy suggests that the number of Kawaii otters remains constant when the population reaches 2000.

(i) Based on Andy's findings, show that $\frac{\mathrm{d}x}{\mathrm{d}t} = k\left(\frac{4-x^2}{x}\right)$, where k is a positive real constant. [3]

(ii) Solve the differential equation in part **(i)** to find an expression for x in terms of t and k. Describe what happens to the number of Kawaii otters if this situation continues over many years. [6]

Ben proposes that the population growth of Kawaii otters can be modelled by the differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{10}{4+t}\ln\left(1+\frac{1}{4}t\right) \text{ for } t > 0.$$

(iii) Based on Ben's proposal, find a general solution for this differential equation. [2]

(iv) Explain if Ben's model is appropriate in modelling the population of the Kawaii otters in the long run. [1]

11 Curved ceilings are used in buildings to achieve acoustics or aesthetic effects. Some architects use mathematical functions to aid them in the design of such curved ceilings. In a small room of an art gallery, an architect designs a curved ceiling for an art exhibition. The cross section of the ceiling can be modelled by curve C_1, which is defined parametrically by

$$x = 2t - \sin 2t, \; y = 5 + 2\sin^2 t \quad \text{for } 0 \le t \le \pi,$$

and x and y are given in metres.

(i) Sketch C_1. You are not required to label the turning point. [2]

(ii) Find $\int \sin^2 t(1 - \cos 2t)\,\mathrm{d}t$. [2]

(iii) The sketch in part **(i)** can be used to model the cross section of the small room, where C_1 is the curved ceiling, the x-axis is the floor and the lines $x = 0$ and $x = 2\pi$ are the walls. A curtain of negligible thickness will be hung from the curved ceiling as part of the art exhibition. Without using a calculator, find the exact smallest amount of curtain required to completely cover the cross section of the room. [4]

A surface of revolution is a surface created by rotating a curve around an axis of rotation. If the curve is rotated by k radians about the x-axis, the area of the surface formed can be found by the formula

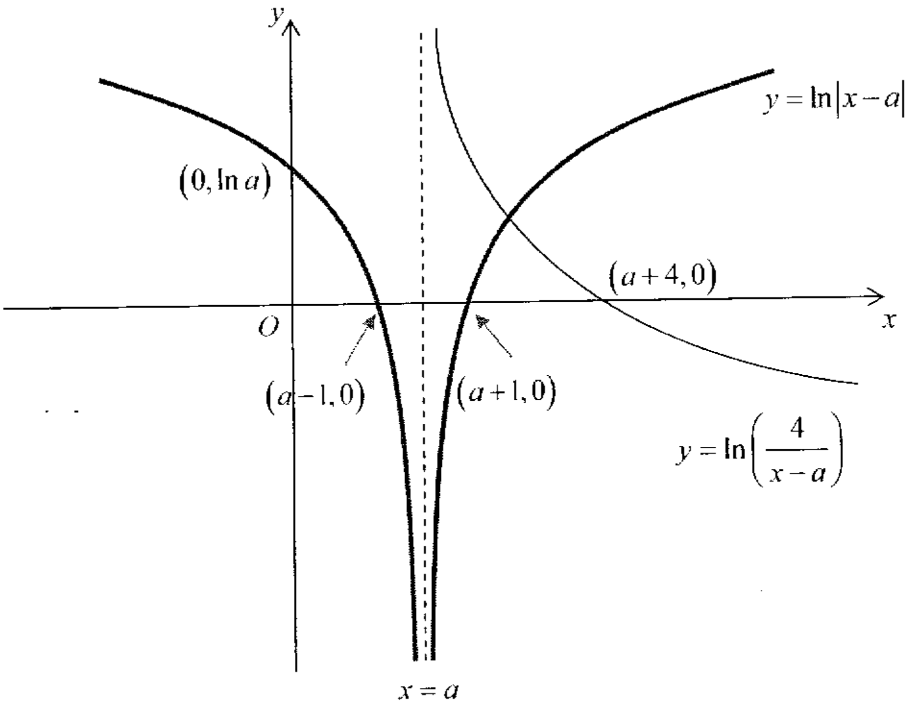
$$k \int_0^{\pi} y \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t.$$

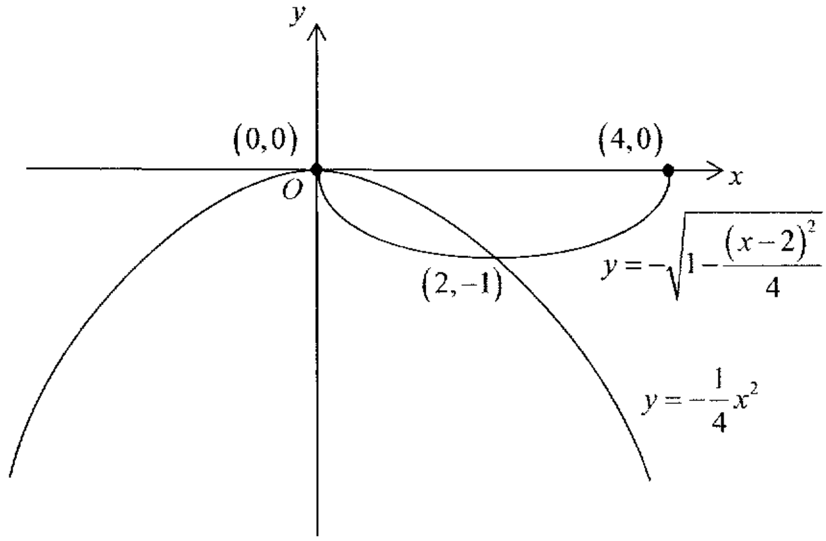
(iv) The surface area of the curved ceiling of the small room is equivalent to the area of the surface formed when the curve C_1 is rotated by $\frac{\pi}{4}$ radians about the x-axis. Without using a calculator, find the exact surface area of the curved ceiling of the small room. [5]

End of Paper

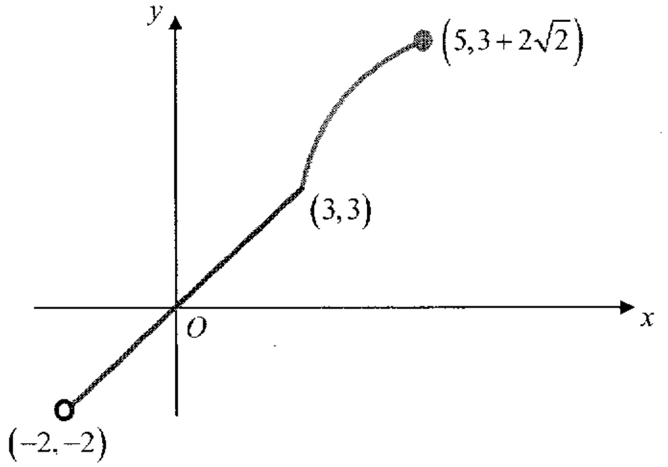
2021 H2 MATH (9758/01) JC 2 PRELIMINARY EXAMINATION – SOLUTION

Qn	Solution	
1	**Equations and Inequalities**	
	$y=a(2x+1)^2+bx+ce^{2x}$ Differentiate wrt x, $\frac{dy}{dx}=2a(2x+1)2+b+2ce^{2x} \ldots (*)$ Sub $(1, 17-e^2)$ into C $9a+b+ce^2=17-e^2 \; \ldots\ldots (1)$ Sub $(0, 1)$ into C $a+0b+c=1 \; \ldots\ldots (2)$ Sub $(0, 1)$ and gradient 5 into $(*)$ $4a+b+2c=5 \; \ldots\ldots (3)$ Using GC, $a=2, b=-1, c=-1$ Equation of C is $y=2(2x+1)^2-x-e^{2x}$	

Qn	Solution	
2	**Graphing Techniques**	
		
	To find intersection between the 2 graphs. $\ln\left(\frac{4}{x-a}\right) = \ln(x-a)$ (since $x > a$) $(x-a)^2 = 4$ $x = a+2$ or $x = a-2$ (rej. $\because x > a$) $\ln\left(\frac{4}{x-a}\right) \geq \ln\lvert x-a\rvert$ $\therefore a < x \leq a+2$	

Qn	Solution	
3	**Definite Integral**	
(i)		
(ii)	Volume generated = $\pi\int_0^2 \left(-\sqrt{1-\frac{(x-2)^2}{4}}\right)^2 - \left(-\frac{1}{4}x^2\right)^2 \mathrm{d}x$ $= \frac{14}{15}\pi$ units^3 or 2.93 units^3 (3 sf)	

Qn	Solution	
4	**Complex Numbers**	
(i)	Since the polynomial $\mathrm{P}(z)$ has real coefficients and $re^{i\theta}$ is a root of $\mathrm{P}(z)=0 \Rightarrow re^{-i\theta}$ is also a root. Thus, a second root for $\mathrm{P}(z)=0$ is $re^{-i\theta}$. A quadratic factor for $\mathrm{P}(z)$ is $\left(z-re^{i\theta}\right)\left(z-re^{-i\theta}\right)$ $=z^2-re^{i\theta}z-re^{-i\theta}z+r^2e^{i\theta}e^{-i\theta}$ $=z^2-rz\left(e^{i\theta}+e^{-i\theta}\right)+r^2$ $=z^2-rz\left(\cos\theta+i\sin\theta+\cos\theta-i\sin\theta\right)+r^2$ $=z^2-rz\left(2\cos\theta\right)+r^2$ $=z^2-\left(2r\cos\theta\right)z+r^2$ (shown)	
(ii)	Since the polynomial $\mathrm{P}(z)$ has real coefficients, $\sqrt{3}e^{-i\frac{\pi}{6}}$ and $\sqrt{2}e^{-i\frac{\pi}{4}}$ are the other 2 roots. From **(i)**, the quadratic factor of $\mathrm{P}(z)$ is $z^2-\left(2r\cos\theta\right)z+r^2$ The solutions $\sqrt{3}e^{i\frac{\pi}{6}}$ and $\sqrt{3}e^{-i\frac{\pi}{6}}$ give the quadratic factor $z^2-2\left(\sqrt{3}\right)\left(\cos\frac{\pi}{6}\right)z+3=z^2-3z+3$ The solutions $\sqrt{2}e^{i\frac{\pi}{4}}$ and $\sqrt{2}e^{-i\frac{\pi}{4}}$ give the quadratic factor $z^2-2\left(\sqrt{2}\right)\left(\cos\frac{\pi}{4}\right)z+2=z^2-2z+2$ Thus, $\mathrm{P}(z)=\left(z^2-3z+3\right)\left(z^2-2z+2\right)$ $=z^4-2z^3+2z^2-3z^3+6z^2-6z+3z^2-6z+6$ $=z^4-5z^3+11z^2-12z+6$ Therefore, $a=-5, b=11, c=-12, d=6$	

Qn	Solution	
5	**Functions**	
(i)	For f to be continuous, $x=\sqrt{(x^2+kx+3)}+3$ at $x=3$ $3=\sqrt{(3^2+k(3)+3)}+3$ $12+3k=0$ $k=-4$ (shown)	
(ii)	 The line $y=a, a\in\mathbb{R}$ cuts the graph of f at most once. Hence f is a one-one function and thus the inverse of f exists.	
(iii)	Let $y=\sqrt{(x^2+kx+3)}+3$ $(y-3)^2=x^2-4x+3$ $(y-3)^2=(x-2)^2-4+3$ $(x-2)^2=(y-3)^2+1$ $x=2\pm\sqrt{(y-3)^2+1}$ Since $3<x\le 5$, $x=2+\sqrt{(y-3)^2+1}$ OR $x=2+\sqrt{y^2-6y+10}$ $\mathrm{f}^{-1}(x)=\begin{cases} x, & \text{for } -2<x\le 3, \\ 2+\sqrt{x^2-6x+10}, & \text{for } 3<x\le 3+2\sqrt{2}. \end{cases}$	
(iv)	$\{x\in\mathbb{R}:-2<x\le 3\}$	

Qn	Solution	
6	**Differentiation Tangents and Normals**	
(i)	$x^3 - 9xy + y^3 = 0$ -------- (1) Differentiate wrt x, $3x^2 - 9\left(y + x\frac{dy}{dx}\right) + 3y^2\frac{dy}{dx} = 0$ $3x^2 - 9y - 9x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$ $3y^2\frac{dy}{dx} - 9x\frac{dy}{dx} = 9y - 3x^2$ $\frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x} = \frac{3y - x^2}{y^2 - 3x}$ At $(4,2)$, gradient of tangent is $\frac{3(2)-(4)^2}{(2)^2-3(4)} = \frac{5}{4}$ Equation of tangent is $y - 2 = \frac{5}{4}(x-4)$ $y = \frac{5}{4}x - 3$	
(ii)	Tangent parallel to the y-axis, $y^2 - 3x = 0$ $x = \frac{y^2}{3}$ Substitute into equation (1), $\left(\frac{y^2}{3}\right)^3 - 9\left(\frac{y^2}{3}\right)y + y^3 = 0$ $\frac{y^6}{27} - 3y^3 + y^3 = 0$ $y^3\left(\frac{y^3}{27} - 2\right) = 0$ $y = 0$ or $y = \sqrt[3]{54} = 3\sqrt[3]{2}$ (Reject) $x = \frac{1}{3}\left(3\sqrt[3]{2}\right)^2 = 3(2^{\frac{2}{3}})$ Point is $\left(3(2^{\frac{2}{3}}), 3(2^{\frac{1}{3}})\right)$	

Qn	Solution	
7	**Arithmetic and Geometric Series**	
(i)	Let b be the first term of the arithmetic progression. $a=b+d \quad -(1)$ $ar^2=b+2d \quad -(2)$ $ar^7=b+4d \quad -(3)$ From equations (1), (2) and (3), $ar^2-a=\frac{ar^7-ar^2}{2}(=d)$ $2r^2-2=r^7-r^2$ $r^7-3r^2+2=0$	
(ii)	Using GC, $r=1$ or $r=0.93725$ or $r=-0.77986$ (5 d.p.) (rejected since d is non-zero) Since $\lvert r\rvert<1$, sum to infinity of geometric progression exists.	
(iii)	The first even-numbered term of the A.P. is $a=12$. $d=ar^2-a=12\left(0.93725^2-1\right)=12\left(-0.12156\right)=-1.4587$ $\lvert E-S_\infty\rvert<1000$ $\left\lvert\frac{n}{2}\left[2a+(n-1)(2d)\right]-\frac{a}{1-r}\right\rvert<1000$ $\left\lvert\frac{n}{2}\left[2(12)+(n-1)(-2.9174)\right]-\frac{12}{1-0.93725}\right\rvert<1000$ $\left\lvert\frac{n}{2}\left[24+(n-1)(-2.9174)\right]-191.23\right\rvert<1000$ Using GC, When $n=28$, LHS $=958<1000$ When $n=29$, LHS $=1028>1000$ Hence largest value of $n=28$.	

Qn	Solution	
8	**Complex Numbers**	
(i)	$z^3-8=(z-2)(z^2+2z+4)=0$ $z=2$ or $z=\frac{-2\pm\sqrt{-12}}{2}=-1\pm i\sqrt{3}$ For $z=-1+i\sqrt{3}$, $\lvert z\rvert=\sqrt{(-1)^2+(\sqrt{3})^2}=2$ $\arg(z)=\pi-\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)=\frac{2}{3}\pi$ For $z=-1-i\sqrt{3}$, $\arg(z)=-\pi+\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)=-\frac{2}{3}\pi$ $\therefore z=2e^{i0}$ or $2e^{i\frac{2\pi}{3}}$ or $2e^{i\left(-\frac{2\pi}{3}\right)}$	
(ii)	$z^6-64=(z^3-8)(z^3+8)=0 \Rightarrow z^3=8$ or $z^3=-8$ For $z^3=-8$, we replace z by $-z$ in $z^3=8$ and we have $z=-2e^{i0}$ or $-2e^{i\frac{2\pi}{3}}$ or $-2e^{i\left(-\frac{2\pi}{3}\right)}$ $=2e^{i\pi}$ or $2e^{i\frac{5\pi}{3}}$ or $2e^{i\frac{\pi}{3}}$ $\equiv 2e^{i\pi}$ or $2e^{i\left(-\frac{\pi}{3}\right)}$ or $2e^{i\frac{\pi}{3}}$ (Since $-re^{i\theta}=-1\times re^{i\theta}=e^{i\pi}\times re^{i\theta}=re^{i(\theta+\pi)}$) Therefore the six roots are $z=2e^{i\left(-\frac{2\pi}{3}\right)}$ or $2e^{i\left(-\frac{\pi}{3}\right)}$ or $2e^{i0}$ or $2e^{i\frac{\pi}{3}}$ or $2e^{i\frac{2\pi}{3}}$ or $2e^{i\pi}$	
(iii)	$f(w)=2^{n+1}+2^n w+2^{n-1}w^2+\ldots+2^2w^{n-1}+2w^n+w^{n+1}$ $=\dfrac{2^{n+1}\left(1-\left(\frac{w}{2}\right)^{n+2}\right)}{1-\frac{w}{2}}$ $=\dfrac{2^{n+1}\left(1-\left(\frac{w}{2}\right)^{2}\right)}{1-\frac{w}{2}}$ $\because w^n=2^n \Rightarrow \left(\frac{w}{2}\right)^n=1$ $=\dfrac{2^{n+1}\left(1-\frac{w}{2}\right)\left(1+\frac{w}{2}\right)}{1-\frac{w}{2}}$ $=2^{n+1}\left(1+\frac{w}{2}\right)$	

(iv)	The roots of $z^n = 2^n$ lie on a circle of radius 2 centred about the origin since $\|z\| = 2$. Therefore, for any root w, $\dfrac{f(w)}{2^{n+1}} = 1 + \dfrac{w}{2}$ will lie on a circle of radius 1, centred about $(1,0)$. Hence the cartesian equation of C is $(x-1)^2 + y^2 = 1$.	

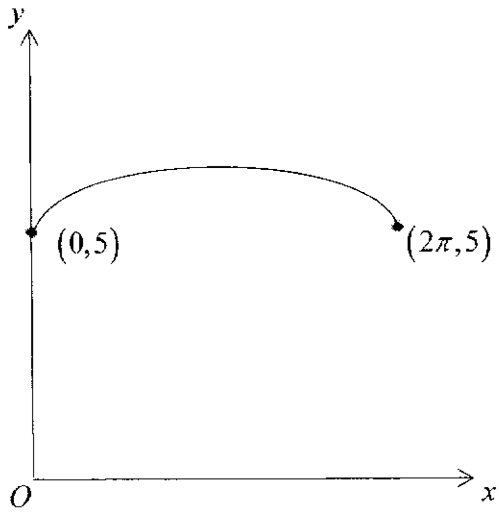
Qn	Solution	
9	**Vectors**	
(i)	$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}2\\-1\\3\end{pmatrix}-\begin{pmatrix}1\\-2\\3\end{pmatrix}=\begin{pmatrix}1\\1\\0\end{pmatrix}$ Normal vector of plane π_1 : $\begin{pmatrix}1\\1\\0\end{pmatrix}\times\begin{pmatrix}4\\0\\-1\end{pmatrix}=\begin{pmatrix}-1\\1\\-4\end{pmatrix}=-\begin{pmatrix}1\\-1\\4\end{pmatrix}$ Equation of plane π_1: $\mathbf{r}\cdot\begin{pmatrix}1\\-1\\4\end{pmatrix}=\begin{pmatrix}1\\-2\\3\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\4\end{pmatrix}=15$ $\therefore$ cartesian equation of plane π_1 is $x-y+4z=15$ (shown).	
(ii)	Equation of line l, $\mathbf{r}=\begin{pmatrix}1\\-2\\3\end{pmatrix}+\lambda\begin{pmatrix}4\\0\\-1\end{pmatrix}$, $\lambda\in\mathbb{R}$ Since F is on l, $\overrightarrow{OF}=\begin{pmatrix}1+4\lambda\\-2\\3-\lambda\end{pmatrix}$ for some $\lambda\in\mathbb{R}$. $\overrightarrow{BF}=\overrightarrow{OF}-\overrightarrow{OB}=\begin{pmatrix}1+4\lambda\\-2\\3-\lambda\end{pmatrix}-\begin{pmatrix}2\\-1\\3\end{pmatrix}=\begin{pmatrix}-1+4\lambda\\-1\\-\lambda\end{pmatrix}$ Since $BF\perp l$, $\overrightarrow{BF}\cdot\mathbf{d}=0$ $\therefore\begin{pmatrix}-1+4\lambda\\-1\\-\lambda\end{pmatrix}\cdot\begin{pmatrix}4\\0\\-1\end{pmatrix}=0$ $-4+16\lambda+\lambda=0$ $\lambda=\frac{4}{17}$ $\therefore\overrightarrow{OF}=\begin{pmatrix}1+4\left(\frac{4}{17}\right)\\-2\\3-\left(\frac{4}{17}\right)\end{pmatrix}$ $\therefore$ coordinates of $F=\left(\frac{33}{17},-2,\frac{47}{17}\right)$	

(iii)	$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=\begin{pmatrix}8\\1\\-3\end{pmatrix}-\begin{pmatrix}1\\-2\\3\end{pmatrix}=\begin{pmatrix}7\\3\\-6\end{pmatrix}$ Shortest distance from C to π_1 $=\dfrac{\left\|\begin{pmatrix}7\\3\\-6\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\4\end{pmatrix}\right\|}{\sqrt{1^2+(-1)^2+4^2}}$ $=\dfrac{20}{\sqrt{18}}$ units or $\dfrac{10\sqrt{2}}{3}$ units		
(iv)	$\overrightarrow{AF}=\overrightarrow{OF}-\overrightarrow{OA}=\begin{pmatrix}\frac{33}{17}\\-2\\\frac{47}{17}\end{pmatrix}-\begin{pmatrix}1\\-2\\3\end{pmatrix}=\begin{pmatrix}\frac{16}{17}\\0\\-\frac{4}{17}\end{pmatrix}$ Area of triangle $ABF=\dfrac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AF}\right\|$ $=\dfrac{1}{2}\left\|\begin{pmatrix}1\\1\\0\end{pmatrix}\times\begin{pmatrix}\frac{16}{17}\\0\\-\frac{4}{17}\end{pmatrix}\right\|$ $=\dfrac{2}{17}\left\|\begin{pmatrix}1\\1\\0\end{pmatrix}\times\begin{pmatrix}4\\0\\-1\end{pmatrix}\right\|$ $=\dfrac{2}{17}\left\|\begin{pmatrix}-1\\1\\-4\end{pmatrix}\right\|$ $=\dfrac{2}{17}\sqrt{(-1)^2+1^2+4^2}$ $=\dfrac{2\sqrt{18}}{17}$ units^2 Volume of $CABF=\dfrac{1}{3}\times\dfrac{2\sqrt{18}}{17}\times\dfrac{10\sqrt{2}}{3}$ $=\dfrac{40}{51}$ units^3 Alternative method Note that triangle ABF is a right angle triangle. $\overrightarrow{AB}=\begin{pmatrix}1\\1\\0\end{pmatrix}$		

	$\left\|\overrightarrow{AF}\right\| = \dfrac{\left\|\begin{pmatrix}1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}4\\0\\-1\end{pmatrix}\right\|}{\sqrt{17}} = \dfrac{4}{\sqrt{17}}$ $\left\|\overrightarrow{BF}\right\| = \sqrt{\left(\sqrt{2}\right)^2 - \left(\dfrac{4}{\sqrt{17}}\right)^2} = \sqrt{2-\dfrac{16}{17}} = \sqrt{\dfrac{18}{17}}$ Area $= \dfrac{1}{2}\left\|\overrightarrow{BF}\right\|\left\|\overrightarrow{AF}\right\|$ $= \dfrac{1}{2}\times\sqrt{\dfrac{18}{17}}\times\dfrac{4}{\sqrt{17}}$ $= \dfrac{2\sqrt{18}}{17}$ Volume of $CABF = \dfrac{1}{3}\times\dfrac{2\sqrt{18}}{17}\times\dfrac{10\sqrt{2}}{3}$ $= \dfrac{40}{51}$ units^3	
(v)	Equation of plane π_1 : $\mathbf{r}\cdot\begin{pmatrix}1\\-1\\4\end{pmatrix} = 15 \Rightarrow \mathbf{r}\cdot\dfrac{1}{\sqrt{18}}\begin{pmatrix}1\\-1\\4\end{pmatrix} = \dfrac{15}{\sqrt{18}}$ $\therefore$ Equation of plane π_2 : $\mathbf{r}\cdot\dfrac{1}{\sqrt{18}}\begin{pmatrix}1\\-1\\4\end{pmatrix} = \dfrac{15}{\sqrt{18}} + \dfrac{20}{\sqrt{18}}$ $\left(\text{from (iii), } -\dfrac{20}{\sqrt{18}} \text{ would contain } C\right)$ $= \dfrac{35}{\sqrt{18}}$ π_2 : $\mathbf{r}\cdot\begin{pmatrix}1\\-1\\4\end{pmatrix} = 35$	

Qn	Solution		
10	**Differential Equations**		
(i)	$\frac{dx}{dt}=\frac{m}{x}-hx,\quad \text{where } m,h>0$ When $x=2, \frac{dx}{dt}=0$. $\frac{m}{2}-2h=0$ $m=4h$ $\frac{dx}{dt}=\frac{m}{x}-hx$ $=\frac{4h}{x}-hx$ $=h\left(\frac{4-x^2}{x}\right)$ $=k\left(\frac{4-x^2}{x}\right),\quad \text{where } k=h$		
(ii)	$\frac{dx}{dt}=k\left(\frac{4-x^2}{x}\right)$ $\int \frac{x}{4-x^2}\,dx=\int k\,dt$ $-\frac{1}{2}\int \frac{-2x}{4-x^2}\,dx=\int k\,dt$ $-\frac{1}{2}\ln\left\|4-x^2\right\|=kt+c,\quad \text{where } c\in\mathbb{R}$ $\ln\left\|4-x^2\right\|=-2kt-2c$ $4-x^2=Ae^{-2kt},\quad A=\pm e^{-2c}$ $x=\sqrt{4-Ae^{-2kt}},\quad \text{since } x\geq 0$ When $t=0, x=1$. $1=\sqrt{4-Ae^{-2k(0)}}\Rightarrow A=3$ $\therefore x=\sqrt{4-3e^{-2kt}}$ The number of Kawaii otters **increases** and **tend towards** 2000.		

(iii)	$\frac{dx}{dt}=\frac{10}{4+t}\ln\left(1+\frac{1}{4}t\right)$ $\int 1\,dx=\int \frac{10}{4+t}\ln\left(1+\frac{1}{4}t\right)dt$ $\int 1\,dx=10\int \frac{1}{4+t}\ln\left(1+\frac{1}{4}t\right)dt$ $\int 1\,dx=10\int \frac{\frac{1}{4}}{1+\frac{1}{4}t}\ln\left(1+\frac{1}{4}t\right)dt$ $x=5\left(\ln\left(1+\frac{1}{4}t\right)\right)^2+D,\quad \text{where } D\in\mathbb{R}$	
(iv)	Ben's model might not be appropriate as his model suggests that the population of Kawaii otters tends towards infinity in the long run, which is not likely to happen due to space constraints and limited resources in Otterland.	

Qn	Solution		
11	**Definite Integral**		
(i)			
(ii)	$\int \sin^2 t\left(1-\cos 2t\right)\,\mathrm{d}t$ $=\frac{1}{2}\int \left(1-\cos 2t\right)^2\,\mathrm{d}t$ $=\frac{1}{2}\int 1-2\cos 2t+\cos^2 2t\,\mathrm{d}t$ $=\frac{1}{2}\int 1-2\cos 2t+\frac{1}{2}\left(1+\cos 4t\right)\mathrm{d}t$ $=\frac{1}{2}\left(\frac{3}{2}t-\sin 2t+\frac{1}{8}\sin 4t\right)+C,\ C\in\mathbb{R}$		
(iii)	$x=2t-\sin 2t$ $\frac{\mathrm{d}x}{\mathrm{d}t}=2-2\cos 2t$ $\text{Area}=\int_0^{2\pi} y\,\mathrm{d}x$ $=\int_0^{\pi}\left(5+2\sin^2 t\right)\left(2-2\cos 2t\right)\mathrm{d}t$ $=2\int_0^{\pi} 5-5\cos 2t+2\sin^2 t\left(1-\cos 2t\right)\mathrm{d}t$ $=2\left[5t-\frac{5}{2}\sin 2t+\frac{3}{2}t-\sin 2t+\frac{1}{8}\sin 4t\right]_0^{\pi}$ (from part (ii)) $=13\pi\ \mathrm{m}^2$ <u>Alternative method</u> $\text{Area}=5\times 2\pi+\int_0^{2\pi} y-5\,\mathrm{d}x$ $=10\pi+\int_0^{\pi}\left(2\sin^2 t\right)\left(2-2\cos 2t\right)\mathrm{d}t$ $=10\pi+2\left[\frac{3}{2}t-\sin 2t+\frac{1}{8}\sin 4t\right]_0^{\pi}$ (from part (ii)) $=13\pi\ \mathrm{m}^2$		

(iv)	$y=5+2\sin^2 t$ $\frac{dy}{dt}=4\sin t\cos t=2\sin 2t$ Surface area $=\frac{\pi}{4}\int_0^{\pi} y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$ $=\frac{\pi}{4}\int_0^{\pi}\left(5+2\sin^2 t\right)\sqrt{\left(2-2\cos 2t\right)^2+\left(2\sin 2t\right)^2}\,dt$ Note that $\sqrt{\left(2-2\cos 2t\right)^2+\left(2\sin 2t\right)^2}=2\sqrt{1-2\cos 2t+\cos^2 2t+\sin^2 2t}$ $=2\sqrt{2-2\cos 2t}$ $=2\sqrt{2-2\left(1-2\sin^2 t\right)}$ $=4\sqrt{\sin^2 t}$ $=4\sin t \quad \left(\text{since } \sin t\geq 0 \text{ for } 0\leq t\leq \pi\right)$ Surface area $=\pi\int_0^{\pi}\left(5+2\sin^2 t\right)\sin t\,dt$ $=\pi\int_0^{\pi}\left(7-2\cos^2 t\right)\sin t\,dt$ $=\pi\int_0^{\pi} 7\sin t-2\sin t\cos^2 t\,dt$ $=\pi\left[-7\cos t+\frac{2}{3}\cos^3 t\right]_0^{\pi}$ $=\pi\left(7-\frac{2}{3}-\left(-7+\frac{2}{3}\right)\right)$ $=\frac{38}{3}\pi \text{ m}^2$		

TAMPINES MERIDIAN JUNIOR COLLEGE

JC2 PRELIMINARY EXAMINATION

CANDIDATE NAME: ____________________

CIVICS GROUP: ____________________

H2 MATHEMATICS — 9758

Paper 2

20 SEPTEMBER 2021

3 hours

Candidates answer on the question paper.

Additional material: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and Civics Group on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

For Examiners' Use	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
Total	

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

This document consists of 29 printed pages and 1 blank pages.

[Turn Over

Section A: Pure Mathematics [40 marks]

1 **(i)** Differentiate $e^{\frac{1}{x^2}}$ with respect to x. [1]

(ii) Hence find the exact value of

$$\int_1^2 \frac{1}{x^3} e^{\frac{1}{x^2}} \, dx \; .$$ [3]

2 **(i)** Show that $\frac{1}{r!} - \frac{1}{(r+1)!} = \frac{r}{(r+1)!}$. [1]

(ii) Hence find $\sum_{r=1}^{n} \frac{2r}{(r+1)!}$ in terms of n. [3]

(iii) Give a reason why the series $\sum_{r=1}^{\infty} \frac{2r}{(r+1)!}$ converges, and write down the value of the sum to infinity. [2]

(iv) Using the result in part **(ii)**, show that

$$\frac{10}{6!} + \frac{12}{7!} + \frac{14}{8!} + \ldots + \frac{50}{26!} < \frac{1}{60}.$$ [2]

3

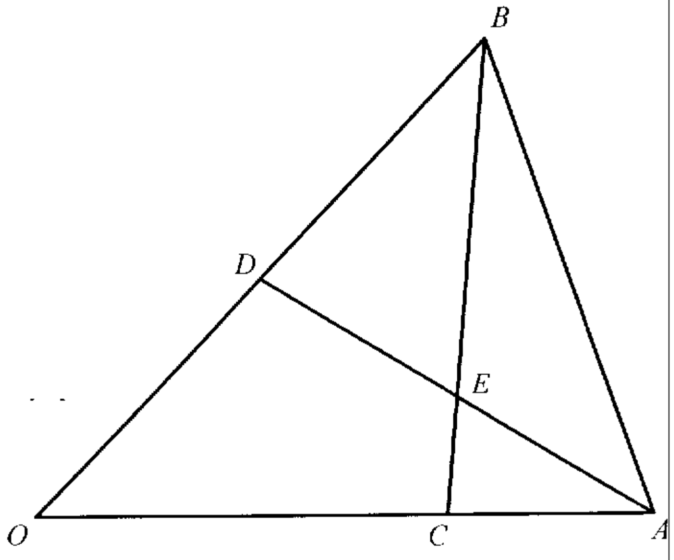


Figure 1

With reference to the origin O, the points A and B are such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. Point D is the mid-point of OB and point C lies on OA such that $2OC = 3CA$. The lines AD and BC intersect at point E (see Figure 1).

(i) Show that the vector equation of the line BC can be written as $\mathbf{r} = \frac{3}{5}\lambda\mathbf{a} + (1-\lambda)\mathbf{b}$, where λ is a real parameter. [1]

(ii) Find the position vector of E in terms of $\mathbf{a}$ and $\mathbf{b}$. [4]

(iii) Point F is such that $\overrightarrow{OF}$ is in the same direction as $\overrightarrow{AB}$. Given that the area of trapezium $OABF$ is $\frac{13}{16}|\mathbf{a}\times\mathbf{b}|$ units^2, find the position vector of F in terms of $\mathbf{a}$ and $\mathbf{b}$. [3]

4 **(a)** Expand $\left(b-\frac{x}{2}\right)^n$ in ascending powers of x, where b is a positive constant and n is a negative constant, up to and including the term in x^2. [2]

It is given that the coefficient of x is four times the coefficient of x^2 and the constant term in the expansion is $\frac{1}{2}$. Find the **integer** values of b and n. [3]

(b) **(i)** Explain why it is not possible to obtain a Maclaurin series for $\ln\left(2x^2\right)$. [1]

(ii) A Taylor series is an expansion of a real function $\mathrm{f}(x)$ about a point $x = a$ and it is defined by

$$\mathrm{f}(x)=\mathrm{f}(a)+\mathrm{f}'(a)(x-a)+\frac{\mathrm{f}''(a)}{2!}(x-a)^2+\ldots+\frac{\mathrm{f}^{(n)}(a)}{n!}(x-a)^n+\ldots$$

where $\mathrm{f}^{(n)}(a)$ is the value of the nth derivative of $\mathrm{f}(x)$ when $x = a$.

Find the first three exact non-zero terms of the Taylor series for $\ln\left(2x^2\right)$ about the point $x = 2$. [You need not simplify your answer.] [3]

5

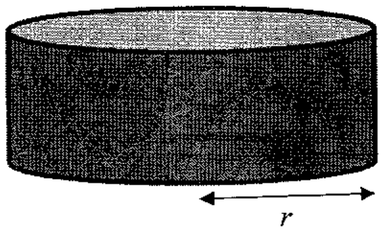


Figure 1

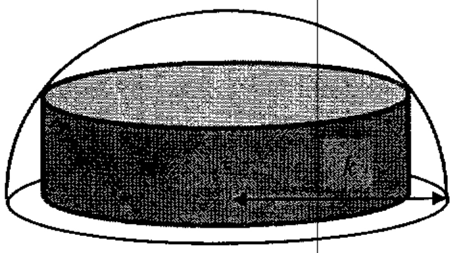


Figure 2

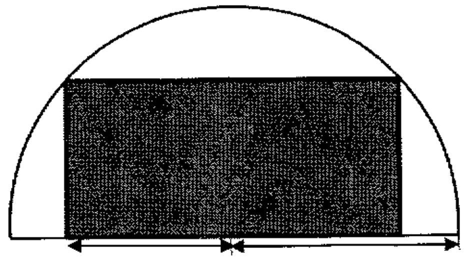


Figure 3

Figure 1 shows a cylinder with radius r cm. As shown in Figure 2, the cylinder can be inscribed in a hemisphere with fixed radius k cm such that the circumference of the top surface of the cylinder is in contact with the curved surface of the hemisphere. A cross-sectional view of the cyclinder and hemisphere is shown in Figure 3.

(i) Show that the volume of the cylinder, V cm^3, is given by $V = \pi r^2\sqrt{k^2 - r^2}$. [1]

(ii) Use differentiation to find, in terms of k, the value of r which gives a maximum value of V, justifying that this value indeed gives a maximum V. Hence write down the exact maximum volume of the cylinder in terms of k. [5]

(iii) Sketch the graph showing the volume of the cylinder as the radius of the cylinder varies. [3]

(iv) It is given that the volume of the cylinder increases at a constant rate of $\sqrt{3}\pi k^2$ cm^3 per minute. Find the rate at which the radius of the cylinder is increasing when $r = 0.5k$. [2]

[Turn Over

Section B: Probability and Statistics [60 marks]

6 A committee of five people is chosen from a group consisting of ten men and eight women. One of the men is Alex and one of the women is Betty.

(i) Find the number of ways such that Alex and Betty are both in the committee. [1]

(ii) It is decided that either Alex or Betty will be in the committee, but not both. Find the number of ways such that there is at most one woman in the committee. [3]

7 A and B are independent events such that $\mathrm{P}(A \mid B)=\frac{1}{5}$ and $\mathrm{P}(A \cup B)=\frac{2}{3}$.

(i) Find the exact value of $\mathrm{P}(A \cap B')$. [4]

For a third event C, it is given that $\mathrm{P}(C)=\frac{2}{7}$ and that A and C are mutually exclusive.

(ii) Find the range of values of $\mathrm{P}(B' \cap C')$. [3]

8 A bag contains 3 white balls and r red balls, where $r \geq 2$. Balls are drawn one at a time, at random from the bag and without replacement. The random variable X is the number of white balls drawn until 2 red balls are drawn.

(i) Show that $\mathrm{P}(X=1)=\frac{6r(r-1)}{(r+3)(r+2)(r+1)}$, and find the probability distribution of X. [4]

(ii) Find the expectation of X and show that the variance of X can be expressed in the form $\frac{6\mathrm{g}(r)}{(r+2)(r+1)^2}$, where $\mathrm{g}(r)$ is a quadratic polynomial to be determined. [5]

(iii) Given that the variance of X is $\frac{11}{16}$, find the value of r. [1]

9 **In this question, you should state clearly the values of the parameters of any normal distribution you use.**

The masses, in grams, of carrots and tomatoes sold at a supermarket have independent normal distributions. The means and standard deviations of these distributions are shown in the following table.

	Mean (grams)	Standard deviation (grams)
Carrots	140	20
Tomatoes	μ	7

(i) Find the probability that the mean mass of four randomly chosen carrots is more than 150 grams. [2]

(ii) If the probability that the total mass of five randomly chosen tomatoes exceeds 520 grams is 0.17, find the value of μ. [3]

For the rest of the question, let $\mu = 100$.

At the supermarket, carrots are sold at \$0.15 per 100 grams and tomatoes are sold at \$0.33 per 100 grams.

(iii) Find the probability that the total cost of four randomly chosen carrots and five randomly chosen tomatoes is at most \$2.50. [5]

10 The manager of a factory producing popcorn claims that the mean mass of packs of popcorn is 500 grams. To test this claim, a random sample of 70 packs of popcorn is weighed and the masses, x grams, of each pack of popcorn is recorded. The results are summarised by

$$\sum(x-500) = -91 \qquad \sum(x-500)^2 = 1830.$$

(i) Find unbiased estimates of the population mean and variance. [3]

(ii) Test, at the 2% significance level, whether the manager's claim is valid. [4]

(iii) Hence, determine the conclusion at the 2% significance level if the manager has overstated the mean mass of packs of popcorn. [2]

(iv) The factory purchases new machinery to pack the popcorn and the mass of packs of popcorn is now known to have a normal distribution. A new random sample of 70 packs of popcorn is taken and found to have mean mass of m grams with standard deviation of 6.1 grams. Find the range of values of m to conclude that the mean mass of packs of popcorn is not overstated at the 2% significance level. Leave your answer to 2 decimal places. [5]

11 A factory manufactures a large number of bulbs every day and they are packed into boxes by thousands for distribution to bulbs suppliers. Due to a manufacturing fault, 1.2% of these bulbs are defective. A random sample of 50 bulbs is taken from each box and tested.

(i) Find the probability that exactly 3 bulbs are defective in a random sample of 50 bulbs. [1]

(ii) Find the probability that in four random samples of 50 bulbs each, there are exactly two samples with exactly 2 defective bulbs each, one sample with exactly 3 defective bulbs, and one sample with fewer than 2 defective bulbs. [3]

(iii) Determine the least number of bulbs tested in a sample such that the probability of having more than 2 defective bulbs is more than 0.25. [2]

A three-step inspection scheme is devised as follows to determine if a box of bulbs can be accepted as satisfactory or rejected:

- Step 1: A random sample of 50 bulbs is tested and if it contains fewer than 3 defective bulbs, the box is accepted as satisfactory. If there is more than 3 defective bulbs in the sample of 50, then the box is rejected. If there is exactly 3 defective bulbs, a second random sample of 25 bulbs is tested.
- Step 2: If the second sample contains no defective bulb, the box is accepted as satisfactory. If there is more than 1 defective bulb in the second sample, then the box is rejected. If there is exactly 1 defective bulb, a third random sample of 25 bulbs is tested.
- Step 3: If the third sample contains no defective bulb, the box is accepted as satisfactory. Otherwise, the box is rejected.

(iv) Find the probability that a randomly chosen box is accepted as satisfactory. [3]

(v) An inspection scheme is considered to be stringent if the likelihood of accepting a box as satisfactory is not high.

(a) Using your answer in part **(iv)**, explain, in context, whether the inspection scheme is stringent. [1]

(b) By considering the expected number of defective bulbs for the first sample, suggest clearly, in context, whether the inspection scheme is stringent. [2]

(vi) Calculate the expected number of bulbs to be tested under this inspection scheme, showing your working clearly. [3]

End of Paper

2021 H2 MATH (9758/02) JC 2 PRELIMINARY EXAMINATION – SOLUTION

Qn	Solution	
1	**Integration Techniques**	
(i)	$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{\frac{1}{x}} = -\frac{1}{x^2}\mathrm{e}^{\frac{1}{x}}$	
(ii)	$\int_1^2 \frac{1}{x^3}\mathrm{e}^{\frac{1}{x}}\,\mathrm{d}x = -\int_1^2 \frac{1}{x}\left(-\frac{1}{x^2}\mathrm{e}^{\frac{1}{x}}\right)\mathrm{d}x$ $= -\left[\frac{1}{x}\mathrm{e}^{\frac{1}{x}}\right]_1^2 + \int_1^2 -\frac{1}{x^2}\mathrm{e}^{\frac{1}{x}}\,\mathrm{d}x$ $= -\left[\frac{1}{2}\mathrm{e}^{\frac{1}{2}} - \mathrm{e}\right] + \left[\mathrm{e}^{\frac{1}{x}}\right]_1^2$ $= -\frac{1}{2}\mathrm{e}^{\frac{1}{2}} + \mathrm{e} + \mathrm{e}^{\frac{1}{2}} - \mathrm{e}$ $= \frac{1}{2}\mathrm{e}^{\frac{1}{2}}$	

Qn	Solution	
2	**Sequences and Series**	
(i)	$\frac{1}{r!}-\frac{1}{(r+1)!}=\frac{(r+1)-1}{(r+1)!}=\frac{r}{(r+1)!}$	
(ii)	$\sum_{r=1}^{n}\frac{2r}{(r+1)!}=2\sum_{r=1}^{n}\left[\frac{1}{r!}-\frac{1}{(r+1)!}\right]$ $=2\left[\frac{1}{1!}-\frac{1}{2!}\right.$ $+\frac{1}{2!}-\frac{1}{3!}$ $+\frac{1}{3!}-\frac{1}{4!}$ $\vdots$ $+\frac{1}{(n-1)!}-\frac{1}{n!}$ $\left.+\frac{1}{n}-\frac{1}{(n+1)!}\right]$ $=2\left(1-\frac{1}{(n+1)!}\right)$	
(iii)	As $n\to\infty$, $\frac{1}{(n+1)!}\to 0$. $\therefore \lim_{n\to\infty}\sum_{r=1}^{n}\frac{2r}{(r+1)!}=\lim_{n\to\infty}\left[2\left(1-\frac{1}{(n+1)!}\right)\right]=2$ Hence the series is convergent. Sum to infinity = 2	
(iv)	$\frac{10}{6!}+\frac{12}{7!}+\frac{14}{8!}+...+\frac{50}{26!}=2\left(\frac{5}{6!}+\frac{6}{7!}+\frac{7}{8!}+...+\frac{25}{26!}\right)$ $=\sum_{r=5}^{25}\frac{2r}{(r+1)!}$ $=\sum_{r=1}^{25}\frac{2r}{(r+1)!}-\sum_{r=1}^{4}\frac{2r}{(r+1)!}$ $=2\left(1-\frac{1}{26!}\right)-2\left(1-\frac{1}{5!}\right)$ $=\frac{1}{60}-\frac{2}{26!}<\frac{1}{60}$	

Qn	Solution	
3	**Vectors**	
(i)	$2OC = 3CA \Rightarrow \frac{OC}{CA} = \frac{3}{2} \Rightarrow \overrightarrow{OC} = \frac{3}{5}\overrightarrow{OA} = \frac{3}{5}\mathbf{a}$ Equation of line BC: $\mathbf{r} = \overrightarrow{OB} + \lambda\overrightarrow{BC}$ $\mathbf{r} = \mathbf{b} + \lambda\left(\frac{3}{5}\mathbf{a} - \mathbf{b}\right)$ $\mathbf{r} = \frac{3}{5}\lambda\mathbf{a} + (1-\lambda)\mathbf{b}$, $\lambda \in \mathbb{R}$ (shown)	
(ii)	$\overrightarrow{OD} = \frac{1}{2}\mathbf{b}$ Equation of line AD: $\mathbf{r} = \overrightarrow{OA} + \mu\overrightarrow{AD}$ $\mathbf{r} = \mathbf{a} + \mu\left(\frac{1}{2}\mathbf{b} - \mathbf{a}\right)$, $\mu \in \mathbb{R}$ To find intersection point E, equate line BC and line AD. $\frac{3}{5}\lambda\mathbf{a} + (1-\lambda)\mathbf{b} = \mathbf{a} + \mu\left(\frac{1}{2}\mathbf{b} - \mathbf{a}\right)$ $\left(1 - \lambda - \frac{1}{2}\mu\right)\mathbf{b} = \left(1 - \mu - \frac{3}{5}\lambda\right)\mathbf{a}$ Since **a** and **b** are non-zero and non-parallel, $1 - \lambda - \frac{1}{2}\mu = 0 \Rightarrow \lambda + \frac{1}{2}\mu = 1$ --- (1) $1 - \mu - \frac{3}{5}\lambda = 0 \Rightarrow \frac{3}{5}\lambda + \mu = 1$ --- (2) Using GC to solve (1) and (2), $\lambda = \frac{5}{7}$ or $\mu = \frac{4}{7}$ Since $\lambda = \frac{5}{7}$, $\overrightarrow{OE} = \mathbf{b} + \frac{5}{7}\left(\frac{3}{5}\mathbf{a} - \mathbf{b}\right) = \frac{3}{7}\mathbf{a} + \frac{2}{7}\mathbf{b}$	
(iii)	Since $\overrightarrow{OF}$ is in the same direction as $\overrightarrow{AB}$, $\overrightarrow{OF} = p\overrightarrow{AB}$ $= p(\mathbf{b} - \mathbf{a})$ for some $p > 0$ Area of trapezium $OABF$ = Area of triangle OAB + Area of triangle OBF = $\frac{13}{16}\lvert\mathbf{a}\times\mathbf{b}\rvert$	

	$\frac{1}{2}\left\|\overrightarrow{OA}\times\overrightarrow{OB}\right\|+\frac{1}{2}\left\|\overrightarrow{OF}\times\overrightarrow{OB}\right\|=\frac{13}{16}\left\|\mathbf{a}\times\mathbf{b}\right\|$ $\frac{1}{2}\left\|\mathbf{a}\times\mathbf{b}\right\|+\frac{1}{2}\left\|p(\mathbf{b}-\mathbf{a})\times\mathbf{b}\right\|=\frac{13}{16}\left\|\mathbf{a}\times\mathbf{b}\right\|$ $\frac{1}{2}\left\|\mathbf{a}\times\mathbf{b}\right\|+\frac{1}{2}\left\|p(\mathbf{b}\times\mathbf{b})-p(\mathbf{a}\times\mathbf{b})\right\|=\frac{13}{16}\left\|\mathbf{a}\times\mathbf{b}\right\|$ $\frac{1}{2}\left\|\mathbf{a}\times\mathbf{b}\right\|+\frac{1}{2}\left\|-p(\mathbf{a}\times\mathbf{b})\right\|=\frac{13}{16}\left\|\mathbf{a}\times\mathbf{b}\right\|$, $\mathbf{b}\times\mathbf{b}=\mathbf{0}$ $\frac{1}{2}\left\|\mathbf{a}\times\mathbf{b}\right\|+\frac{1}{2}p\left\|(\mathbf{a}\times\mathbf{b})\right\|=\frac{13}{16}\left\|\mathbf{a}\times\mathbf{b}\right\|$, $p>0$ $\frac{1}{2}+\frac{1}{2}p=\frac{13}{16}$ $p=\frac{5}{8}$ $\therefore \overrightarrow{OF}=\frac{5}{8}(\mathbf{b}-\mathbf{a})$ <u>Alternative method</u> Let G lie on OF extended such that $OABG$ is a parallelogram with area $\left\|\mathbf{a}\times\mathbf{b}\right\|$. Let $\overrightarrow{OF}=p\overrightarrow{AB}$ for some $p>0$. $\frac{1+p}{2}=\frac{13}{16}$ $p=\frac{5}{8}$ $\therefore \overrightarrow{OF}=\frac{5}{8}(\mathbf{b}-\mathbf{a})$	

Qn	Solution	
4	**Maclaurin Series**	
(a)	$\left(b-\frac{x}{2}\right)^n=b^n\left(1-\frac{x}{2b}\right)^n$ $=b^n\left(1-\frac{xn}{2b}+\frac{n(n-1)}{2!}\left(\frac{x}{2b}\right)^2+..\right)$ $=b^n\left(1-\frac{n}{2b}x+\frac{n(n-1)}{8b^2}x^2+..\right)$ Since the coefficient of x is four times the coefficient of x^2, $-\frac{n}{2b}=\frac{4n(n-1)}{8b^2}$ $-1=\frac{n-1}{b}$ $-b=n-1$ $n=1-b$ Since the constant term in the expansion is $\frac{1}{2}$, $b^n=\frac{1}{2}$ Sub $n=1-b$ $b^{1-b}=\frac{1}{2}$ Using GC, $b=0.346$ (rejected because b is an integer) or $b=2$ $\therefore\ b=2$ and $n=-1$	
(bi)	Let $\mathrm{f}(x)=\ln\left(2x^2\right)$. As $\mathrm{f}(0)$ is undefined, it is not possible to obtain a Maclaurin series for $\ln\left(2x^2\right)$.	
(bii)	$\mathrm{f}(x)=\ln\left(2x^2\right)$ $\mathrm{f}'(x)=\frac{4x}{2x^2}=\frac{2}{x}$ $\mathrm{f}''(x)=-\frac{2}{x^2}$ When $x=2$, $\mathrm{f}(2)=\ln 8$, $\mathrm{f}'(2)=1$, $\mathrm{f}''(2)=-\frac{1}{2}$ $\therefore \ln\left(2x^2\right)=\ln 8+1(x-2)+\frac{\left(\frac{-1}{2}\right)}{2!}(x-2)^2+...$ $=\ln 8+(x-2)-\frac{1}{4}(x-2)^2+...$	

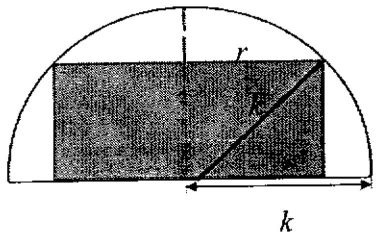
Qn	Solution	
5	**Differentiation Max/Min Problems**	

(i)

Height of cylinder, $h=\sqrt{k^2-r^2}$

$V=\pi r^2\left(\text{height}\right)$

$\quad=\pi r^2\sqrt{k^2-r^2}$



(ii)

$$\frac{\mathrm{d}V}{\mathrm{d}r}=\pi\left[r^2.\frac{1}{2}\left(k^2-r^2\right)^{-\frac{1}{2}}\left(-2r\right)+\sqrt{k^2-r^2}\left(2r\right)\right]$$

$$=\frac{\pi r\left[-r^2+2\left(k^2-r^2\right)\right]}{\sqrt{x^2-r^2}}$$

$$=\frac{\pi r\left[2k^2-3r^2\right]}{\sqrt{k^2-r^2}}$$

Let $\frac{\mathrm{d}V}{\mathrm{d}r}=0$

$2k^2-3r^2=0 \quad (\because r\neq 0)$

$3r^2=2k^2$

$r=\sqrt{\frac{2}{3}}k \quad (\because r>0)$

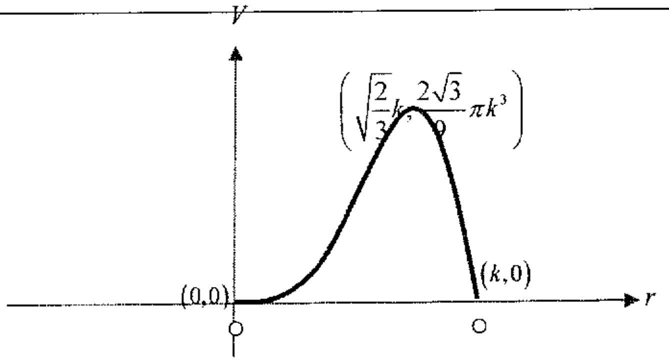
To show maximum V:

1st derivative Test

r	$\left(\sqrt{\frac{2}{3}}k\right)^-$	$\sqrt{\frac{2}{3}}k$	$\left(\sqrt{\frac{2}{3}}k\right)^+$
Sign of $\frac{\mathrm{d}V}{\mathrm{d}r}$	+ve	0	-ve

Hence V is a maximum at $r=\sqrt{\frac{2}{3}}k$.

Maximum volume $=\pi\frac{2\sqrt{3}}{9}k^3$ cm^3

(iii)		
(iv)	$\frac{dr}{dt}=\frac{dr}{dV}\times\frac{dV}{dt}$ $=\frac{\sqrt{k^2-(0.5k)^2}}{\pi(0.5k)\left[2k^2-3(0.5k)^2\right]}\left(\sqrt{3}\pi k^2\right)$ $=2.4$ cm/min	

Qn	Solution	
6	**Permutation and Combination**	
(i)	Number of ways = ${}^{16}C_3$ $= 560$	
(ii)	Number of ways if Alex is in = ${}^{9}C_4 + {}^{7}C_1\,{}^{9}C_3$ $= 714$ Number of ways if Betty is in = ${}^{9}C_4$ $= 126$ Total number of ways = 840	

Qn	Solution
7	**Probability**

(i) Since A and B are independent,

$$\mathrm{P}(A \mid B)=\mathrm{P}(A)=\frac{1}{5} \quad -(1)$$

$$\mathrm{P}(A \cap B)=\frac{1}{5}\mathrm{P}(B) \quad -(2)$$

$$\mathrm{P}(A \cup B)=\frac{2}{3}$$

$$\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A \cap B)=\frac{2}{3} \quad -(3)$$

Sub (1) and (2) into (3),

$$\frac{1}{5}+\mathrm{P}(B)-\frac{1}{5}\mathrm{P}(B)=\frac{2}{3}$$

$$\frac{4}{5}\mathrm{P}(B)=\frac{7}{15}$$

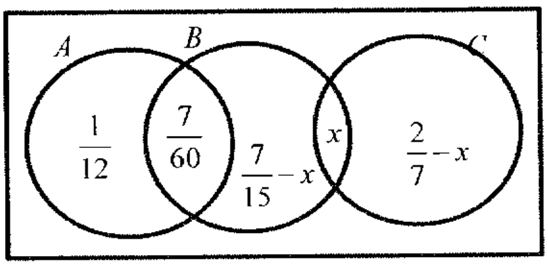
$$\mathrm{P}(B)=\frac{7}{12}$$

$$\mathrm{P}(A \cap B')=\mathrm{P}(A \cup B)-\mathrm{P}(B)$$

$$=\frac{2}{3}-\frac{7}{12}$$

$$=\frac{1}{12}$$

(ii)



Let $x=\mathrm{P}(B \cap C)$.

Since total probability is 1,

$$\mathrm{P}(B' \cap C')+\frac{7}{60}+\frac{7}{15}-x+x+\frac{2}{7}-x=1$$

$$\mathrm{P}(B' \cap C')=\frac{11}{84}+x$$

Since $0 \le x \le \frac{2}{7}$,

$$\frac{11}{84} \le \frac{11}{84}+x \le \frac{5}{12}$$

$$\frac{11}{84} \le \mathrm{P}(B' \cap C') \le \frac{5}{12}$$

Qn	Solution	
8	**Discrete Random Variable**	
(i)	X can take values 0, 1, 2, 3	

$$P(X=0)=\frac{r(r-1)}{(r+3)(r+2)}$$

$$P(X=1)=\frac{3r(r-1)\times 2}{(r+3)(r+2)(r+1)}=\frac{6r(r-1)}{(r+3)(r+2)(r+1)}$$

$$P(X=2)=\frac{3\times 2\times r(r-1)\times 3}{(r+3)(r+2)(r+1)r}=\frac{18(r-1)}{(r+3)(r+2)(r+1)}$$

$$P(X=3)=\frac{3\times 2\times 1\times r(r-1)\times 4}{(r+3)(r+2)(r+1)r(r-1)}=\frac{24}{(r+3)(r+2)(r+1)}$$

x	0	1	2	3
$P(X=x)$	$\frac{r(r-1)}{(r+3)(r+2)}$	$\frac{6r(r-1)}{(r+3)(r+2)(r+1)}$	$\frac{18(r-1)}{(r+3)(r+2)(r+1)}$	$\frac{24}{(r+3)(r+2)(r+1)}$

(ii)

$$E(X)=\frac{6r(r-1)}{(r+3)(r+2)(r+1)}+\frac{36(r-1)}{(r+3)(r+2)(r+1)}+\frac{72}{(r+3)(r+2)(r+1)}$$

$$=\frac{6(r^2+5r+6)}{(r+3)(r+2)(r+1)}$$

$$=\frac{6(r+2)(r+3)}{(r+3)(r+2)(r+1)}$$

$$=\frac{6}{r+1}$$

$$E(X^2)=\frac{6r(r-1)}{(r+3)(r+2)(r+1)}+\frac{72(r-1)}{(r+3)(r+2)(r+1)}+\frac{216}{(r+3)(r+2)(r+1)}$$

$$=\frac{6(r^2+11r+24)}{(r+3)(r+2)(r+1)}$$

$$=\frac{6(r+3)(r+8)}{(r+3)(r+2)(r+1)}$$

$$=\frac{6(r+8)}{(r+2)(r+1)}$$

$$\text{Var}(X)=E(X^2)-[E(X)]^2$$

$$\text{Var}(X)=\frac{6(r+8)}{(r+2)(r+1)}-\left(\frac{6}{r+1}\right)^2$$

$$=\frac{6(r+8)(r+1)-36(r+2)}{(r+2)(r+1)^2}$$

$$=\frac{6(r^2+3r-4)}{(r+2)(r+1)^2}\quad\text{(shown)}$$

$$\therefore g(r)=r^2+3r-4$$

(iii)	Given that $\text{Var}(X)=\frac{11}{16}$ $\frac{6(r^2+3r-4)}{(r+2)(r+1)^2}=\frac{11}{16}$ Solve using GC, $r=7$		

Qn	Solution	
9	**Normal Distribution**	
(i)	Let X be the mass of a randomly chosen carrot in g $\bar{X} = \dfrac{X_1 + X_2 + X_3 + X_4}{4}$ $E(\bar{X}) = 140, \quad Var(\bar{X}) = \dfrac{20^2}{4}$ $\bar{X} \sim N\left(140, \dfrac{20^2}{4}\right)$ $P(\bar{X} > 150) = 0.159$ (3 s.f)	
(ii)	Let Y be the mass of a randomly chosen tomato in g Let $S = Y_1 + Y_2 + Y_3 + Y_4 + Y_5$ $S \sim N\left(5\mu, 5\left(7^2\right)\right)$ <u>Method 1</u> $P(S > 520) = 0.17$ $P(Z > \dfrac{520 - 5\mu}{\sqrt{5(7^2)}}) = 0.17$ $\dfrac{520 - 5\mu}{\sqrt{5(49)}} = 0.95417$ $520 - 5\mu = 0.95417\sqrt{5(49)}$ $\mu = \dfrac{520 - 0.95417\sqrt{5(49)}}{5}$ $= 101$ (3 s.f.) <u>Method 2</u> $P(S > 520) = 0.17$ Using GC, $\mu = 101$ (3 s.f.)	

(iii)	Let C be the total cost of four randomly chosen carrots and five randomly chosen tomatoes $C=\frac{0.15}{100}\left(X_1+X_2+X_3+X_4\right)+\frac{0.33}{100}\left(Y_1+Y_2+Y_3+Y_4+Y_5\right)$ $\mathrm{E}\left(C\right)=\frac{0.15}{100}(4)(140)+\frac{0.33}{100}\left(5\right)(100)$ $=2.49$ $\mathrm{Var}\left(C\right)=\left(\frac{0.15}{100}\right)^2(4)(20^2)+\left(\frac{0.33}{100}\right)^2\left(5\right)(7^2)$ $=0.00626805$ $C\sim\mathrm{N}\left(2.49, 0.00626805\right)$ $\mathrm{P}(C\le 2.50)=0.550$ (to 3 s.f)	

Qn	Solution	
10	**Hypothesis Testing**	

(i)

Unbiased estimate of population mean is $\bar{x} = \dfrac{-91}{70} + 500$

$= 498.7$ (exact)

Unbiased estimate of population variance is $s^2 = \dfrac{1}{69}\left(1830 - \dfrac{(-91)^2}{70}\right)$

$= \dfrac{17117}{690}$

$= 24.807$

$= 24.8$ (3 s.f.)

(ii)

Let X be the mass of a randomly chosen pack of popcorn (in grams).
Let μ denote the population mean mass of the packs of popcorn (in grams).

$H_0 : \mu = 500$

$H_1 : \mu \neq 500$

Under H_0, since $n = 70$ is large, by Central Limit Theorem,

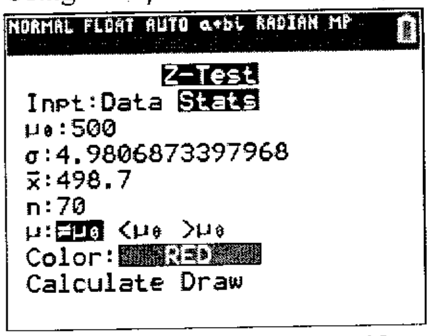
$\bar{X} \sim N\left(500, \dfrac{24.807}{70}\right)$ approximately.

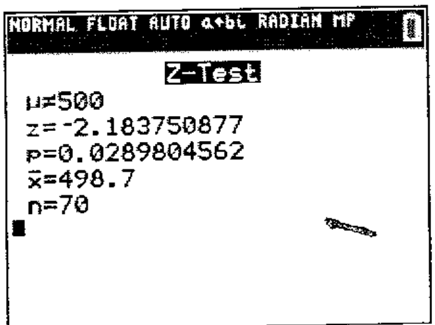
Test statistics: $Z = \dfrac{\bar{X} - 500}{\sqrt{\dfrac{24.807}{70}}}$

Level of significance: 2%

Reject H_0 if p – value < 0.02

Using GC, p – value = 0.0290 (3 s.f)





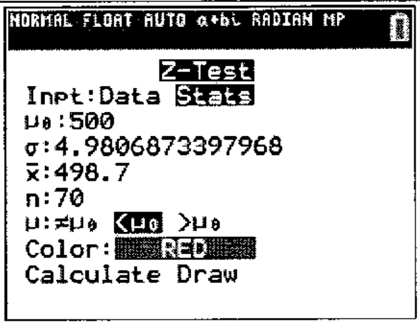
Since p – value = 0.0290 > 0.02 we do not reject H_0 and conclude that there is insufficient evidence at 2% significance level, that population mean mass of packs of popcorn is not 500 grams.

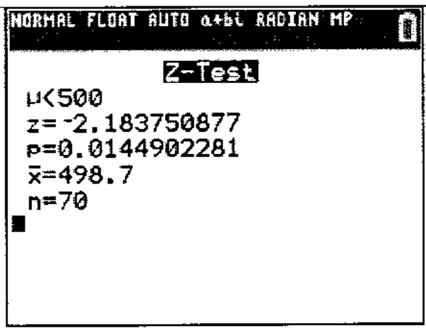
Thus the manager's claim is valid at 2% level of significance.

(iii)

$H_0 : \mu = 500$

$H_1 : \mu < 500$





From two-tailed to one-tailed test, the new

$p-\text{value}=\frac{1}{2}(0.0290)=0.0145.$

Since $p-\text{value}=0.0145<0.02$ we reject H_0 and conclude that there is sufficient evidence at 2% significance level, that population mean mass of packs of popcorn is less than 500 grams. Thus the manager has overstated his claim at 2% level of significance.

(iv)

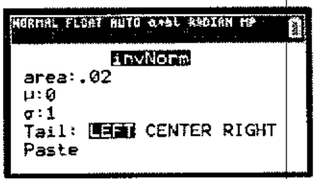
$s^2=\frac{n}{n-1}(\text{sample variance})=\frac{70}{69}(6.1^2)=37.749 \quad (5 \text{ s.f.})$

$H_0: \mu=500$

$H_1: \mu<500$

Under H_0, since $X \sim N\left(500, \frac{70}{69}(6.1^2)\right)$,

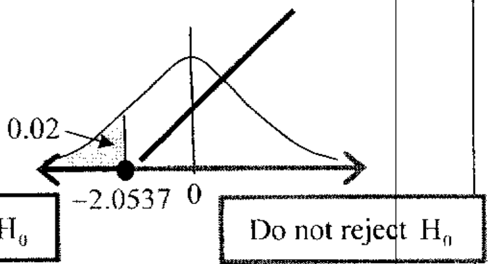
$\overline{X} \sim N\left(500, \frac{1}{70}\left(\frac{70}{69}(6.1^2)\right)\right)$.



Test statistics: $Z=\frac{\overline{X}-500}{\frac{6.1}{\sqrt{69}}}$

Level of significance: 2%

Reject H_0 if $z-\text{value}<-2.0537$



$z-\text{value}=\frac{m-500}{\frac{6.1}{\sqrt{69}}}$

Since mean mass of packs of popcorn is not overstated at 2% level of significance, there is insufficient evidence that the mean mass of packs of popcorn is less than 500 grams. Thus, H_0 is not rejected

$z-\text{value}>-2.0537$

$\frac{m-500}{\frac{6.1}{\sqrt{69}}}>-2.0537$

$m>498.48936$

Therefore, $m>498.49$ (2 d.p.)

Qn	Solution	
11	**Binomial Distribution**	
(i)	Let X be the number of bulbs, out of 50, that are defective. $X \sim B(50, 0.012)$ $P(X=3) = 0.019203$ $= 0.0192$ (3 s.f.)	
(ii)	Required Prob $= P(X=2)^2 \times P(X=3) \times P(X \le 1) \times \frac{4!}{2!}$ $= 0.0019777$ $= 0.00198$ (3 s.f.)	
(iii)	Let Y be the number of bulbs, out of n, that are defective. $Y \sim B(n, 0.012)$ Given $P(Y > 2) > 0.25 \quad \Rightarrow \quad 1 - P(Y \le 2) > 0.25$, Using GC, when $n = 144, 1 - P(Y \le 2) = 0.2497 < 0.25$ when $n = 145, 1 - P(Y \le 2) = 0.2529 > 0.25$ Least number of bulbs = 145	
(iv)	Let W be the number of bulbs, out of 25, that are defective. $W \sim B(25, 0.012)$ $P(\text{box is accepted as satisfactory})$ $= P(X \le 2) + P(X=3) \times P(W_1 = 0) + P(X=3) \times P(W_1 = 1) \times P(W_2 = 0)$ $= 0.99511$ $= 0.995$ (3 s.f.)	
(v)(a)	Since the probability of accepting a box as satisfactory is 0.99511 is high, the inspection scheme is not stringent.	
(v)(b)	Since the expected number of defective bulbs in the first sample is $50 \times 0.012 = 0.6$ is significantly lower than the rejection criterion of 4 or more defective bulbs (and the criterion for the testing of the 2nd sample), the probability of accepting a box as satisfactory would be high regardless of the outcomes of the 2nd or 3rd samples. Thus, the inspection scheme is considered not stringent.	
(vi)	Let N be the number of bulbs tested. $P(N=50) = 1 - P(X=3) = 0.9807967$ $P(N=75) = P(X=3) \times (1 - P(W=1)) = 0.0148914$ $P(N=100) = P(X=3) \times P(W=1) = 0.0043119$ $E(N) = 50 \times P(N=50) + 75 \times P(N=75) + 100 \times P(N=100)$ $= 50.588$ $= 50.6$ (3 s.f.)	