7

Name:______________________ Class:__________

JURONG PIONEER JUNIOR COLLEGE

JC2 Preliminary Examination 2021

MATHEMATICS
Higher 2

Paper 1

9758/01

17 September 2021

3 hours

Candidates answer on the Question Paper.

Additional materials: List of Formulae (MF 26)

READ THESE INSTRUCTIONS FIRST

Write your name and civics class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given by [ ] at the end of each question or part question.

For Candidate's Use	For Examiner's Use
Question Number	**Marks Obtained**
1	
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6	
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8	
9	
10	
Total Marks	/ 100

This document consists of **6** printed pages.

[Turn over

1 A household's monthly utility bill comprises the charges incurred for electricity, gas and water consumption based on the prevailing unit cost for each type of utility. The table below shows the electricity, gas and water consumption of Mrs Ow's household for the months of June, July and August in 2021.

Month	Consumption of electricity (in kWh)	Consumption of gas (in kWh)	Consumption of water (in m^3)
June	316	97	15.6
July	320	86	15.7
August	330	80	15.8

Mrs Ow's monthly utility bills for June, July and August are \$130.84, \$134.61 and \$136.02 respectively. It is given that the unit cost for electricity has increased by \$0.015/kWh with effect from 1 July 2021. Find the unit cost for electricity, gas and water from 1 July 2021, giving your answers correct to 3 decimal places. [4]

2 Referred to the origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ such that

$$\mathbf{a}=\mathbf{i}+\mathbf{j}+\mathbf{k} \quad \text{and} \quad \mathbf{b}=\mathbf{i}+2\mathbf{j}+2\mathbf{k}.$$

(i) Find the size of angle OAB. [2]

The point C has position vector $\mathbf{c}$ given by $\mathbf{c}=\lambda\mathbf{a}+\mu\mathbf{b}$, where λ and μ are positive constants. Given that the area of triangle OAC is twice that of triangle OBC,

(ii) find μ in terms of λ, [3]

(iii) hence, if $OC=\sqrt{118}$, find the position vector $\mathbf{c}$. [4]

3 The parametric equations of a curve are

$$x=2t+1, \quad y=\frac{1}{t^2}, \quad t\neq 0.$$

(i) Find the equation of the tangent to the curve at the point $\left(2p+1, \frac{1}{p^2}\right)$, simplifying your answer. [3]

(ii) The tangent at point A where $p=1$, meets the curve again at point B. Find the coordinates of A and B. [4]

(iii) There exists a variable point C such that AC is perpendicular to BC. Find a Cartesian equation for the path traced by C. [3]

4 **(a)** Given that P(z) is a polynomial with real coefficients and the equation P(z) = 0 has a root $re^{i\theta}$, where $r > 0$ and $0 < \theta < \pi$, write down a second root in terms of r and θ. Hence, show that a quadratic factor of P(z) is $z^2 - 2rz\cos\theta + r^2$. [3]

Let $P(z) = 3z^3 + az^2 + z + 2$, where a is a real number. One of the roots of the equation P(z) = 0 is $e^{i\left(\frac{\pi}{3}\right)}$. By expressing P(z) as a product of two factors with real coefficients, find a and the other roots of P(z) = 0. [4]

(b) It is given that $w = -p + pi$, where p is a positive real constant.

Find the two smallest positive integers of n such that $\dfrac{w^n}{w^*}$ is purely imaginary. [3]

5 **(i)** By sketching the curves $y = \sin x$ and $y = \cos x$, or otherwise, solve the inequality $\sin x > \cos x$, for $0 < x < \dfrac{\pi}{2}$. [2]

Hence, find $\int_0^n |\sin x - \cos x|\, dx$, in terms of n, where $\dfrac{\pi}{4} < n < \dfrac{\pi}{2}$. [4]

(ii) The region bounded by the curve $y = |\sin x - \cos x|$, the x-axis, the y-axis and the line $x = \dfrac{\pi}{2}$ is rotated completely about the x-axis. Find the exact volume of the solid formed. [4]

6 It is given that

$$f : x \mapsto \begin{cases} -a\sqrt{1-\dfrac{x^2}{4}}, & \text{for } -2 \le x < 2 \\ 3x-6, & \text{for } \quad 2 \le x < 3 \end{cases}$$

and that $f(x) = f(x + 5)$ for all real values of x, where a is a positive constant.

(i) Find f(2021). [2]

(ii) Sketch the graph of $y = f(x)$ for $-5 \le x \le 7$, stating the coordinates of the maximum points, end-points and points where the curve crosses the x- and y-axes. [4]

(iii) On a separate diagram, sketch the curve with equation $y = 3f(x+2)$ for $-2 \le x \le 3$, stating the coordinates of the points where the curve crosses the x- and y-axes. [2]

(iv) Another function g is defined by

$$g : x \mapsto \sqrt{x} + 2, \quad \text{for } 0 \le x < 1.$$

State the range of g and find fg(x), stating its domain. [3]

7 Relative to the origin O, the points A, B and C have position vectors given by

$$\mathbf{a} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}, \ \mathbf{b} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k} \text{ and } \mathbf{c} = \mathbf{i} - \mathbf{j} + \mathbf{k}.$$

(i) Find the equation of plane p_1, in scalar product form, which contains A, B and C. [3]

The line l has equation $\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$, where t is a parameter.

(ii) Show that l is perpendicular to p_1. [1]

(iii) The point D lies on l and plane p_2 such that the perpendicular distance from p_2 to p_1 is 6 units. Find the possible coordinates of D. [6]

8

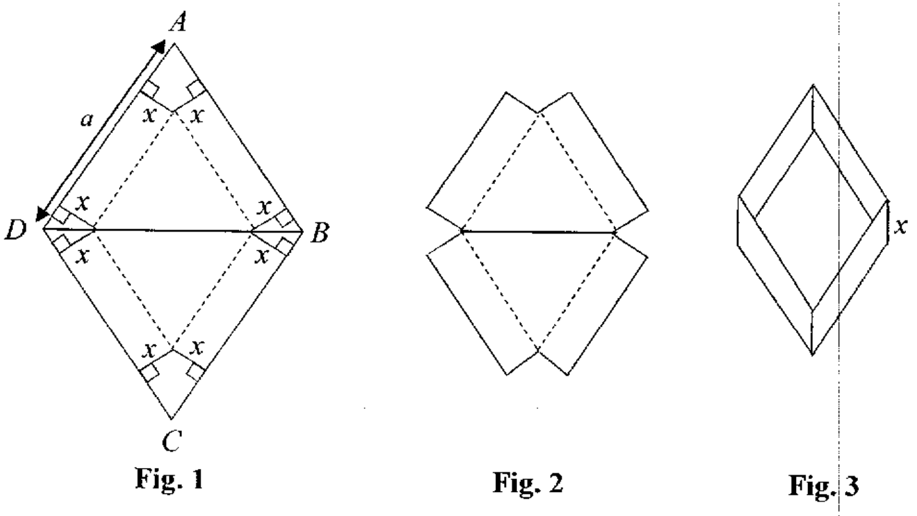

Fig. 1 shows a piece of card, $ABCD$, in the form of a rhombus of sides a cm. The card is made up of 2 identical equilateral triangles ABD and CBD. Kite shapes are cut from each corner, to give the shape shown in Fig. 2. The remaining card in Fig. 2 is folded along the dotted lines, to form an open rhombus shaped prism of height x shown in Fig. 3.

(i) Show that the volume V of the prism is given by $V = \frac{\sqrt{3}x}{2}\left(a - \frac{4x}{\sqrt{3}}\right)^2$. [3]

(ii) Use differentiation to find, in terms of a, the maximum value of V, proving that it is a maximum. [6]

The prism is then filled with sand, evenly distributed through a sieve, at a constant rate of $\sqrt{3}$ cm^3/s. Find in terms of a, the rate of increase of the depth of the sand when the depth of the sand in the prism is $\sqrt{3}a$ cm. [3]

9 Two expedition teams are to climb a vertical distance of 8500 m from the foot to the peak of a mountain over a period of time.

(i) Team A plans to cover a vertical distance of 400 m on the first day. On each subsequent day, the vertical distance covered is 5 m less than the vertical distance covered in the previous day. Find the number of days required for Team A to reach the peak. [2]

(ii) Team B plans to cover a vertical distance of 800 m on the first day. On each subsequent day, the vertical distance covered is 90% of the vertical distance covered in the previous day. On which day will Team A overtake Team B? [3]

(iii) Explain why Team B will never be able to reach the peak. [2]

(iv) At the end of the 15th day, Team B decided to modify their plan, such that on each subsequent day, the vertical distance covered is 95% of the vertical distance covered in the previous day. Which team will be the first to reach the peak of the mountain? Justify your answer. [5]

10 The function f is defined by

$$\mathrm{f} : x \mapsto \frac{x^2+x+1}{x+1} \qquad \text{for } x \in \mathbb{R}, \quad x \neq -1.$$

(i) Find algebraically the range of f. [4]

(ii) Sketch the graph of $y = \mathrm{f}(x)$, showing clearly the equations of any asymptotes and coordinates of any turning points and axial intercepts. [3]

(iii) By adding a suitable graph to your sketch in **(ii)**, deduce the range of values of k for which the equation

$$\left(x^2+x+1\right)^2 + (x+1)^2(x+2)^2 = k^2(x+1)^2$$

has exactly one positive real root and exactly one negative real root. [5]

Name:________________________________ Class:__________

JURONG PIONEER JUNIOR COLLEGE

JC2 Preliminary Examination 2021

MATHEMATICS
Higher 2

Paper 2

9758/02
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Total Marks	/ 100

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Section A : Pure Mathematics [40 Marks]

1 In this question, you may use expansions from the List of Formula (MF26).

(i) Find the Maclaurin expansion of $\ln(2-2\sin x)$ in ascending powers of x, up to and including the term in x^3. [4]

(ii) Use your series in (i) to estimate $\int_1^2 \frac{\ln(2-2\sin x)}{x^2}\,dx$, correct to 4 decimal places. With reference to the values of x, comment on the accuracy of your approximation. [2]

2 Use the substitution $u=3x+y$ to find the particular solution of the differential equation $\frac{dy}{dx}=(3x+y)^2-12$ given that $y=-9$ when $x=0$. Express your answer in the form $y=f(x)$. [7]

3 A curve C has equation $y=3-x|1+x|$.

(i) Describe 2 transformations which transform the graph of C on to the graph of $y=x|x-1|$. [2]

(ii) Sketch C, giving the coordinates of the axial intercepts and turning point. [2]

(iii) Using your sketch in (ii), solve the inequality $x|1+x|\geq x+1$. [2]

4 A sequence $v_1, v_2, v_3, \ldots$ is such that

$$v_n=(2-n)2^{n-1} \text{ and } \quad v_{n+1}=v_n-n(2)^{n-1}.$$

(i) By considering $\sum_{n=1}^{m}(v_n-v_{n+1})$, find $\sum_{n=1}^{m} n2^n$. [3]

(ii) Determine whether the series in (i) converges, stating your reason clearly. [1]

(iii) Using the result in (i), find $\sum_{n=3}^{m}(n+2)2^n$, leaving your answer in the form of $2^{m+1}(a+m)+b$, where a and b are constants to be determined. [4]

5 (a) Find $\int \frac{x}{x^2-2x+5}\,dx$. [3]

(b) State the derivative of $\tan x^3$. Hence, find $\int x^5 \sec^2\left(x^3\right)dx$. [4]

(c) Find the exact value of $\int_1^{\sqrt{3}} \left(3+x^2\right)^{-2} dx$, using the substitution $x=\sqrt{3}\cot\theta$. [6]

Section B : Statistics [60 Marks]

6 A 4-sided biased die has its face marked with the numbers 3, 5, 7 and 9. The probability of getting 3, 5, 7 and 9 are p, q, $\frac{1}{5}$, $\frac{2}{5}$ respectively.

(i) If the mean number obtained is 6.8, find p and q. [2]

A game consists of a player throwing two such 4-sided biased dice simultaneously. The number on each die is recorded. The score X is defined as the absolute difference of the two numbers.

(ii) Obtain the probability distribution of X. [2]

(iii) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. [3]

A player pays $\$a$ to play this game. The player receives an amount (in dollars) corresponding to thrice his score obtained. Determine the range of values of a if the player is expected to win. [2]

7 In this question you should state clearly the values of the parameters of any normal distribution you use.

The masses, in grams, of apples, pears and empty boxes follow normal distributions with means and standard deviations as shown in the following table.

	Mean	Standard deviation
Apple	80	5
Pear	150	10
Empty Box	k	15

(i) Find the probability that the total mass of 4 randomly chosen apples differs from twice the mass of a randomly chosen pear by at most 20 grams. [3]

4 randomly chosen apples and 4 randomly chosen pears are packed into an empty box to form a fruit hamper.

(ii) Find, to 1 decimal place, the range of values of k such that more than 95% of the fruit hampers have a mass of at most 1.8kg. [4]

(iii) State, in the context of the question, an assumption required in your calculation in **(i)** and **(ii)**. [1]

8 The events A, B and C are such that $P(A) = 0.26$, $P(B) = 0.3$ and $P(C) = c$. It is given that A and B are independent and that A and C are mutually exclusive.

(i) Find $P(A \mid B)$. [1]

(ii) Find $P(A \cup B)$. Hence find $P(B \cap C)$ in terms of c if it is given that $P(A' \cap B' \cap C') = 0.196$. [3]

(iii) Given that $P(C) = 0.55$, find the maximum and minimum values of $P(B \cap C)$. [4]

9 Find the number of ways in which the letters of the word PASSIONATE can be arranged if

(i) there are no restrictions, [1]

(ii) P and N must not be next to each other, [2]

(iii) consonants (P, N, T, S) and vowels (I, O, E, A) must alternate, [3]

(iv) between P and N, there are exactly two other letters and at least one of them must be a S. [4]

10 The management of a theme park claims that a visitor spends more than 9 hours in the theme park. The time, x hours, spent by 100 randomly chosen visitors are recorded and summarised as follows:

$$\sum x = 920, \ \sum x^2 = 8668.$$

(i) Test at the 10% significance level whether the management's claim is valid. [6]

(ii) Explain why the use of Central Limit Theorem is necessary in your calculation in **(i)**. [1]

The management now claims that a visitor spends more than μ_0 hours in the theme park. A second random sample of 60 visitors is taken. The time, y hours, spent are recorded and the results are summarised as follows:

$$\sum y = 550, \quad \sum y^2 = 5120.$$

(iii) Combining the two samples into a single sample and using an algebraic method, calculate the set of values of μ_0 for the management's new claim not to be accepted at the 10% level of significance. (Answers obtained by trial and improvement from a calculator will obtain no marks.) [5]

11 Mr Goh sells Hainanese chicken rice at a hawker centre. His stall is open everyday except Sundays. It is observed that, on average, one in p customers uses e-payment for their chicken rice orders.

(i) State, in context, two assumptions needed for the number of customers who use e-payment to be well modelled by a binomial distribution. [2]

Assuming that these assumptions do in fact hold. On each day when Mr Goh's stall is open, 50 customers order chicken rice from him during lunch time.

(ii) Given that the modal number of customers who use e-payment is 8, write down the inequalities involving p. Hence find the integer value of p. [4]

For the rest of this question, use $p = 6$.

(iii) Find the probability that on a particular day, more than 9 and no more than 15 customers use e-payment during lunch time. [2]

(iv) Find the probability that in a particular week, there are more than 2 days with more than 9 and no more than 15 customers use e-payment during lunch time. [2]

(v) Find the probability that in a particular 6-weeks period, the average number of customers who use e-payment during lunch time each day exceeds 8. [3]

Jurong Pioneer Junior College
H2 Mathematics
JC2 Preliminary Examination Paper 1 (Solution) **JC2 – 2021**

Q1

Let the unit cost **from 1 July 2021** for electricity be $$x$ per kWh, gas be $$y$ per kWh and water be $$z$ per m^3.

June: $(x-0.015)316+97y+15.6z=130.84$
$316x+97y+15.6z=135.58\cdots\cdots(1)$

July: $320x+86y+15.7z=134.61\cdots\cdots(2)$

August: $330x+80y+15.8z=136.02\cdots\cdots(3)$

From GC, $x\approx 0.232,\quad y\approx 0.198,\quad z\approx 2.763$
The unit cost from 1 July 2021 for electricity is \$0.232 per kWh, gas is \$0.198 per kWh and water is \$2.763 per m^3.

Alternative
Let the unit cost **before 1 July 2021** for electricity be $$x$ per kWh, gas be $$y$ per kWh and water be $$z$ per m^3.

June: $316x+97y+15.6z=130.84\cdots\cdots(1)$

July: $320(x+0.015)+86y+15.7z=134.61$
$320x+86y+15.7z=129.81\cdots\cdots(2)$

August: $330(x+0.015)+80y+15.8z=136.02$
$330x+80y+15.8z=131.07\cdots\cdots(3)$
From GC, $x\approx 0.21696,\quad y\approx 0.198,\quad z\approx 2.763$
The unit cost from 1 July 2021 for electricity is \$0.232 per kWh, gas is \$0.198 per kWh and water is \$2.763 per m^3.

Q2

(i)

$$\overrightarrow{BA}=\begin{pmatrix}1\\1\\1\end{pmatrix}-\begin{pmatrix}1\\2\\2\end{pmatrix}=\begin{pmatrix}0\\-1\\-1\end{pmatrix}$$

$$\cos\measuredangle OAB=\frac{\overrightarrow{OA}\cdot\overrightarrow{BA}}{\left|\overrightarrow{OA}\right|\left|\overrightarrow{BA}\right|}=\frac{\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}0\\-1\\-1\end{pmatrix}}{\sqrt{3}\sqrt{2}}=\frac{-2}{\sqrt{6}}$$

$\measuredangle OAB=144.7^\circ$

(ii)

Area of $OAC = \frac{1}{2}\left|\underset{\sim}{a}\times(\lambda\underset{\sim}{a}+\mu\underset{\sim}{b})\right| = \frac{\mu}{2}\left|\underset{\sim}{a}\times\underset{\sim}{b}\right|$ since $\mu > 0$

Area of $OBC = \frac{1}{2}\left|\underset{\sim}{b}\times(\lambda\underset{\sim}{a}+\mu\underset{\sim}{b})\right| = \frac{\lambda}{2}\left|\underset{\sim}{b}\times\underset{\sim}{a}\right|$ since $\lambda > 0$

Given area of triangle OAC is twice that of triangle OBC,

$\frac{\mu}{2}\left|\underset{\sim}{a}\times\underset{\sim}{b}\right| = 2\frac{\lambda}{2}\left|\underset{\sim}{b}\times\underset{\sim}{a}\right|$

since $\left|\underset{\sim}{a}\times\underset{\sim}{b}\right| = \left|\underset{\sim}{b}\times\underset{\sim}{a}\right|$

$\therefore \mu = 2\lambda$

(iii)

$OC = \sqrt{118}$

$\left|\lambda\underset{\sim}{a}+\mu\underset{\sim}{b}\right| = \sqrt{118}$

$\left|\lambda\underset{\sim}{a}+2\lambda\underset{\sim}{b}\right| = \sqrt{118}$ since $\mu = 2\lambda$

$\left|\begin{pmatrix}\lambda+2\lambda\\ \lambda+4\lambda\\ \lambda+4\lambda\end{pmatrix}\right| = \sqrt{118}$

$(3\lambda)^2+(5\lambda)^2+(5\lambda)^2 = 118$

$9\lambda^2+25\lambda^2+25\lambda^2 = 118$

$59\lambda^2 = 118$

$\lambda = \pm\sqrt{2}$

Since $\lambda > 0$

$\lambda = \sqrt{2}$, $\underset{\sim}{c} = \begin{pmatrix}3\sqrt{2}\\ 5\sqrt{2}\\ 5\sqrt{2}\end{pmatrix}$

Or

$\left|\begin{pmatrix}3\lambda\\ 5\lambda\\ 5\lambda\end{pmatrix}\right| = \sqrt{118}$

$\lambda\left|\begin{pmatrix}3\\ 5\\ 5\end{pmatrix}\right| = \sqrt{118}$ since $\lambda > 0$

$\lambda\sqrt{59} = \sqrt{118}$

$\lambda = \sqrt{2}$

$\underset{\sim}{c} = \begin{pmatrix}3\sqrt{2}\\ 5\sqrt{2}\\ 5\sqrt{2}\end{pmatrix}$

Q3

(i)

$\frac{dx}{dt}=2 \qquad \frac{dy}{dt}=-\frac{2}{t^3}$

$\frac{dy}{dx}=-\frac{1}{t^3}$

At $t=p, \frac{dy}{dx}=-\frac{1}{p^3}$

$y-\frac{1}{p^2}=-\frac{1}{p^3}(x-2p-1)$

$yp^3-p=2p+1-x$

$y=\frac{3}{p^2}+\frac{1}{p^3}-\frac{x}{p^3}$

(ii)

At $p=1$

Tangent:

$y=4-x$

Intersection:

$\frac{1}{t^2}=4-(2t+1)$

$2t^3-3t^2+1=0$

From GC:

$t=1$ or $-\frac{1}{2}$

$A(3,1)$ and $B(0,4)$

(iii)

Method 1

Circumference of a circle with AB as diameter

Mid point of $AB=\left(\frac{3}{2},\frac{5}{2}\right)$

Length of $AB=\sqrt{3^2+(1-4)^2}=\sqrt{18}$

Equation of Path traced by C

$\left(x-\frac{3}{2}\right)^2+\left(y-\frac{5}{2}\right)^2=\left(\frac{\sqrt{18}}{2}\right)^2$

$\frac{1}{4}(2x-3)^2+\frac{1}{4}(2y-5)^2=\frac{18}{4}$

$(2x-3)^2+(2y-5)^2=18$

Method 2

Let coordinates of C be (x,y).
By Pythagoras Theorem,

$$AC^2 + BC^2 = AB^2$$
$$(x-3)^2 + (y-1)^2 + x^2 + (y-4)^2 = (3-0)^2 + (1-4)^2$$
$$x^2 - 6x + 9 + y^2 - 2y + 1 + x^2 + y^2 - 8y + 16 = 18$$
$$2x^2 - 6x + 2y^2 - 10y + 26 = 18$$
$$x^2 - 3x + y^2 - 5y + 13 = 9$$
$$x^2 - 3x + y^2 - 5y + 4 = 0$$

Method 3

Let coordinates of C be (x, y)

$$\overrightarrow{AC} = \begin{pmatrix} x-3 \\ y-1 \end{pmatrix} \qquad \overrightarrow{BC} = \begin{pmatrix} x \\ y-4 \end{pmatrix}$$
$$\overrightarrow{AC} \bullet \overrightarrow{BC} = 0$$
$$\begin{pmatrix} x-3 \\ y-1 \end{pmatrix} \bullet \begin{pmatrix} x \\ y-4 \end{pmatrix} = 0$$
$$x^2 - 3x + y^2 - 5y + 4 = 0$$

Method 4

Let coordinates of C be (x, y).
Since AC is perpendicular to BC, $m_1 m_2 = -1$

$$\frac{y-1}{x-3} \times \frac{y-4}{x-0} = -1$$
$$(y-1)(y-4) = -x(x-3)$$
$$y^2 - 5y + x^2 - 3x + 4 = 0$$

Q4

(a)

Since P(z) contains only real coefficient and $re^{i\theta}$ is a root, a second root is $re^{-i\theta}$.

Quadratic factor of P(z) = $\left(z - re^{i\theta}\right)\left(z - re^{-i\theta}\right)$

$$= z^2 - \left(re^{i\theta} + re^{-i\theta}\right)z + \left(re^{i\theta}\right)\left(re^{-i\theta}\right)$$
$$= z^2 - r\left(e^{i\theta} + e^{-i\theta}\right)z + r^2$$
$$= z^2 - r\left(\cos\theta + i\sin\theta + \cos\theta - i\sin\theta\right)z + r^2$$
$$= z^2 - \left(2r\cos\theta\right)z + r^2 \quad \text{(shown)}$$

Since root of the equation is $e^{i\left(\frac{\pi}{3}\right)}$, $r=1$ and $\theta=\frac{\pi}{3}$.

Quadratic factor of $P(z)=z^2-2(1)\left(\cos\frac{\pi}{3}\right)z+1^2=z^2-z+1$

$\therefore\ 3z^3+az^2+z+2=(z^2-z+1)(3z+2)$

By comparing z^2 term, $a=2-3=-1$

Hence, roots of the equation $3z^3-z^2+z+2=(z^2-z+1)(3z+2)=0$ are $e^{i\left(\frac{\pi}{3}\right)}$, $e^{i\left(-\frac{\pi}{3}\right)}$ and $-\frac{2}{3}$

(b)

$w=-p+pi=\sqrt{2}pe^{i\left(\frac{3\pi}{4}\right)}$

$$\frac{w^n}{w^*}=\frac{\left(\sqrt{2}p\right)^n e^{i\left(\frac{3n\pi}{4}\right)}}{\sqrt{2}pe^{i\left(-\frac{3\pi}{4}\right)}}$$

$$=\left(\sqrt{2}p\right)^{n-1}e^{i\left(\frac{3n\pi}{4}+\frac{3\pi}{4}\right)}$$

$$=\left(\sqrt{2}p\right)^{n-1}e^{i\frac{3\pi}{4}(n+1)}$$

For $\frac{w^n}{w^*}$ to be purely imaginary,

$\arg\left(\frac{w^n}{w^*}\right)=\pm\frac{\pi}{2},\pm\frac{3\pi}{2},\pm\frac{5\pi}{2},\pm\frac{7\pi}{2},\pm\frac{9\pi}{2},\ldots$ or $\cos\left[\frac{3\pi}{4}(n+1)\right]=0$

$\frac{3\pi}{4}(n+1)=\pm\frac{\pi}{2},\pm\frac{3\pi}{2},\pm\frac{5\pi}{2},\pm\frac{7\pi}{2},\pm\frac{9\pi}{2},\ldots$

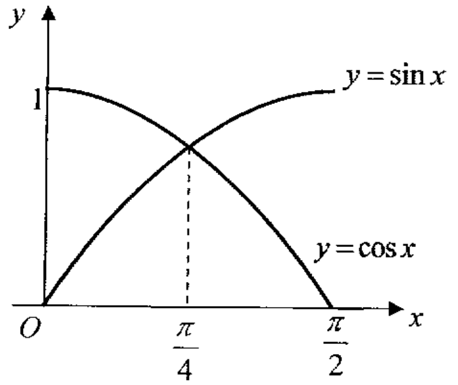
$n+1=\pm\frac{2}{3},\pm2,\pm\frac{10}{3},\pm\frac{14}{3},\pm6,\ldots$

$n=\ \ldots,\ 1,\ \ \ldots\ \ ,\ 5,\ldots$

The two smallest positive integers of n are 1 and 5.

Q5

(i)



From the graph, $\sin x > \cos x$ when $\frac{\pi}{4} < x < \frac{\pi}{2}$.

From above, when $\frac{\pi}{4} < x < \frac{\pi}{2}$, $\sin x > \cos x \Rightarrow \sin x - \cos x > 0$

$\int_0^n |\sin x - \cos x| \, dx$

$= -\int_0^{\frac{\pi}{4}} (\sin x - \cos x) \, dx + \int_{\frac{\pi}{4}}^n \sin x - \cos x \, dx$

$= \int_0^{\frac{\pi}{4}} \cos x - \sin x \, dx + \int_{\frac{\pi}{4}}^n \sin x - \cos x \, dx$

$= [\sin x + \cos x]_0^{\frac{\pi}{4}} + [-\cos x - \sin x]_{\frac{\pi}{4}}^n$

$= (\sqrt{2} - 1) - (\cos n + \sin n - \sqrt{2})$

$= 2\sqrt{2} - 1 - \cos n - \sin n$

(ii)

Volume generated

$= \pi \int_0^{\frac{\pi}{2}} (|\sin x - \cos x|)^2 \, dx$

$= \pi \int_0^{\frac{\pi}{2}} (\sin x - \cos x)^2 \, dx$

$= \pi \int_0^{\frac{\pi}{2}} (\sin^2 x - 2\sin x \cos x + \cos^2 x) \, dx$

$= \pi \int_0^{\frac{\pi}{2}} (1 - \sin 2x) \, dx$

$= \pi \left[x + \frac{\cos 2x}{2} \right]_0^{\frac{\pi}{2}}$

$= \pi \left[\left(\frac{\pi}{2} - \frac{1}{2} \right) - \left(\frac{1}{2} \right) \right]$

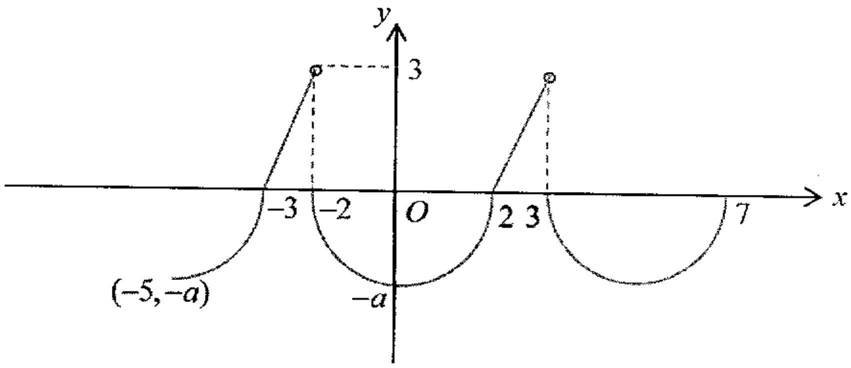
$= \pi \left(\frac{\pi}{2} - 1 \right)$

Q6

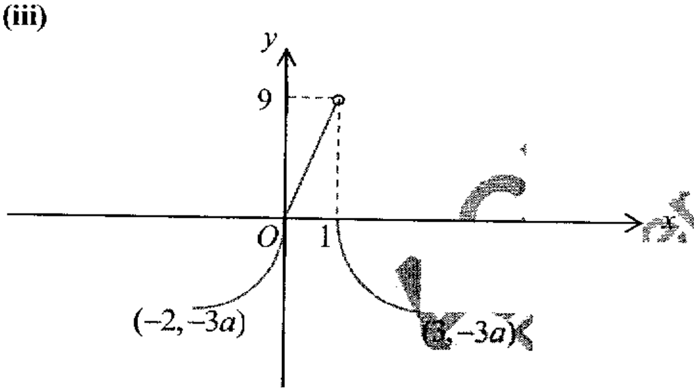
(i)

$f(2021) = f(1) = -a\sqrt{1-\frac{(1)^2}{4}} = -\frac{\sqrt{3}}{2}a$

(ii)



(iii)



(iv)

$R_g = [2,3)$

For fg to exist, $R_g \subseteq D_f$. Hence, $f(x) = 3x - 6$

$fg(x) = 3(\sqrt{x}+2) - 6 = 3\sqrt{x}$, $0 \le x < 1$.

Q7

(i)

$$\underset{\sim}{a}=\begin{pmatrix}1\\-3\\2\end{pmatrix},\ \underset{\sim}{b}=\begin{pmatrix}3\\1\\2\end{pmatrix},\ \underset{\sim}{c}=\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

$$\overrightarrow{AB}=\begin{pmatrix}2\\4\\0\end{pmatrix},\ \overrightarrow{AC}=\begin{pmatrix}0\\2\\-1\end{pmatrix}$$

$$\underset{\sim}{n}=\begin{pmatrix}2\\2\\0\end{pmatrix}\times\begin{pmatrix}0\\2\\-1\end{pmatrix}=\begin{pmatrix}-4\\2\\4\end{pmatrix}$$

$$\underset{\sim}{r}\bullet\begin{pmatrix}-4\\2\\4\end{pmatrix}=\begin{pmatrix}1\\-3\\2\end{pmatrix}\bullet\begin{pmatrix}-4\\2\\4\end{pmatrix}$$

$$\therefore \underset{\sim}{r}\bullet\begin{pmatrix}-2\\1\\2\end{pmatrix}=-1$$

(ii)

Since normal of p_1 is parallel to l, l is perpendicular to p_1

(iii)

Method 1

Since D lies on l, $\overrightarrow{OD}=\begin{pmatrix}-2t\\1+t\\2+2t\end{pmatrix}$

$$\overrightarrow{AD}=\begin{pmatrix}-2t\\1+t\\2+2t\end{pmatrix}-\begin{pmatrix}1\\-3\\2\end{pmatrix}=\begin{pmatrix}-1-2t\\4+t\\2t\end{pmatrix}$$

$$\left|\overrightarrow{AD}\bullet\underset{\sim}{\hat{n}}\right|=6$$

$$\left|\begin{pmatrix}-1-2t\\4+t\\2t\end{pmatrix}\bullet\begin{pmatrix}-2\\1\\2\end{pmatrix}\right|\frac{1}{3}=6$$

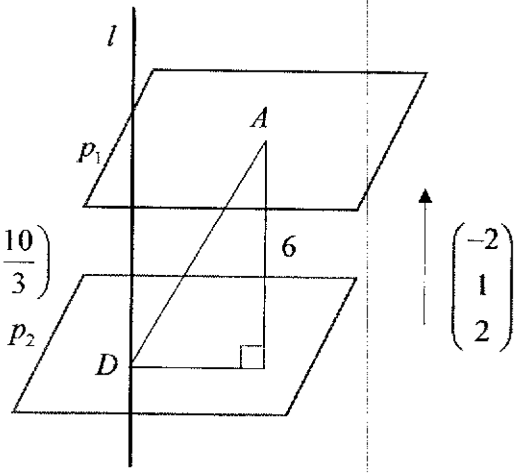
$$|2+4t+4+t+4t|=18$$

$$|9t+6|=18$$

$9t+6=18 \quad \text{or} \quad -18$

$t=\frac{4}{3} \text{ or } -\frac{8}{3}$

$\overrightarrow{OD}=\begin{pmatrix} -8/3 \\ 7/3 \\ 14/3 \end{pmatrix} \text{ or } \begin{pmatrix} 16/3 \\ -5/3 \\ -10/3 \end{pmatrix}$



Coordinates of D are $\left(-\frac{8}{3},\frac{7}{3},\frac{14}{3}\right)$ or $\left(\frac{16}{3},-\frac{5}{3},-\frac{10}{3}\right)$

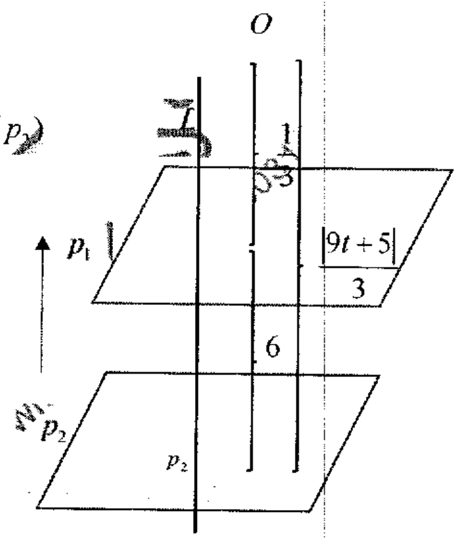
Method 2

Since D lies on l, $\overrightarrow{OD}=\begin{pmatrix} -2t \\ 1+t \\ 2+2t \end{pmatrix}$

Eqn of p_2

$\underset{\sim}{r}\bullet\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}=\begin{pmatrix} -2t \\ 1+t \\ 2+2t \end{pmatrix}\bullet\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}=9t+5$ $\quad (\because p_1 // p_2)$

$\underset{\sim}{r}\bullet\frac{1}{3}\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}=\frac{9t+5}{3}$



Eqn of p_1

$\underset{\sim}{r}\bullet\frac{1}{3}\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}=-\frac{1}{3}$

Distant between both planes

$\left|\frac{9t+5}{3}-\left(-\frac{1}{3}\right)\right|=6$

$|9t+6|=18$

$9t+6=18 \quad \text{or} \quad -18$

$t=\frac{4}{3} \text{ or } -\frac{8}{3}$

$\overrightarrow{OD}=\begin{pmatrix} -8/3 \\ 7/3 \\ 14/3 \end{pmatrix} \text{ or } \begin{pmatrix} 16/3 \\ -5/3 \\ -10/3 \end{pmatrix}$

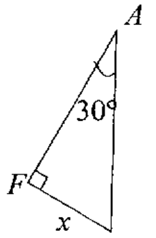
Coordinates of D are $\left(-\frac{8}{3},\frac{7}{3},\frac{14}{3}\right)$ or $\left(\frac{16}{3},-\frac{5}{3},-\frac{10}{3}\right)$

Q8

Consider Vertex A

$\tan 30^\circ = \dfrac{x}{AF}$

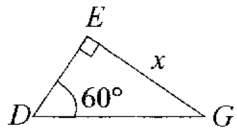
$AF = \dfrac{x}{\tan 30^\circ}$



Consider Vertex D

$\tan 60^\circ = \dfrac{x}{DE}$ or $\sin 60^\circ = \dfrac{x}{DG}$

$DE = \dfrac{x}{\tan 60^\circ}$ $\quad DG = \dfrac{x}{\sin 60^\circ}$



Length of box:

$$l = a - \frac{x}{\tan 30^\circ} - \frac{x}{\tan 60^\circ} = a - \sqrt{3}x - \frac{x}{\sqrt{3}} = a - \frac{4x}{\sqrt{3}} \qquad \text{or} \qquad l = a - 2\left(\frac{x}{\sin 60^\circ}\right) = a - 2\left(\frac{2x}{\sqrt{3}}\right)$$

$$V = 2 \times \frac{1}{2} l^2 \sin 60^\circ \times x = \frac{\sqrt{3}x}{2}\left(a - \frac{4x}{\sqrt{3}}\right)^2$$

$$\frac{dV}{dx} = \frac{\sqrt{3}}{2}\left(a - \frac{4x}{\sqrt{3}}\right)^2 - 4x\left(a - \frac{4x}{\sqrt{3}}\right) = \frac{1}{2}\left(a - \frac{4x}{\sqrt{3}}\right)\left(\sqrt{3}a - 12x\right)$$

At Stat pt, $\dfrac{dV}{dx} = 0$

$$\therefore a - \frac{4x}{\sqrt{3}} = 0 \quad \text{or} \quad \sqrt{3}a - 12x = 0$$

$$x = \frac{\sqrt{3}a}{4} \text{ or } \frac{\sqrt{3}a}{12}$$

$$\frac{d^2V}{dx^2} = \frac{1}{2}\left(-\frac{4}{\sqrt{3}}\left(\sqrt{3}a - 12x\right) - 12\left(a - \frac{4x}{\sqrt{3}}\right)\right) = \frac{48}{\sqrt{3}}x - 8a$$

$$x = \frac{\sqrt{3}a}{4}, \quad \frac{d^2V}{dx^2} = \frac{48}{\sqrt{3}}\left(\frac{\sqrt{3}a}{4}\right) - 8a = 4a > 0 \rightarrow \text{minimum, since } a > 0$$

$$x = \frac{\sqrt{3}a}{12}, \quad \frac{d^2V}{dx^2} = \frac{48}{\sqrt{3}}\left(\frac{\sqrt{3}a}{12}\right) - 8a = -4a < 0 \rightarrow \text{maximum, since } a > 0$$

$$\max V = \frac{\sqrt{3}}{2}\left(\frac{\sqrt{3}a}{12}\right)\left(a - \frac{4}{\sqrt{3}}\left(\frac{a\sqrt{3}}{12}\right)\right)^2 = \frac{a^3}{18}$$

Let the volume and depth of the sand in the prism be V_s and x_s

$$\frac{dV_s}{dt} = \frac{dV_s}{dx_s} \times \frac{dx_s}{dt}$$

$$\frac{dx_s}{dt} = \frac{dV_s}{dt} \div \left.\frac{dV_s}{dx_s}\right|_{x_s = \sqrt{3}a}$$

$$\frac{dx_s}{dt} = \sqrt{3} \div \left(\frac{1}{2}\left(a - \frac{4\sqrt{3}a}{\sqrt{3}}\right)\left(\sqrt{3}a - 12\sqrt{3}a\right)\right)$$

$$\frac{dx_s}{dt} = \frac{2}{33a^2}$$

Q9

(i)

AP with $a=400,\ d=-5$
$S_n=8500$

$\frac{n}{2}[2(400)+(n-1)(-5)]=8500$

$5n^2-805n+17000=0$
$n=25$ or $n=136$ (rejected as already reached on 25th day)

(ii)

GP with $a=800,\quad r=0.9$
$S_{n(AP)}>S_{n(GP)}$

$\frac{n}{2}[2(400)+(n-1)(-5)]>\frac{800(1-0.9^n)}{1-0.9}$

$805n-5n^2>16000\left(1-0.9^n\right)$

Method 1

Using GC

n	$805n-5n^2$		$16000\left(1-0.9^n\right)$
19	6745	<	6919
20	7050	>	7027
21	7350	>	7124

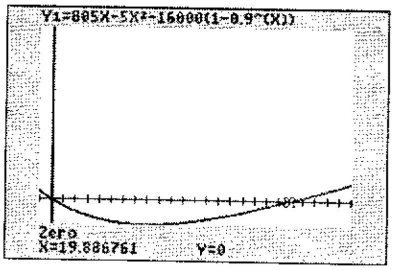
$n\geq 20$
A will overtake B on the 20th day.

Method 2

$805n-5n^2-16000\left(1-0.9^n\right)>0$

Using GC,

Sketch $y=805x-5x^2-16000\left(1-0.9^x\right)$



From graph, $n<0$ (rejected as $n\in\mathbb{Z}^+$) or $n>19.9$

Hence, A will overtake B on the 20th day.

(iii)

$$S_\infty = \frac{800}{1-0.9} = 8000(<8500)$$

Hence, Team B will never be able to reach the peak.

(iv)

$$T_{15} = 800\left(0.9^{15-1}\right) = 183.014$$

$$S_{15} = \frac{800\left(1-0.9^{15}\right)}{1-0.9} = 6352.871$$

Remaining distance $=8500-6352.871=2147.129$

First term of new GP $=183.014\times0.95=173.864$

$S_{n(New\,GP)} = 2147.129$

$$\frac{173.864\left(1-0.95^n\right)}{1-0.95} = 2147.129$$

$0.95^n = 0.38253$

$n = 18.7$

Team B will take $15+19=34$ days

Hence, Team A will reach the peak first.

Q10

(i)

$$y = \frac{x^2+x+1}{x+1}$$

$y(x+1) = x^2+x+1$

$x^2+(1-y)x+(1-y)=0$

For quadratic equation to have real roots,

discriminant ≥ 0

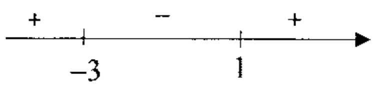
$(1-y)^2-4(1)(1-y)\geq 0$

$1-2y+y^2-4+4y\geq 0$

$y^2+2y-3\geq 0$

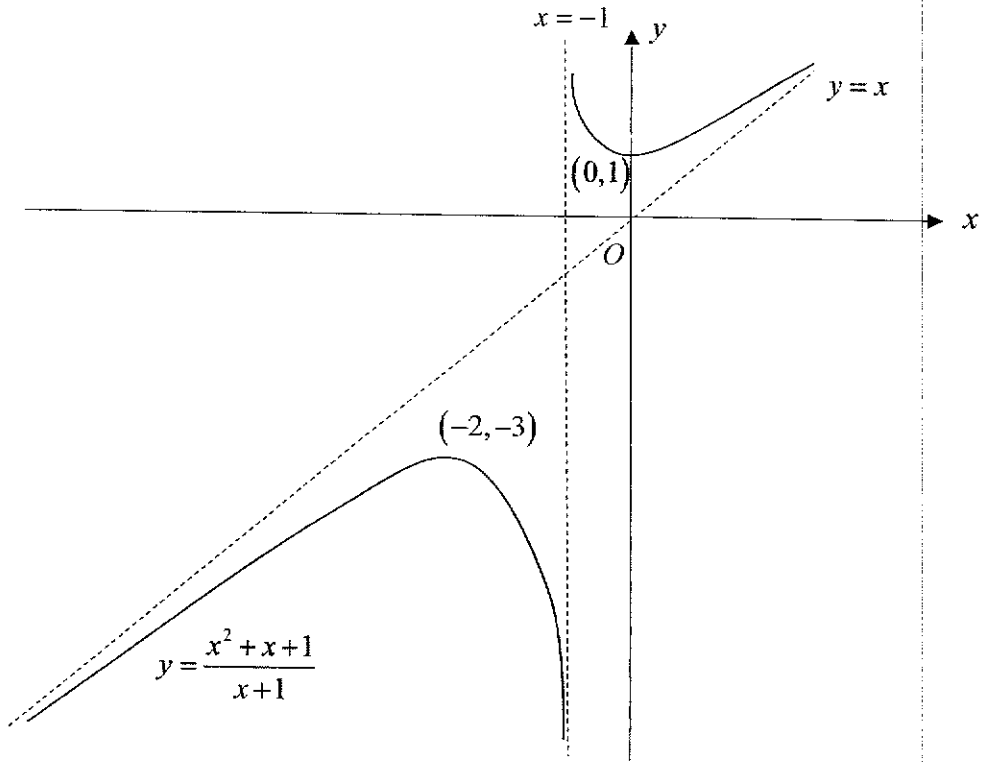
$(y+3)(y-1)\geq 0$

$y\leq -3$ or $y\geq 1$



$\therefore R_f = (-\infty,-3]\cup[1,\infty)$

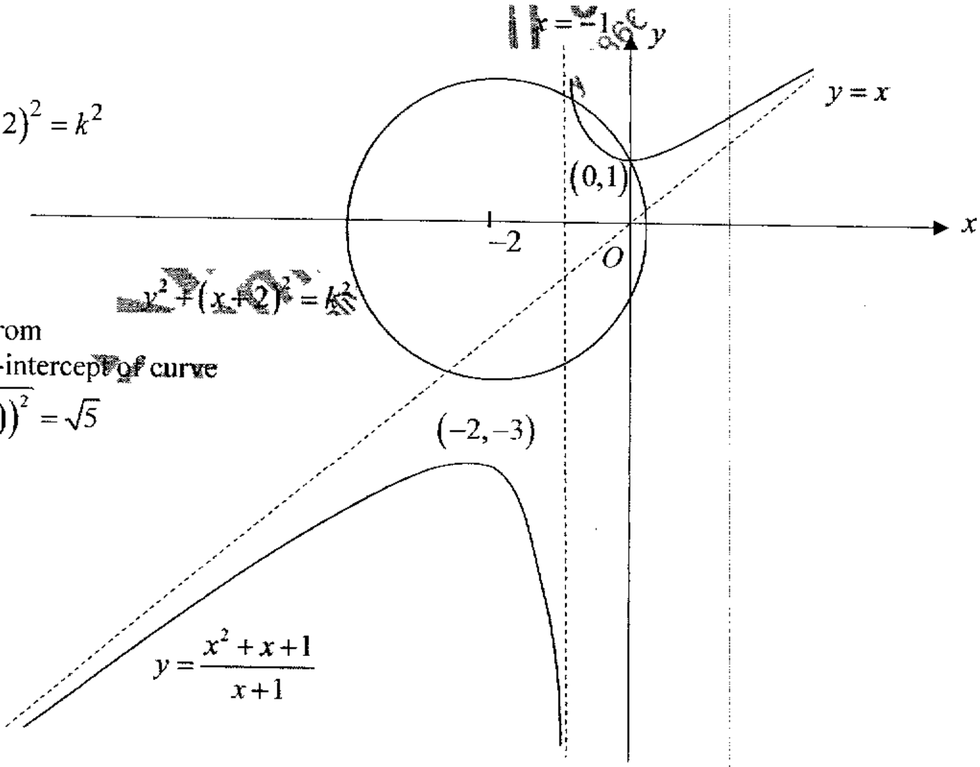
(ii)



(iii)

$$\frac{\left(x^2+x+1\right)^2}{(x+1)^2}+(x+2)^2=k^2$$

$$y^2+(x+2)^2=k^2$$



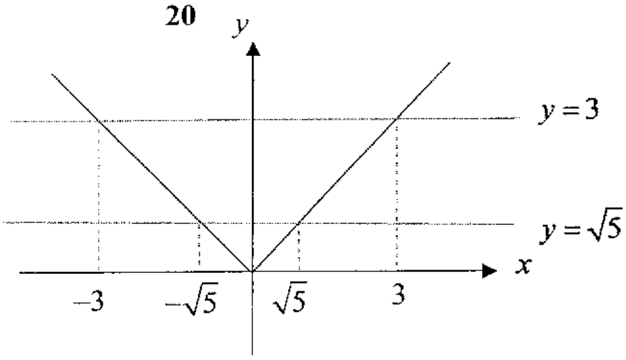
Consider distance from
centre of circle to y-intercept of curve

$$\sqrt{(1-0)^2+(0-(-2))^2}=\sqrt{5}$$

$$\sqrt{5}<|k|<3$$

$-3 < k < -\sqrt{5}$ or $\sqrt{5} < k < 3$

20



Jurong Pioneer Junior College
H2 Mathematics
JC2 Preliminary Examination Paper 2 (Solution)

JC2 – 2021

Q1

(i)

Method 1

$$
\begin{aligned}
&\ln(2-2\sin x)\\
&= \ln 2(1-\sin x)\\
&= \ln 2+\ln(1-\sin x)\\
&= \ln 2+\left[(-\sin x)-\frac{(-\sin x)^2}{2}+\frac{(-\sin x)^3}{3}+\ldots\right]\\
&=\ln 2+\left[-\left(x-\frac{x^3}{3!}\right)-\frac{1}{2}\left(x-\frac{x^3}{3!}\right)^2-\frac{1}{3}\left(x-\frac{x^3}{3!}\right)^3+\ldots\right]\\
&= \ln 2+\left[-\left(x-\frac{x^3}{6}\right)-\frac{1}{2}\left(x^2-\ldots\right)-\frac{1}{3}\left(x^3-\ldots\right)+\ldots\right]\\
&\approx \ln 2-x-\frac{x^2}{2}-\frac{x^3}{6}
\end{aligned}
$$

Method 2

$$
\begin{aligned}
&\ln(2-2\sin x)\\
&= \ln 2(1-\sin x)\\
&= \ln 2+\ln(1-\sin x)\\
&\approx \ln 2+\ln\left[1-\left(x-\frac{x^3}{3!}\right)\right]\\
&=\ln 2+\ln\left[1+\left(\frac{x^3}{3!}-x\right)\right]\\
&=\ln 2+\left(\frac{x^3}{3!}-x\right)-\frac{\left(\frac{x^3}{3!}-x\right)^2}{2}+\frac{\left(\frac{x^3}{3!}-x\right)^3}{3}-\ldots\\
&=\ln 2+\left[\frac{x^3}{6}-x-\frac{1}{2}\left(\ldots+x^2\right)+\frac{1}{3}\left(\ldots-x^3\right)-\ldots\right]\\
&\approx \ln 2-x-\frac{x^2}{2}-\frac{x^3}{6}
\end{aligned}
$$

(ii)

$$\int_1^2 \frac{\ln(2-2\sin x)}{x^2}\,dx \approx \int_1^2 \frac{1}{x^2}\left(\ln 2 - x - \frac{x^2}{2} - \frac{x^3}{6}\right) dx \approx -1.0966$$

As the limits of integration are $x=1$ and $x=2$,
which are far away from zero,
the graphs of $y=\frac{\ln(2-2\sin x)}{x^2}$ and $y=\frac{1}{x^2}\left(\ln 2 - x - \frac{x^2}{2} - \frac{x^3}{6}\right)$
deviate from each other. Hence the approximation is inaccurate.

Q2

$$\frac{dy}{dx} = (3x+y)^2 - 12 \qquad\qquad u = 3x+y \Rightarrow \frac{du}{dx} = 3 + \frac{dy}{dx}$$

$$\frac{du}{dx} - 3 = u^2 - 12$$

$$\frac{du}{dx} = u^2 - 9$$

$$\int \frac{1}{u^2-9}\,du = \int 1\,dx$$

$$\frac{1}{2(3)}\ln\left|\frac{u-3}{u+3}\right| = x + c$$

$$\left|\frac{u-3}{u+3}\right| = e^{6x+k}, k = 6c$$

$$\frac{u-3}{u+3} = Ae^{6x}, A = \pm e^k$$

$$\frac{3x+y-3}{3x+y+3} = Ae^{6x}$$

Given that $y=-9$ when $x=0$, $\frac{3(0)-9-3}{3(0)-9+3} = Ae^{6(0)} \Rightarrow A = 2$

$$\frac{3x+y-3}{3x+y+3} = 2e^{6x}$$

$$3x+y-3 = 2e^{6x}(3x+y+3)$$

$$y\left(1-2e^{6x}\right) = 2e^{6x}(3x+3) - 3x + 3$$

$$y = \frac{-3x\left(1-2e^{6x}\right) + 3\left(1+2e^{6x}\right)}{1-2e^{6x}}$$

$$= \frac{3\left(1+2e^{6x}\right)}{1-2e^{6x}} - 3x$$

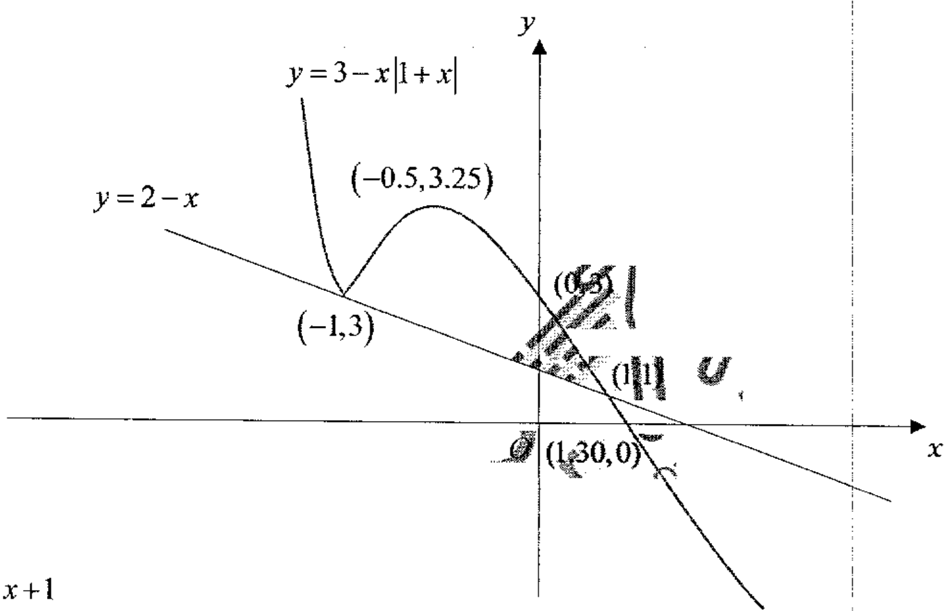
Q3

(i)

$$y = 3 - x|1+x| \xrightarrow{A} y = 3 + x|1-x| = 3 + x|x-1| \xrightarrow{B} y = x|x-1|$$

A: Reflection in the y-axis
B: Translation of 3 units in the negative y direction
(The sequence of the transformations does not matter)

(ii)



$x|1+x| \geq x+1$

$-x|1+x| \leq -x-1$

$3 - x|1+x| \leq 2 - x$

From the graphs, $x \geq 1$ or $x = -1$

Q4

(i)

$$\sum_{n=1}^{m} n2^n = 2\sum_{n=1}^{m} n2^{n-1} = 2\sum_{n=1}^{m}\left(v_n - v_{n+1}\right)$$

$$= 2\begin{bmatrix} v_1 - v_2 \\ + v_2 - v_3 \\ + v_3 - v_4 \\ + \ldots \\ + v_{m-2} - v_{m-1} \\ + v_{m-1} - v_m \\ + v_m - v_{m+1} \end{bmatrix}$$

$$= 2\left[v_1 - v_{m+1}\right]$$

$$= 2\left[(2-1)2^0 - (2-m-1)2^m\right]$$

$$= 2\left[1-(1-m)2^m\right]$$

(ii)

As $m \to \infty$, $(1-m)2^m \to -\infty$, $2\left[1-(1-m)2^m\right] \to \infty$, $\therefore \sum_{n=1}^{m} n2^n$ does not converge.

(iii)

Method 1

$$\sum_{n=3}^{m}(n+2)2^n = \sum_{n=3}^{m} n2^n + 2\sum_{n=3}^{m} 2^n$$

$$= \sum_{n=1}^{m} n2^n - \sum_{n=1}^{2} n2^n + 2\sum_{n=3}^{m} 2^n$$

$$= 2\left[1-(1-m)2^m\right] - 2 - 2(2)^2 + 2\left[\frac{2^3(1-2^{m-2})}{1-2}\right]$$

$$= 2 - 2^{m+1}(1-m) - 10 - 2(8 - 2^{m+1})$$

$$= 2^{m+1}(m-1) - 8 - 16 + 2^{m+1}(2)$$

$$= 2^{m+1}(m-1+2) - 24$$

$$= 2^{m+1}(1+m) - 24 \quad \text{where } a = 1,\ b = -24$$

Method 2

Let $n = k-2$

$$\sum_{n=3}^{m}(n+2)2^n = \sum_{k=5}^{m+2} k2^{k-2}$$

$$= \frac{1}{4}\sum_{k=5}^{m+2} k2^k$$

$$= \frac{1}{4}\sum_{k=1}^{m+2} k2^k - \frac{1}{4}\sum_{k=1}^{4} k2^k$$

$$= \frac{1}{4}(2)\left[1-(1-(m+2))2^{m+2}\right] - \frac{1}{4}(2)\left[1-(1-4)2^4\right] \quad \text{(or use GC to get } \frac{49}{2}\text{)}$$

$$= \frac{1}{2} + (m+1)2^{m+1} - \frac{49}{2}$$

$$= 2^{m+1}(1+m) - 24 \quad \text{where } a = 1,\ b = -24$$

Q5

(a)

$$\int \frac{x}{x^2-2x+5}\,dx$$

$$= \int \frac{\frac{1}{2}(2x-2)+1}{x^2-2x+5}\,dx$$

$$= \frac{1}{2}\int \frac{2x-2}{x^2-2x+5}\,dx + \int \frac{1}{(x-1)^2+2^2}\,dx$$

$$= \frac{1}{2}\ln\left(x^2-2x+5\right) + \frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right) + c, \text{ since } x^2-2x+5>0$$

(b)

$$\frac{d}{dx}\tan x^3 = 3x^2\sec^2\left(x^3\right)$$

$u = x^3$ $\qquad$ $\frac{dv}{dx} = 3x^2\sec^2\left(x^3\right)$

$\frac{du}{dx} = 3x^2$ $\qquad$ $v = \tan\left(x^3\right)$

$$\int x^5\sec^2\left(x^3\right)dx$$

$$= \frac{1}{3}\int x^3\left[3x^2\sec^2\left(x^3\right)\right]dx$$

$$= \frac{1}{3}\left[x^3\tan\left(x^3\right) - \int 3x^2\tan\left(x^3\right)dx\right]$$

$$= \frac{1}{3}\left[x^3\tan\left(x^3\right) - \ln\left|\sec\left(x^3\right)\right|\right] + c$$

(c)

$$\int_1^{\sqrt{3}} \left(3+x^2\right)^{-2} dx$$

$$x = \sqrt{3}\cot\theta$$

$$\frac{dx}{d\theta} = -\sqrt{3}\operatorname{cosec}^2\theta$$

when $x = \sqrt{3}$, $\cot\theta = 1 \Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

when $x = 1$, $\cot\theta = \frac{1}{\sqrt{3}} \Rightarrow \tan\theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$

$$= \int_{\pi/3}^{\pi/4} \left(3+3\cot^2\theta\right)^{-2} \left(-\sqrt{3}\operatorname{cosec}^2\theta\right) d\theta$$

$$= \int_{\pi/4}^{\pi/3} (3)^{-2} \left(1+\cot^2\theta\right)^{-2} (3)^{\frac{1}{2}} \left(\operatorname{cosec}^2\theta\right) d\theta$$

$$= \frac{\sqrt{3}}{9} \int_{\pi/4}^{\pi/3} \left(\operatorname{cosec}^2\theta\right)^{-2} \left(\operatorname{cosec}^2\theta\right) d\theta$$

$$= \frac{\sqrt{3}}{9} \int_{\pi/4}^{\pi/3} \left(\operatorname{cosec}\theta\right)^{-2} d\theta$$

$$= \frac{\sqrt{3}}{9} \int_{\pi/4}^{\pi/3} \sin^2\theta \, d\theta$$

$$= \frac{\sqrt{3}}{9} \int_{\pi/4}^{\pi/3} \frac{1-\cos 2\theta}{2} d\theta$$

$$= \frac{\sqrt{3}}{18} \left[\theta - \frac{\sin 2\theta}{2}\right]_{\pi/4}^{\pi/3}$$

$$= \frac{\sqrt{3}}{18} \left[\frac{\pi}{3} - \frac{\sin\frac{2\pi}{3}}{2} - \left(\frac{\pi}{4} - \frac{\sin\frac{\pi}{2}}{2}\right)\right]$$

$$= \frac{\sqrt{3}}{18} \left(\frac{\pi}{12} - \frac{\sqrt{3}}{4} + \frac{1}{2}\right)$$

Q6

(i)

$$p + q + \frac{1}{5} + \frac{2}{5} = 1$$

$$p + q = \frac{2}{5} \quad \text{-----------(1)}$$

Given mean number = 6.8

$$3p + 5q + 7\left(\frac{1}{5}\right) + 9\left(\frac{2}{5}\right) = 6.8$$

$$3p + 5q = 1.8 \quad \text{-----------(2)}$$

Solving, $p = \frac{1}{10}, q = \frac{3}{10}$

(ii)

X	3	5	7	9
3	0	2	4	6
5	2	0	2	4
7	4	2	0	2
9	6	4	2	0

$$P(X=0)=\left(\frac{1}{10}\right)^2+\left(\frac{3}{10}\right)^2+\left(\frac{1}{5}\right)^2+\left(\frac{2}{5}\right)^2=0.3$$

$$P(X=2)=2\left(\frac{3}{10}\right)\left(\frac{1}{10}\right)+2\left(\frac{3}{10}\right)\left(\frac{1}{5}\right)+2\left(\frac{1}{5}\right)\left(\frac{2}{5}\right)=0.34$$

$$P(X=4)=2\left(\frac{1}{5}\right)\left(\frac{1}{10}\right)+2\left(\frac{3}{10}\right)\left(\frac{2}{5}\right)=0.28$$

$$P(X=6)=2\left(\frac{2}{5}\right)\left(\frac{1}{10}\right)=0.08$$

Probability Distribution of X:

x	0	2	4	6
$P(X=x)$	0.3	0.34	0.28	0.08

(iii)

$E(X)=2(0.34)+4(0.28)+6(0.08)=2.28$

$E(X^2)=2^2(0.34)+4^2(0.28)+6^2(0.08)=8.72$

$\text{Var}(X)=E(X^2)-(E(X))^2=8.72-(2.28)^2=3.5216$

(Last part)

$E(3X-a)>0$

$3E(X)-a>0$

$\therefore 0<a<6.84$

Q7

(i)

Let X be the mass of an apple. Then $X\sim N(80, 5^2)$

Let Y be the mass of a pear. Then $Y\sim N(150, 10^2)$

$X_1+X_2+X_3+X_4-2Y\sim N(20, 500)$

$P(|X_1+X_2+X_3+X_4-2Y|\le 20)=P(-20\le X_1+X_2+X_3+X_4-2Y\le 20)$

≈ 0.463

(ii)

Let W be the mass of an empty box. Then $W \sim \mathrm{N}(k, 15^2)$

$X_1 + X_2 + X_3 + X_4 + Y_1 + Y_2 + Y_3 + Y_4 + W \sim \mathrm{N}(920+k, 725)$

$\mathrm{P}(X_1 + X_2 + X_3 + X_4 + Y_1 + Y_2 + Y_3 + Y_4 + W \leq 1800) > 0.95$

$\mathrm{P}\left(Z \leq \dfrac{1800-(920+k)}{\sqrt{725}}\right) > 0.95$

$\dfrac{1800-(920+k)}{\sqrt{725}} > 1.6449$ (correct to 5 s.f.)

$880 - k > 1.6449\sqrt{725}$

$\therefore\ k < 835.7$ (correct to 1 d.p)

$0 < k < 835.7$

(iii)

The masses of apples, pears and empty boxes are independent of one another.

Q8

(i)

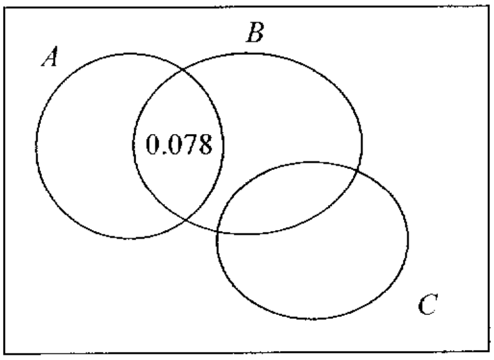
Since A and B are independent,

$\mathrm{P}(A|B) = \mathrm{P}(A) = 0.26$

(ii)

Since A and B are independent,

$\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B) = (0.26)(0.3) = 0.078$



$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$

$= 0.26 + 0.3 - 0.078$

$\mathrm{P}(A \cup B) = 0.482$

Since A and C are mutually exclusive,

$\mathrm{P}(A \cap C) = 0$ and $\mathrm{P}(A \cap B \cap C) = 0$

$\mathrm{P}(A' \cap B' \cap C') = 0.196$

$1 - \mathrm{P}(A \cup B \cup C) = 0.196$

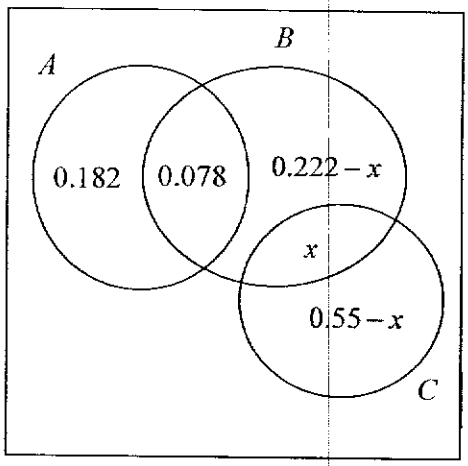
$\mathrm{P}(A \cup B \cup C) = 1 - 0.196$

$\mathrm{P}(A \cup B) + \mathrm{P}(C) - \mathrm{P}(B \cap C) = 0.804$

$0.482 + c - \mathrm{P}(B \cap C) = 0.804$

$\mathrm{P}(B \cap C) = c - 0.322$

(iii)



Let $P(B \cap C) = x$
From the Venn diagram,
$\text{Max } x = \max P(B \cap C)$
$= P(B) - P(A \cap B)$
$= 0.3 - 0.078$
$= 0.222$

$P(A \cup B \cup C)$
$= P(A \cup B) + P(C) - P(B \cap C)$
$= 0.482 + 0.55 - x$
$= 1.032 - x$
Since $P(A \cup B \cup C) \leq 1$
$1.032 - x \leq 1$
$x \geq 0.032$
$\text{Min } x = \min P(B \cap C) = 0.032$

Q9

(i)

P I O N T E 2A 2S
no. of ways $= \dfrac{10!}{2!2!} = 907200$

(ii)

Method 1 (Slotting):
The 8 letters excluding P & N can be arranged in $\dfrac{8!}{2!2!}$ ways.
The letters P & N can be arranged in the 9 spaces available in ${}^{9}C_2 \times 2!$ ways.
Hence total no. of ways $= \dfrac{8!}{2!2!} \times {}^{9}C_2 \times 2! = 725760$

Method 2 (Complement):
No. of ways where P & N are together $= \dfrac{9!}{2!2!} \times 2! = 181440$
Hence no. of ways where P & N are not next to each other = 907200 − 181440 = 725760

(iii)

The consonants and vowels must be arranged like this:
C V C V C V C V C V
or V C V C V C V C V C

The 5 consonants can be arranged in $\frac{5!}{2!} = 60$ ways.

The 5 vowels can be arranged in $\frac{5!}{2!} = 60$ ways.

Hence total no. of ways = $60 \times 60 \times 2 = 7200$.

(iv)

Case 1: 2 S between (i.e. P S S N)

Group "PSSN" as 1 unit and arrange with the other 6 letters (including 2As)

no. of ways = $\frac{7!}{2!} \times 2!$ (to arrange P and N) = 5040

Case 2: one S and one other letter (I, O, T or E) between, without A
No. of ways to choose the other letter to be between P and N = ${}^{4}C_{1} = 4$
Group "P_SN" as 1 unit and arrange with the other 6 letters (including 2As)

no. of ways = $4 \times \frac{7!}{2!} \times 2!$ (to arrange P and N) $\times 2!$ (to arrange S and the other letter) = 40320

Case 3: one S and one A (i.e. PSAN)
Group "PSAN" as 1 unit and arrange with the other 6 letters (including only 1 A)

no. of ways = $7! \times 2!$ (to arrange P and N) $\times 2!$ (to arrange S and A) = 20160
Hence total no. of ways = $5040 + 40320 + 20160 = 65520$

Q10

(i)

$\bar{x} = \frac{920}{100} = 9.2$

Unbiased estimate of the population variance = $s^2 = \frac{1}{99}\left[\sum x^2 - \frac{\left(\sum x\right)^2}{100}\right]$

$$= \frac{1}{99}\left[8668 - \frac{(920)^2}{100}\right] \approx 2.0606$$

Let X = time taken by each visitor
μ = mean time taken by each visitor

$H_0: \mu = 9$
$H_1: \mu > 9$

Under H_0, since $n = 100$ is large, by Central Limit Theorem,
$\bar{X} \sim N\left(9, \frac{2.0606}{100}\right)$ approximately

Test statistic, $z = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} = \frac{9.2 - 9}{\sqrt{\frac{2.0606}{100}}} \approx 1.3933 \approx 1.39$

From GC, p-value = $0.081770 \approx 0.0818$
Level of significance, $\alpha = 0.1$

Since p-value < α, we reject H_0. There is sufficient evidence, at the 10% level of significance, to conclude that the management's claim is **valid**.

(ii)

Since sample size $n = 100$ is large, Central Limit Theorem is used to approximate the distribution of the sample mean, $\bar{X} \sim N\left(9, \frac{2.0606}{100}\right)$ approximately. This is necessary in order to conduct the z-test since the distribution of the population is unknown.

(iii)

Let W = time taken by each visitor in the combined sample

$\sum w = \sum x + \sum y = 1470$
$\sum w^2 = \sum x^2 + \sum y^2 = 13788$
$n = 100 + 60 = 160$

New sample mean $\bar{w} = \frac{\sum w}{160} \approx 9.1875$

Unbiased estimate $s^2 = \frac{1}{159}\left\{13788 - \frac{(1470)^2}{160}\right\} \approx 1.7759$

$H_0: \mu = \mu_0$
$H_1: \mu > \mu_0$

Under H_0, since $n = 160$ is large, by Central Limit Theorem,
$\bar{W} \sim N\left(\mu_0, \frac{1.7759}{160}\right)$ approximately

Test statistic, $z = \dfrac{\overline{w}-\mu_0}{\dfrac{s}{\sqrt{n}}} = \dfrac{9.1875-\mu_0}{\sqrt{\dfrac{1.7759}{160}}} = \dfrac{\sqrt{160}\,(9.1875-\mu_0)}{\sqrt{1.7759}}$

Level of significance: 10%.

Critical region: $z > 1.2816$

Since H_0 is not rejected,

$$\frac{\sqrt{160}\,(9.1875-\mu_0)}{\sqrt{1.7759}} < 1.2816$$

$$9.1875-\mu_0 < 0.13502$$

$$\mu_0 > 9.05$$

Q11

(i)

The event that a customer uses e-payment for his chicken rice order is independent of any other customer.

The probability that a customer uses e-payment for his chicken rice order is $\dfrac{1}{p}$, which is constant.

(ii)

Let X = Number of customers who use e-payment for their chicken rice orders out of 50.

Then $X \sim \mathrm{B}\left(50, \dfrac{1}{p}\right)$.

Consider $\quad P(X=8) > P(X=7)$

$$^{50}C_8\left(\frac{1}{p}\right)^8\left(1-\frac{1}{p}\right)^{42} > {}^{50}C_7\left(\frac{1}{p}\right)^7\left(1-\frac{1}{p}\right)^{43}$$

$$536878650\left(\frac{1}{p}\right)^8\left(1-\frac{1}{p}\right)^{42} > 99884400\left(\frac{1}{p}\right)^7\left(1-\frac{1}{p}\right)^{43}$$

$$\frac{43}{8}\left(\frac{1}{p}\right) > 1-\frac{1}{p} \qquad \text{since } \frac{1}{p}>0,\ 1-\frac{1}{p}>0$$

$$p < \frac{51}{8}$$

Consider $P(X=8) > P(X=9)$

$$^{50}C_8\left(\frac{1}{p}\right)^8\left(1-\frac{1}{p}\right)^{42} > {}^{50}C_9\left(\frac{1}{p}\right)^9\left(1-\frac{1}{p}\right)^{41}$$

$$536878650\left(\frac{1}{p}\right)^8\left(1-\frac{1}{p}\right)^{42} > 2505433700\left(\frac{1}{p}\right)^9\left(1-\frac{1}{p}\right)^{41}$$

$$\frac{14}{3}\left(\frac{1}{p}\right) < 1-\frac{1}{p} \qquad \text{since } \frac{1}{p}>0,\ 1-\frac{1}{p}>0$$

$p > \frac{17}{3}$

$\frac{17}{3} < p < \frac{51}{8}$ $(\text{or } 5.6667 < p < 6.375)$

Therefore, integer value of $p = 6$.

(iii)

$X \sim \text{B}\left(50, \frac{1}{6}\right)$

$\text{P}(9 < X \leq 15) = \text{P}(X \leq 15) - \text{P}(X \leq 9) \approx 0.31122 \approx 0.311$

(iv)

Let Y = Number of days with more than 9 and no more than 15 customers using e-payment during lunch time out of 6 days.

$Y \sim \text{B}(6, 0.31122)$.

$\text{P}(Y > 2) = 1 - \text{P}(Y \leq 2) \approx 0.27674 \approx 0.277$

(v)

$\text{E}(X) = \frac{50}{6}$, $\text{Var}(X) = \frac{250}{36}$

Since sample size $n = 36$ is large, by Central Limit Theorem,

$\overline{X} \sim \text{N}\left(\frac{50}{6}, \frac{125}{648}\right)$ approximately where $\overline{X} = \frac{X_1 + X_2 + \ldots + X_{36}}{36}$

$\text{P}(\overline{X} > 8) \approx 0.776$